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AN ANALYTIC EXTENSION OF A SPACELIKE MAXIMAL SURFACE

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It is shown that a spacelike maximal surface in the three dimensional Lorentz-Minkowski space can be extended analytically if it meets a spacelike plane at a constant hyperbolic angle.

Let $\mathbb{L}^3$ be the three dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric tensor $\langle\cdot,\cdot\rangle$ given by $\langle\cdot,\cdot\rangle = dx^2+dy^2-dt^2$ where (x,y,t) are the canonical coordinate of $\mathbb{R}^3$. An immersed surface $\Sigma\subset\mathbb{L}^3$ is called *spacelike* if the induced metric on Σ is a Riemannian metric, which is equivalent to the fact that the unit normal vector field η to Σ is a timelike vector field. If the trace of the map $d\eta : T\Sigma \to T\Sigma$ is zero everywhere on Σ, the surface Σ is called a *maximal* surface. It is well knows that for a spacelike maximal surface Σ the coordinate functions $x, y, t : \Sigma \to \mathbb{R}$ are harmonic functions and hence it admits a Weierstrass representation [4], similar to minimal surfaces in the three dimensional Euclidean space $\mathbb{E}^3$. But it is very different from the minimal surfaces in $\mathbb{E}^3$ in that it has naturally arising singularities due to the geometry of the unit normal vector field η.

If a spacelike maximal surface Σ has no singular point, the unit normal vector field can be considered as a map $\eta : \Sigma \to \mathbb{H}^2 = \{(x,y,t) : x^2+y^2-t^2=-1\}$. Let $\sigma : \mathbb{C} - \{|z|=1\} \to \mathbb{H}^2$ be the stereographic projection defined by

$$\sigma(z) = \Big(\frac{2\operatorname{Re}(z)}{1-|z|^2}, \frac{2\operatorname{Im}(z)}{1-|z|^2}, \frac{1+|z|^2}{1-|z|^2}\Big), \quad \sigma(\infty) = (0,0,-1),$$

that is, $\sigma(z)$ is the intersection of $\mathbb{H}^2$ and the line joining the point $(\operatorname{Re}(z), \operatorname{Im}(z), 0)$ and "the south pole" $(0,0,-1)$ of $\mathbb{H}^2$. It is well known that σ is conformal in the natural manner. Then one has a complex-valued conformal Gauss map $\sigma^{-1}\circ\eta : \Sigma \to \mathbb{C} - \{|z| = 1\}$. If, moreover, Σ is connected (which is assumed in this paper), one has by the connectivity either $\eta : \Sigma \to \mathbb{H}^2_+ = \{(x,y,t) : x^2+y^2-t^2=-1,\ t>0\}$ and consequently $|\sigma^{-1}\circ\eta(p)| < 1$ for every $p\in\Sigma$ or $\eta : \Sigma \to \mathbb{H}^2_- = \{(x,y,t) : x^2+y^2-t^2=-1,\ t<0\}$

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and consequently $\left|\sigma^{-1} \circ \eta(p)\right| > 1$ for every $p \in \Sigma$. Hence, if Σ has a singular point, say $q \in \Sigma$, then one has $\left|\sigma^{-1} \circ \eta(q)\right| = 1$ and vice versa.

On the other hand, it has long been known that in the three dimensional Euclidean space $\mathbb{E}^3$ one can reflect a minimal surface across a part of its boundary if the minimal surface meets a plane at a constant angle (not necessarily 90 degrees) along the boundary [**3**]. The proof of this fact makes use of H.A. Schwarz's reflection principle for holomorphic functions.

Then, since a spacelike maximal surface in $\mathbb{L}^3$ is represented with holomorphic data, similar to a minmal surface in $\mathbb{E}^3$, one may expect that a spacelike maximal surface in $\mathbb{L}^3$ has the same reflection property. In fact, the argument in [**3**] gives the following theorem.

Theorem 1. *Let $\Sigma \subset \mathbb{L}^3$ be a spacelike maximal surface (possibly with singular points) and let Π be a spacelike plane. Suppose that $L \subset \Sigma \cap \Pi$ is a C^1-curve, Σ is C^1 along L and at all points of L the tangent plane to Σ makes a constant hyperbolic angle $\theta > 0$ with Π. Then Σ can be analytically extended across L to a spacelike maximal surface $\overline{\Sigma}$ satisfying the following properties:*

(i) $\overline{\Sigma} = \Sigma \cup \Sigma^*$ *where Σ^* is the set of all images p^* of $p \in \Sigma$ under the analytic extension map $*$.*

(ii) *Two points p and p^* are separated by Π in such a way that*

$$d(p, \Pi) = d(p^*, \Pi)$$

where d is the Lorentzian distance.

(iii) *The Gauss map $g : \overline{\Sigma} \to \mathbb{C}$ satisfies*

$$\overline{g(p)}g(p^*) = \tanh^2(\theta/2).$$

For the definition of the hyperbolic angle, see for example, [**2**, p. 57].

Remark 1. Since the hypothesis on L requires that the tangent plane is defined at every point of L, it is assumed implicitly in Theorem 1 that L contains no singular point of Σ.

Remark 2. For a given *real analytic* curve γ in the spacelike plane Π, the existence of the spacelike maximal surface Σ which meets Π along γ at a constant hyperbolic angle $\theta \neq 0$ is guaranteed by the Björling's representation formula [**1**].

Proof: We may assume that $\Pi = \{(x, y, t,) : t = 0\}$ and $\eta(\Sigma) \subset \mathbb{H}^2_+$. Since x, y, t are harmonic functions on the spacelike maximal surface Σ, one can find the conjugate harmonic (possibly multi-valued) functions $\overline{x}, \overline{y}, \overline{t}$ to x, y, t respectively on Σ. Then

$$u = x + i\overline{x},\ v = y + i\overline{y},\ w = t + i\overline{t}$$

are holomorphic (possibly multi-valued) functions on Σ and

$$du = dx + id\overline{x},\ dv = dy + id\overline{y},\ dw = dt + id\overline{t}$$

are holomorphic 1-forms on Σ. Introduce $t, \bar{t}$ as conformal parameters on Σ. Then Σ can be recaptured by setting

$$x = \operatorname{Re} \int^w du, \ y = \operatorname{Re} \int^w dv, \ t = \operatorname{Re} \int^w dw.$$

From the holomorphicity of u, v, w, it follows that

$$du^2 + dv^2 - dw^2 \equiv 0.$$

Define a meromorphic function g on Σ by

$$g = \frac{dw}{du - idv}$$

Then we have

$$\text{(1)} \qquad \begin{aligned} x &= \operatorname{Re} \int^w \frac{1}{2}\Big(g + \frac{1}{g}\Big)\, dw, \\ y &= \operatorname{Re} \int^w -\frac{i}{2}\Big(g - \frac{1}{g}\Big)\, dw, \\ t &= \operatorname{Re} \int^w dw \end{aligned}$$

Put

$$-\Sigma = \big\{(x, y, -t) : (x, y, t) \in \Sigma\big\}$$

and define a surface

$$\tilde{\Sigma} = \Sigma \cup (-\Sigma).$$

For any $p \in \Sigma$, let $-p = (x, y, -t) \in -\Sigma$. Since $t = 0$ on $L \subset \Sigma \cap (-\Sigma)$, we can extend the conformal parameters $t, \bar{t}$ over $\tilde{\Sigma}$ across L by the usual reflection with respect to Π; that is

$$t(-p) = -t(p), \ \bar{t}(-p) = \bar{t}(p)$$

for any $-p \in (-\Sigma)$. Hence we see that dw is a well-defined holomorphic 1-form on the surface $\tilde{\Sigma}$.

On the other hand, it is well known that g is the same as the complex-valued Gauss map, $g = \sigma^{-1} \circ \eta$. Then the constant hyperbolic angle hypothesis implies

$$|g(p)| = \frac{\sinh \theta}{\cosh \theta + 1} = \tanh(\theta/2)(\neq 1)$$

for all $p \in L$. That is, g maps L into a circle of radius $\neq 1$ in $\mathbb{C}$. Since Σ is C^1 along L and L plays the same role in the surface $\tilde{\Sigma}$ as a line does in $\mathbb{C}$, we can extend g holomorphically over $\tilde{\Sigma}$ across L as follows:

Define the extension of g, still called g, by

$$\text{(2)} \qquad g(-p) = \tanh^2(\theta/2)\overline{g(p)}^{-1}, \ -p \in (-\Sigma).$$

Clearly g is holomorphic on $-\Sigma$ and continuous on $\widetilde{\Sigma}$. Let $h : \mathbb{C} \to \mathbb{C}$ be a linear fractional transformation which maps the circle $|w| = \tanh(\theta/2)$ onto the imaginary axis of $\mathbb{C}$. Then the real part of $h \circ g$ is continuous on $\widetilde{\Sigma}$ and harmonic on Σ and $-\Sigma$. Moreover, we have

$$\begin{aligned} \operatorname{Re}\bigl[h \circ g(-p)\bigr] &= \operatorname{Re}\bigl[h \circ g(p)\bigr] = 0 \quad \text{for} \quad p \in L, \\ \operatorname{Re}\bigl[h \circ g(-p)\bigr] &= -\operatorname{Re}\bigl[h \circ g(p)\bigr] \qquad \text{for} \quad -p \in (-\Sigma). \end{aligned}$$

Hence by the reflection principle we conclude that $h \circ g$ is holomorphic on $\widetilde{\Sigma}$ and so is g.

Using this extended map g, the extended 1-form dw and the representation formula (1), we can define the analytic extension map $*$ as follows:

> For any $p \in \Sigma$, p^* is determined by integrating (1) over a contour on $\widetilde{\Sigma}$ from a fixed point to $-p$.

Then we can obtain the extended spacelike maximal surface $\overline{\Sigma} = \Sigma \cup \Sigma^*$. This completes the proof of (i).

Conclusion (ii) follows from symmetry of $-\Sigma$ to Σ and the formula for t in (1).

Conclusion (iii) follows from (2). $\square$

Let Σ be the spacelike maximal surface in Theorem 1. For any real number $0 < r < 1$, let us denote by Σ_r the spacelike maximal surface in $\mathbb{L}^3$ defined by the formula

$$\begin{aligned} x &= \operatorname{Re} \int^w \frac{1}{2}\Bigl(rg + \frac{1}{rg}\Bigr)\, dw, \\ y &= \operatorname{Re} \int^w -\frac{i}{2}\Bigl(rg - \frac{1}{rg}\Bigr)\, dw, \\ t &= Re \int^w dw. \end{aligned}$$

Then we see that Σ can be deformed into a 1-parameter family of spacelike maximal surfaces and that this deformation preserves t-coordinates and multiplies g by r. As a corollary of the proof, we have the following theorem:

THEOREM 2. *Let $\Sigma \subset \mathbb{L}^3$ be a spacelike maximal surface with nonempty boundary $\partial\Sigma$ which makes a constant angle $\theta > 0$ with the spacelike plane Π along $\partial\Sigma \cap \Pi$. For any $\alpha > 0$, there exists an $r > 0$ such that the spacelike maximal surface Σ_r makes a constant angle α with Π along $\partial\Sigma_r \cap \Pi$.*

PROOF: We have $t = 0$ on every point of $\partial\Sigma \cap \Pi$. Since the deformation preserves the t-coordinate, we have $t = 0$ as well on every point of $\partial\Sigma_r \cap \Pi$. Now take

$$r = \tanh(a/2)\bigl[\tanh(\theta/2)\bigr]^{-1}.$$

Then we have

$$\{t = 0\} = \bigl\{|g| = \tanh(\theta/2)\bigr\} = \bigl\{|rg| = \tanh(a/2)\bigr\},$$

which implies that, since rg is the complex-valued Gauss map if Σ_r, the surface Σ_r makes a constant angle α with Π along $\partial\Sigma_r \cap \Pi$. ∎

The following example shows a behaviour of the singular point under the analytic extension, which cannot happen in the case of minimal surfaces in $\mathbb{E}^3$.

EXAMPLE. Let Σ be an elliptic catenoid $\Sigma = \{(x, y, t) : x^2 + y^2 - \sinh^2 t = 0\}$ which has the singular point $(0,0,0)$. Now consider $\Sigma_{[a,b)} = \{(x, y, t) : x^2 + y^2 - \sinh^2 t = 0, 0 \neq a \leqslant t < b\} \subset \Sigma$ which meets the spacelike plane $\Pi_a = \{(x, y, t) : t = a\}$ at a constant hyperbolic angle. Note that Σ is a surface of rotation whose axis of rotation is the t-axis. Then by (ii) of Theorem 1, the extended surface $\overline{\Sigma}_{[a,b)}$ of $\Sigma_{[a,b)}$ is

$$\overline{\Sigma}_{[a,b)} = \Sigma_{(2a-b,b)} = \{(x, y, t) : x^2 + y^2 - \sinh^2 t = 0,\ 2a - b < t < b\}.$$

We first consider the case when $a > 0$.

(i) If $b < 2a$, since $2a - b > 0$, neither $\Sigma_{[a,b)}$ nor the extended surface $\overline{\Sigma}_{[a,b)}$ have singular points.

(ii) If $b > 2a$, the surface $\Sigma_{[a,b)}$ has no singular point but the extended surface $\overline{\Sigma}_{[a,b)}$ contains a singular point $(0,0,0)$ since $2a - b < 0 < b$. In fact, every point $(x, y, 2a) \in \Sigma_{[a,b)}$ reflects to the singular point $(0,0,0)$. This happens because the set $\{(x, y, 2a)\} \subset \Sigma_{[a,b)}$ is parameterised by $\{|g| = c\}$ for a constant $c \neq 0, 1$ which reflects to the set parametrised by $\{|g| = 1\}$, which is the (singular) parametrisation of the singular point $(0,0,0)$.

(iii) If $b = \infty$, the surface $\Sigma_{[a,b)}$ extends to make the whole elliptic catenoid Σ.

We next consider the case when $a < 0$.

(iv) If $b < 0$, the same case as (i) or (ii) occurs.

(v) If $b > 0$, the surface $\Sigma_{[a,b)}$ contains the singular point $(0,0,0)$ and the singular point $(0,0,0)$ reflects to the whole $\{(x, y, 2a)\}$. The reason of this result is the same as the case (ii).

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