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A strengthening of the GL(2) converse theorem

Andrew R. Booker and M. Krishnamurthy

Dedicated to Ilya Piatetski-Shapiro (1929–2009)

Abstract

We generalize the method of A. R. Booker (*Poles of Artin L-functions and the strong Artin conjecture*, Ann. of Math. (2) **158** (2003), 1089–1098; MR 2031863(2004k:11082)) to prove a version of the converse theorem of Jacquet and Langlands with relaxed conditions on the twists by ramified idèle class characters.

Contents

1. Introduction

The 'converse theorem' in the theory of automorphic forms has a long history, beginning with the work of Hecke [Hec36] and a paper of Weil [Wei67] relating the automorphy relations satisfied by classical holomorphic modular forms f to analytic properties of the twisted L-functions $L(s, f \times \chi)$ for Dirichlet characters χ. Soon after, the classical theory was recast in the modern

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setting of automorphic representations by Jacquet and Langlands [JL70], who generalized Weil's result to GL_2 representations π over a global field, characterizing them in terms of their twists $L(s, \pi \otimes \omega)$ by idèle class characters ω. The converse theorem has since been a cornerstone of the theory, with its generalizations responsible for some of the most striking results of the Langlands program.

In this paper we show that one may relax the hypotheses of the Jacquet and Langlands theorem over number fields by allowing the twisted L-functions $L(s, \pi \otimes \omega)$ to have essentially arbitrary poles for all ω that are ramified at a finite place. Our approach is based on that of [Boo03], where it was shown that twists can be eliminated altogether in the converse theorem for two-dimensional complex Galois representations ρ over $\mathbb{Q}$; in other words, whenever $L(s, \rho)$ is holomorphic, as predicted by Artin's conjecture, there is a modular form f (of either holomorphic or Maass type) such that $L(s, \rho) = L(s, f)$, as predicted by Langlands' functoriality conjecture. The possibility of extending this method to a version of the converse theorem with restrictions on the twists was alluded to in [Boo03], and the present paper is an attempt to realize that goal in the case of GL_2 over a number field.

The precise result that we prove here is the following.

Theorem 1.1. *Let F be a number field, $\mathbb{A}_F$ its ring of adèles and $\pi = \bigotimes_v \pi_v$ an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central idèle class character ω_π, such that π_v is unitary for all archimedean places v. For every (unitary) idèle class character ω, suppose that the complete L-functions*

$$\Lambda(s, \pi \otimes \omega) = \prod_v L(s, \pi_v \otimes \omega_v) \quad \text{and} \quad \Lambda(s, \widetilde{\pi} \otimes \omega^{-1}) = \prod_v L(s, \widetilde{\pi}_v \otimes \omega_v^{-1})$$

(i) *converge absolutely and define analytic functions in some right half-plane $\Re(s) > \sigma$;*

(ii) *continue meromorphically to ratios of entire functions of finite order;*

(iii) *satisfy the functional equation*

$$\Lambda(s, \pi \otimes \omega) = \epsilon(s, \pi \otimes \omega)\Lambda(1 - s, \widetilde{\pi} \otimes \omega^{-1}),$$

where $\epsilon(s, \pi \otimes \omega)$ is as in [JL70, Theorem 11.3];

(iv) *are entire whenever ω is unramified at every non-archimedean place.*

Then π is an automorphic representation.

Corollary 1.2. *Let $\rho : W_F \to \mathrm{GL}_2(\mathbb{C})$ be a representation of the Weil group of F. Suppose that the associated L-functions $\Lambda(s, \rho \otimes \omega)$ are entire for all idèle class characters ω that are unramified at every non-archimedean place. Then ρ is automorphic, i.e. there is an automorphic representation π such that π_v corresponds to ρ_v under the local Langlands correspondence for each place v.*

Proof. For each place v, let π_v be the representation corresponding to ρ_v under the local Langlands correspondence, and form $\pi = \bigotimes \pi_v$. The analytic properties of the Weil L-functions $\Lambda(s, \rho \otimes \omega) = \Lambda(s, \pi \otimes \omega)$, including hypotheses (i) to (iii) of Theorem 1.1, are summarized in [JL70, ch. 2, § 12]. In particular, (ii) follows from Brauer's theory of induced characters and the properties of Hecke L-functions, and (iii) was shown by Langlands [Lan69] and Deligne [Del73]. □

Remarks. (i) Note that π need not be cuspidal, since we allow poles. The cuspidality criterion is Theorem 10.10 of Jacquet and Langlands [JL70].

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(ii) Theorem 1.1 is a direct generalization of Theorem 11.3 of Jacquet and Langlands [JL70], except for the mild assumption that π_v be unitary for archimedean places v. Although this assumption is unlikely to present an obstacle in applications, one naturally wonders whether it is necessary. We remark that even without this hypothesis, our method would show that the *finite* L-functions

$$L(s,\pi\otimes\omega)=\prod_{v<\infty}L(s,\pi_v\otimes\omega_v)\quad\text{and}\quad L(s,\widetilde{\pi}\otimes\omega^{-1})=\prod_{v<\infty}L(s,\widetilde{\pi}_v\otimes\omega_v^{-1})$$

are entire for all idèle class characters ω that have no finite places of ramification in common with π; thus, the unitary hypothesis is only needed to preclude the possibility of poles arising from the L-factors at archimedean places.

(iii) A different approach to weakening the hypotheses of the Jacquet and Langlands theorem was taken by Piatetski-Shapiro [Pja75] and later by Li [Li79]. In particular, both authors showed that the automorphy of a given representation is determined by analytic properties of twists by characters unramified outside of a finite set S of places. In some cases one may take S to be the set of infinite places, and thus give a stronger result than Theorem 1.1. In fact, Piatetski-Shapiro's paper shows that this is the case if $F\neq\mathbb{Q}$ has class number 1 and at least one real embedding. However, it does not seem possible to reduce S to the set of infinite places in general.

(iv) Analogous results are known in some cases for representations of GL_n for $n>2$. For instance, Cogdell and Piatetski-Shapiro [CKM04] proved a version of the converse theorem for GL_n over a number field F assuming analytic properties of all twists by unramified GL_{n-1} representations, provided that F has class number 1. It seems likely that the methods of this paper will generalize and make it possible to remove the class number restriction; we will investigate this in future papers.

The outline of the paper is as follows. We conclude the introduction by recalling the standard notation and conventions for number fields and their rings of adèles. Next, as the arguments are rather technical in nature, in § 2 we provide a sketch of the proof in classical notation for holomorphic modular forms over $\mathbb{Q}$; this is based on a version of the argument of [Boo03] due to Sarnak [Sar02]. There are a few noteworthy differences between the result over $\mathbb{Q}$ and the general case. First, it is not immediately obvious what should play the role of the 'additive twists' from classical analytic number theory, which are important in the proof; we explain the relevant notion in § 3, as well as its connection to the 'multiplicative twists' by Größencharakters (or, equivalently, idèle class characters). Second, while there are no non-trivial unramified idèle class characters over $\mathbb{Q}$, there are infinitely many such characters over a number field; thus, unlike the result of [Boo03], it does not seem possible in general (by these methods) to deduce automorphy from analytic properties of a single L-function.[1] We recall some of the background on Whittaker functions and Fourier expansions in § 4 before establishing the connection between analytic properties of twists by unramified characters and automorphy relations in § 5.2. The proof of Theorem 1.1 takes up the remainder of § 5, relying on some combinatorial identities presented in the appendix.

1.1 Notation

Let F be a number field and $\mathfrak{o}_F$ its ring of integers. For each place v of F, we denote by F_v the completion of F at v. To avoid confusion at archimedean places, we will use the symbol $\|\cdot\|_v$ to

[1] This is in line with Sarnak's suggestion that over a number field it is most natural to consider a representation π together with its twists $\pi\otimes\omega$ by unramified idèle class characters. For a recent example of this phenomenon, see the results of Diaconu and Garrett [DG09] on moments of GL_2 L-functions.

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denote the absolute value on F_v, and reserve $|\cdot|$ for the usual absolute value of real or complex numbers. Let S_∞ denote the set of archimedean places of F, and define

$$S_{\mathbb{C}} = \{v \in S_\infty : F_v = \mathbb{C}\}, \quad S_{\mathbb{R}} = \{v \in S_\infty : F_v = \mathbb{R}\}.$$

We will write $v|\infty$ and $v<\infty$ to mean $v \in S_\infty$ and $v \notin S_\infty$, respectively. For $v<\infty$, we let $\mathfrak{o}_v$ denote the ring of integers of F_v, $\mathfrak{p}_v$ the unique prime ideal of $\mathfrak{o}_v$, $\mathfrak{o}_v^\times$ the group of local units and q_v the cardinality of $\mathfrak{o}_v/\mathfrak{p}_v$. We write $\mathbb{A}_F$ for the ring of adèles of F and $\mathbb{A}_F^\times$ for its group of idèles. The symbol $\mathbb{A}_{F,f}$ will denote the ring of finite adèles and F_∞ will denote $\prod_{v|\infty} F_v$, so that $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$; we write x_∞ and x_f for the corresponding components of $x \in \mathbb{A}_F$.

We regard F as a subring of $\mathbb{A}_F$, embedded diagonally, and fix an additive character $\psi = \bigotimes_v \psi_v$ of $F\backslash\mathbb{A}_F$ whose conductor is the inverse different $\mathfrak{d}^{-1}$ of F. Namely, $\psi = \psi_{\mathbb{Q}} \circ \mathrm{tr}$, where tr is the trace map from $\mathbb{A}_F$ to $\mathbb{A}_{\mathbb{Q}}$, and $\psi_{\mathbb{Q}}$ is the additive character of $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ which is unramified at all finite primes and whose restriction to $\mathbb{R}$ is the exponential function $e(x) = e^{2\pi i x}$. Further, let d be a finite idèle such that $(d) = \mathfrak{d}$, where, for any $t \in \mathbb{A}_F^\times$, we write (t) to denote the fractional ideal $(t) = \prod_{v<\infty} (\mathfrak{p}_v \cap F)^{\mathrm{ord}_v(t_v)}$.

Suppose that $v|\infty$ and $\chi_v : F_v^\times \to \mathbb{C}^\times$ is a continuous quasi-character. Then, for $v \in S_{\mathbb{R}}$, χ_v may be written uniquely in the form $\chi_v(y) = \|y\|_v^{\nu(\chi_v)} \operatorname{sgn}_v(y)^{\epsilon(\chi_v)}$, where $\operatorname{sgn}_v : F_v^\times \to \{\pm 1\}$ is the local sign character, $\nu(\chi_v) \in \mathbb{C}$ and $\epsilon(\chi_v) \in \{0,1\}$; similarly, for $v \in S_{\mathbb{C}}$, we have $\chi_v(y) = \|y\|_v^{\nu(\chi_v)} \theta_v(y)^{k(\chi_v)}$, where $\theta_v(y) = y\|y\|_v^{-1/2}$, $\nu(\chi_v) \in \mathbb{C}$ and $k(\chi_v) \in \mathbb{Z}$.

We fix our choice of Haar measure on the idèle class group as follows. For each finite place v of F, let $d^\times y_v$ be the Haar measure on $F_v^\times$ such that the volume of $\mathfrak{o}_v^\times$ is 1. For $v|\infty$, let dy_v be the ordinary Lebesgue measure; we then set $d^\times y_v = dy_v/2\|y_v\|_v$ for $v \in S_{\mathbb{R}}$ and $d^\times y_v = dy_v/\pi\|y_v\|_v$ for $v \in S_{\mathbb{C}}$. This choice of local Haar measures yields a unique Haar measure $d^\times y$ on $\mathbb{A}_F^\times$ such that the volume of $\prod_{v<\infty} \mathfrak{o}_v^\times$ is 1. Moreover, the Mellin inversion formula at an archimedean place v takes the form

$$\widetilde{f}(s,\epsilon) = \int_{F_v^\times} f(y) \operatorname{sgn}_v(y)^\epsilon \|y\|_v^{s-\frac{1}{2}}\, d^\times y \iff f(y) = \sum_{\epsilon \in \{0,1\}} \int \widetilde{f}(s,\epsilon) \operatorname{sgn}_v(y)^\epsilon \|y\|_v^{\frac{1}{2}-s} \frac{ds}{2\pi i}$$

for $v \in S_{\mathbb{R}}$ and

$$\widetilde{f}(s,k) = \int_{F_v^\times} f(y)\theta_v(y)^k \|y\|_v^{s-\frac{1}{2}}\, d^\times y \iff f(y) = \sum_{k \in \mathbb{Z}} \int \widetilde{f}(s,k)\theta_v(y)^{-k} \|y\|_v^{\frac{1}{2}-s} \frac{ds}{2\pi i}$$

for $v \in S_{\mathbb{C}}$, where in each case the s-integral is taken along a vertical line to the right of any poles of the integrand.

Recall that a *Größencharakter* of conductor $\mathfrak{q}$ is a multiplicative function χ of non-zero integral ideals satisfying $\chi(a\mathfrak{o}_F) = \chi_f(a)\chi_\infty(a)$ for associated characters $\chi_f : (\mathfrak{o}_F/\mathfrak{q})^\times \to S^1$ and $\chi_\infty : F_\infty^\times \to S^1$, with χ_f primitive and χ_∞ continuous, and all $a \in \mathfrak{o}_F$ relatively prime to $\mathfrak{q}$. By convention, we set $\chi(\mathfrak{a}) = 0$ for any ideal $\mathfrak{a}$ with $(\mathfrak{a},\mathfrak{q}) \neq 1$. The Größencharakters are in one-to-one correspondence with idèle class characters $\omega : F^\times\backslash\mathbb{A}_F^\times \to S^1$, and the correspondence is such that $\chi_\infty = \omega_\infty^{-1}$. (Note that by a *character* we always mean a unitary character, and use the word *quasi-character* for the more general notion.)

Let Cl_F be the class group of F, and h its order. We fix a set $\{t_j\}_{j=1}^h$ of finite idèles such that the fractional ideals (t_j) represent the ideal classes of F.

Finally, we recall the standard notation of analytic number theory; we write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ to mean that there is a number $C \geqslant 0$ such that $|f(x)| \leqslant Cg(x)$ for all values of x, where the set of possible x is taken from context. We write O_y or $\ll_y$ to indicate that the number C implied by the notation depends on the variable y; the reader should beware, however, that this is mainly for emphasis, and we will not necessarily indicate every variable on which a particular expression depends.

2. The classical case

In this section we sketch the proof of Theorem 1.1 in the classical case of holomorphic modular forms for $\Gamma_0(N)$. For notational simplicity, we consider only trivial nebentypus character, though that is not essential to the argument.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers satisfying $a_n = O(n^{\sigma})$ for some $\sigma > 0$, and define $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ for z in the upper half-plane. By Weil's converse theorem [Wei67], $f \in M_k(\Gamma_0(N))$ for some $k \in 2\mathbb{Z}_{>0}$, $N \in \mathbb{Z}_{>0}$ if and only if the complete twisted L-functions

$$\Lambda(s, f \times \chi) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} a_n \chi(n) n^{-s},$$

which are initially defined for $\Re(s) > \sigma + 1$, continue to entire functions of finite order and satisfy the functional equation

$$\Lambda(s, f \times \chi) = \epsilon\chi(N) \frac{\tau(\chi)^2}{q} (q^2 N)^{k/2-s} \Lambda(k - s, f \times \overline{\chi}) \tag{2.1}$$

for some $\epsilon \in \{\pm 1\}$ and every primitive Dirichlet character χ of conductor q relatively prime to N. We relax these assumptions by allowing $\Lambda(s, f \times \chi)$ to have poles when $q > 1$, provided that all twists, including those with $(q, N) \neq 1$, extend meromorphically to ratios of entire functions of finite order, and that all poles are confined to a fixed vertical strip $\{s \in \mathbb{C} : \Re(s) \in [\sigma_1, \sigma_2]\}$. This follows from the hypotheses of Theorem 1.1 in the general setting.

We also consider the additively twisted L-functions

$$\Lambda(s, f, \alpha) = (2\pi|\alpha|)^{-s}\Gamma(s) \sum_{n=1}^{\infty} a_n e(n\alpha) n^{-s}$$

for $\alpha \in \mathbb{Q}^{\times}$. These are likewise defined initially for $\Re(s) > \sigma + 1$ but, by Proposition 3.1, they can be expressed in terms of the multiplicative twists $\Lambda(s, f \times \chi)$; thus, they also have meromorphic continuation to $\mathbb{C}$, with poles possible only for $\Re(s) \in [\sigma_1, \sigma_2]$, and are ratios of entire functions of finite order. Conversely, if χ is a primitive Dirichlet character of conductor q, then

$$\chi(n) = \frac{\tau(\chi)}{q} \sum_{a=1}^{q} \overline{\chi(-a)} e\left(\frac{an}{q}\right),$$

from which it follows that

$$\Lambda(s, f \times \chi) = \frac{\tau(\chi)}{q} \sum_{a=1}^{q} \overline{\chi(-a)} \left(\frac{a}{q}\right)^s \Lambda\left(s, f, \frac{a}{q}\right).$$

Hence, if we show that all additive twists $\Lambda(s, f, \alpha)$ are holomorphic for $\Re(s) \leqslant k/2$, then this, together with (2.1), implies that $\Lambda(s, f \times \chi)$ is entire. Since any entire function that can be expressed as a ratio of functions of finite order must itself have finite order, our hypotheses thus reduce to those of Weil's theorem, and the result follows.

Our starting point is an argument that goes back to Hecke. We have assumed that $\Lambda(s, f \times \chi)$ is entire when χ is the trivial character, and hence it must also have finite order, as remarked above. Hecke proved that these analytic properties, together with the functional equation, are equivalent to the automorphy relation

$$f(z) = \epsilon\left(\frac{i}{\sqrt{N}z}\right)^k f\left(-\frac{1}{Nz}\right). \tag{2.2}$$

Now, let $\alpha \in \mathbb{Q}^\times$ and set $\beta = -1/N\alpha$. We consider both sides of (2.2) evaluated at $z = \beta + i|\beta|y$ for a small $y > 0$. Using the Mellin transform identity $e^{-t} = (1/2\pi i)\int_{\Re(s)=\sigma'} \Gamma(s)t^{-s}\,ds$, valid for any $t, \sigma' > 0$, we have

$$f(z) = \sum_{n=1}^{\infty} a_n e(n\beta) e^{-2\pi n|\beta|y} = \sum_{n=1}^{\infty} a_n e(n\beta) \frac{1}{2\pi i}\int_{\Re(s)=\sigma+2} \Gamma(s)(2\pi n|\beta|y)^{-s}\,ds.$$

In view of the bound on a_n, we may change the order of sum and integral, to get

$$f(z) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma+2} (2\pi|\beta|y)^{-s}\Gamma(s)\sum_{n=1}^{\infty} a_n e(n\beta) n^{-s}\,ds = \frac{1}{2\pi i}\int_{\Re(s)=\sigma+2} \Lambda(s, f, \beta) y^{-s}\,ds.$$

On the other hand, by (2.2), we have

$$f(z) = \epsilon\left(\frac{i}{\sqrt{N}(\beta + i|\beta|y)}\right)^k f\left(-\frac{1}{N(\beta + i|\beta|y)}\right)$$

and, substituting

$$\frac{-1}{N(\beta + i|\beta|y)} = \alpha + i|\alpha|y - \frac{\alpha y^2}{1 - i\,\mathrm{sgn}(\alpha)y},$$

this yields

$$f(z) = \epsilon(-N\alpha^2)^{k/2}\sum_{n=1}^{\infty} a_n e(n\alpha) e^{-2\pi n|\alpha|y}(1 - i\,\mathrm{sgn}(\alpha)y)^{-k} e\left(-\frac{n\alpha y^2}{1 - i\,\mathrm{sgn}(\alpha)y}\right).$$

Note that if not for the correction factor $(1 - i\,\mathrm{sgn}(\alpha)y)^{-k} e(-n\alpha y^2/(1 - i\,\mathrm{sgn}(\alpha)y))$, which is approximately 1 for small y, the right-hand side could again be expressed simply in terms of $\Lambda(s, f, \alpha)$. Our strategy is to expand this factor in a Taylor series and treat each term separately; to that end, we have

$$\begin{aligned}
(1 - i\,\mathrm{sgn}(\alpha)y)^{-k} e\left(-\frac{n\alpha y^2}{1 - i\,\mathrm{sgn}(\alpha)y}\right) &= \sum_{j=0}^{\infty} (i\,\mathrm{sgn}(\alpha)y)^j (1 - i\,\mathrm{sgn}(\alpha)y)^{-j-k}\frac{(-2\pi n|\alpha|y)^j}{j!} \\
&= \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{j+k+\ell-1}{\ell}(i\,\mathrm{sgn}(\alpha)y)^{j+\ell}\frac{(-2\pi n|\alpha|y)^j}{j!} \\
&= \sum_{m=0}^{\infty}(i\,\mathrm{sgn}(\alpha)y)^m\sum_{j=0}^{m}\binom{m+k-1}{m-j}\frac{(-2\pi n|\alpha|y)^j}{j!},
\end{aligned}$$

converging absolutely for $y < 1$. Note further that for any $M, K \in \mathbb{Z}_{\geqslant 0}$ we have

$$\begin{aligned}
&\left|\sum_{m=M}^{\infty}(i\,\mathrm{sgn}(\alpha)y)^m\sum_{j=0}^{m}\binom{m+k-1}{m-j}\frac{(-2\pi n|\alpha|y)^j}{j!}\right| \\
&\quad\leqslant (2\pi n|\alpha|y)^{-K}K!\sum_{m=M}^{\infty} y^m\sum_{j=0}^{m}\binom{m+k-1}{m-j}\binom{j+K}{j}\frac{(2\pi n|\alpha|y)^{j+K}}{(j+K)!}
\end{aligned}$$

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$$\leqslant (\pi n|\alpha|y)^{-K} K!(3/2)^{k-1} \sum_{m=M}^{\infty} (3y)^m e^{2\pi n|\alpha|y}$$
$$\ll_{k,\alpha,M,K} y^M (ny)^{-K} e^{2\pi n|\alpha|y},$$

for $y < 1/4$, say. Hence, we have

$$f(z) = \epsilon(-N\alpha^2)^{k/2} \sum_{m=0}^{M-1} (i \operatorname{sgn}(\alpha))^m \sum_{j=0}^{m} \binom{m+k-1}{m-j} \frac{y^{j+m}}{j!} \frac{d^j}{dy^j} \sum_{n=1}^{\infty} a_n e(n\alpha) e^{-2\pi n|\alpha|y} + O\biggl(y^{M-K} \sum_{n=1}^{\infty} |a_n| n^{-K}\biggr).$$

Choosing $K = \lfloor\sigma\rfloor + 2$, the error term converges and gives the estimate $O(y^{M-K})$.

Applying Mellin inversion as before, we have

$$\frac{y^{j+m}}{j!} \frac{d^j}{dy^j} \sum_{n=1}^{\infty} a_n e(n\alpha) e^{-2\pi n|\alpha|y} = \frac{y^{j+m}}{j!} \frac{d^j}{dy^j} \frac{1}{2\pi i} \int_{\Re(s)=\sigma+m+2} \Lambda(s, f, \alpha) y^{-s}\, ds$$
$$= \frac{1}{2\pi i} \int_{\Re(s)=\sigma+2} \binom{-s-m}{j} \Lambda(s+m, f, \alpha) y^{-s}\, ds.$$

Moreover, by the Chu–Vandermonde identity, we have

$$\sum_{j=0}^{m} \binom{m+k-1}{m-j} \binom{-s-m}{j} = \binom{-s+k-1}{m} = (-1)^m \binom{s+m-k}{m}.$$

Putting everything together, we arrive at

$$f(z) = \epsilon(-N\alpha^2)^{k/2} \sum_{m=0}^{M-1} (-i \operatorname{sgn}(\alpha))^m \frac{1}{2\pi i} \int_{\Re(s)=\sigma+2} \binom{s+m-k}{m} \Lambda(s+m, f, \alpha) y^{-s}\, ds + O(y^{M-K}).$$

Although we have assumed that $y < 1/4$ throughout this analysis, from the exponential decay of $f(z)$ and by shifting the contour of the above to the right, we see that the error term decays rapidly as $y \to \infty$. Hence, we can take the Mellin transform of both sides, to get

$$\Lambda(s, f, \beta) = \epsilon(-N\alpha^2)^{k/2} \sum_{m=0}^{M-1} (-i \operatorname{sgn}(\alpha))^m \binom{s+m-k}{m} \Lambda(s+m, f, \alpha) + H_{\alpha,M}(s),$$

where $H_{\alpha,M}(s)$ is holomorphic for $\Re(s) > K - M$. Next, suppose that $\alpha = a/q$ with $a, q \in \mathbb{Z}$, $(a, q) = 1$, and let λ be a non-zero integer $\equiv 1 \pmod{q}$. Then $e(n\alpha) = e(n\lambda\alpha)$ for all $n \in \mathbb{Z}$, so that $\Lambda(s, f, \lambda\alpha) = |\lambda|^{-s} \Lambda(s, f, \alpha)$. Replacing α in the above by $\lambda\alpha$, we have

$$|\lambda|^{s-k} \Lambda(s, f, \lambda^{-1}\beta) = \epsilon(-N\alpha^2)^{k/2} \sum_{m=0}^{M-1} \lambda^{-m} (-i \operatorname{sgn}(\alpha))^m \binom{s+m-k}{m} \times \Lambda(s+m, f, \alpha) + |\lambda|^{s-k} H_{\lambda\alpha,M}(s).$$

Now, let T be a finite subset of $(1+q\mathbb{Z})\cap\mathbb{Q}^\times$, and let $c_\lambda\in\mathbb{C}$ for each $\lambda\in T$. Then

$$\epsilon(-N\alpha^2)^{k/2}\sum_{m=0}^{M-1}\biggl(\sum_{\lambda\in T}c_\lambda\lambda^{-m}\biggr)(-i\,\mathrm{sgn}(\alpha))^m\binom{s+m-k}{m}\Lambda(s+m,f,\alpha)\\ -\sum_{\lambda\in T}c_\lambda|\lambda|^{s-k}\Lambda(s,f,\lambda^{-1}\beta)$$

is holomorphic for $\Re(s)>K-M$.

Fix a non-negative integer $m_0<M$. From the Vandermonde determinant, we see that as long as T has cardinality at least M, we may choose the c_λ so that

$$\sum_{\lambda\in T}c_\lambda\lambda^{-m}=\begin{cases}1 & \text{if } m=m_0,\\ 0 & \text{if } m\neq m_0\end{cases}$$

for all non-negative integers $m<M$. Therefore,

$$\epsilon(-N\alpha^2)^{k/2}(-i\,\mathrm{sgn}(\alpha))^{m_0}\binom{s+m_0-k}{m_0}\Lambda(s+m_0,f,\alpha)-\sum_{\lambda\in T}c_\lambda|\lambda|^{s-k}\Lambda(s,f,\lambda^{-1}\beta)$$

is holomorphic for $\Re(s)>K-M$.

Recall that $\Lambda(s+m_0,f,\alpha)$ can only have poles in the strip $\{s\in\mathbb{C}:\Re(s)\in[\sigma_1-m_0,\sigma_2-m_0]\}$, while the sum over $\lambda\in T$ is holomorphic outside of $\{s\in\mathbb{C}:\Re(s)\in[\sigma_1,\sigma_2]\}$. Choosing $m_0>\sigma_2-\sigma_1$ and $M>m_0+\max(0,K-\sigma_1)$, we see that $\binom{s-k}{m_0}\Lambda(s,f,\alpha)$ must be entire. In particular, since $\binom{s-k}{m_0}$ has zeros only at integers $\geqslant k>k/2$, $\Lambda(s,f,\alpha)$ is holomorphic for $\Re(s)\leqslant k/2$, and this concludes the proof.

3. Additive and multiplicative twists

In this section we describe the functions that play the role of the additive twists in the preceding argument, and the transition between those and the multiplicative twists by Größencharakters. Let λ be a function of integral ideals satisfying $\lambda(\mathfrak{a})=O(N(\mathfrak{a})^K)$ for some $K>0$, so that the corresponding L-series $L(s,\lambda)=\sum_{\mathfrak{a}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s}$ defines an analytic function for $\Re(s)>K+1$. Given a Größencharakter χ of conductor $\mathfrak{q}$, the *multiplicative twist* of $L(s,\lambda)$ by χ is the series

$$L(s,\lambda\times\chi)=\sum_{\mathfrak{a}}\lambda(\mathfrak{a})\chi(\mathfrak{a})N(\mathfrak{a})^{-s}.$$

Next, let $\Gamma_{\mathfrak{q}}=\{\epsilon\in\mathfrak{o}_F^\times:\epsilon\equiv1\ (\mathrm{mod}\ \mathfrak{q})\}$ and note that, for any $z\in\mathfrak{q}^{-1}\mathfrak{d}^{-1}\cap F^\times$, the function $\psi_\infty(z)\chi_\infty(z)$ is unchanged if z is replaced by ϵz for any $\epsilon\in\Gamma_{\mathfrak{q}}$. Thus, we may define

$$e_{\mathfrak{q}}((z),\chi_\infty)=\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}}]}\sum_{\eta\in\Gamma_{\mathfrak{q}}\backslash\mathfrak{o}_F^\times}\psi_\infty(\eta z)\chi_\infty(\eta z),$$

which is a function of principal fractional ideals generated by such z. Let $\{\mathfrak{a}_j\}_{j=1}^h$ be a fixed set of fractional ideals representing the ideal classes of F. Then, by an *additive twist*, we mean a series of the form

$$L_{\mathfrak{a}_j}(s,\lambda,\alpha,\chi_\infty)=\sum_{\mathfrak{a}\sim\mathfrak{a}_j}\lambda(\mathfrak{a})e_{\mathfrak{q}}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)N(\mathfrak{a})^{-s},$$

where $\alpha\in\mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1}\cap F^\times$ and the sum runs over non-zero integral ideals $\mathfrak{a}$ in the class of $\mathfrak{a}_j$. Note that for given α and χ_∞, if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are ideals such that $\alpha\in\mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}_i^{-1}\cap F^\times$ and χ_∞ is

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trivial on $\Gamma_{\mathfrak{q}_i}$ for $i = 1, 2$ then $e_{\mathfrak{q}_1}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha), \chi_\infty) = e_{\mathfrak{q}_2}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha), \chi_\infty)$; hence, we are justified in omitting $\mathfrak{q}$ from the above notation.

The precise result that we prove in this section is the following.

PROPOSITION 3.1. *Let λ be as above, and suppose that $L(s, \lambda)$ factors into an Euler product of the form $\prod_{\mathfrak{p}}(1/f_{\mathfrak{p}}(N(\mathfrak{p})^{-s}))$, where $f_{\mathfrak{p}}$ is a polynomial for each prime ideal $\mathfrak{p}$. If $\chi = \chi_f\chi_\infty$ is a Größencharakter then the multiplicative twist $L(s, \lambda \times \chi)$ is a $\mathbb{C}$-linear combination $\sum_{i=1}^m c_i L_{\mathfrak{a}_{j_i}}(s, \lambda, \alpha_i, \chi_\infty)$ of additive twists. Conversely, any $L_{\mathfrak{a}_j}(s, \lambda, \alpha, \chi_\infty)$ may be expressed in the form $\sum_{i=1}^m D_i(s)L(s, \lambda \times \chi_i)$ for certain finite Dirichlet series D_i and Größencharakters $\chi_i = \chi_{i,f}\chi_\infty$.*

3.1 Multiplicative to additive twists

Let χ be a Größencharakter, as above. For a fixed ideal class j, we choose $u_j \in \mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1} - \bigcup_{\mathfrak{p}|\mathfrak{q}} \mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{p}$ and $v_j \in \mathfrak{a}_j - \bigcup_{\mathfrak{p}|\mathfrak{q}} \mathfrak{a}_j\mathfrak{p}$. Let $\mathfrak{a}$ be an integral ideal in the class of $\mathfrak{a}_j$, so that $\mathfrak{a} = (\gamma)\mathfrak{a}_j$ for some $\gamma \in F^\times$; then $\gamma = v_j^{-1}b$ for some $b \in \mathfrak{o}_F$. Similarly, multiplication by u_j defines an $\mathfrak{o}_F$-module isomorphism between $\mathfrak{o}_F/\mathfrak{q}$ and $\mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1}/\mathfrak{a}_j\mathfrak{d}^{-1}$. Consider the sum

$$\sum_{\substack{\alpha \in \mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1}/\mathfrak{a}_j\mathfrak{d}^{-1} \\ \text{coset reps. } \alpha \in F^\times}} \overline{\chi((u_j^{-1}\alpha))} e_{\mathfrak{q}}((\gamma\alpha), \chi_\infty).$$

It will be clear from the discussion below that the summand does indeed factor through $\mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1}/\mathfrak{a}_j\mathfrak{d}^{-1}$; for now, we may just consider it as running through any particular set of non-zero coset representatives. By the above remarks, the sum equals

$$\sum_{x \in \mathfrak{o}_F/\mathfrak{q}} \overline{\chi_\infty(x)\chi_f(x)} e_{\mathfrak{q}}((\gamma u_j x), \chi_\infty) = \sum_{x \in (\mathfrak{o}_F/\mathfrak{q})^\times} \frac{\overline{\chi_\infty(x)\chi_f(x)}}{[\mathfrak{o}_F^\times : \Gamma_{\mathfrak{q}}]} \sum_{\eta \in \Gamma_{\mathfrak{q}}\backslash\mathfrak{o}_F^\times} \psi_\infty(\eta\gamma u_j x)\chi_\infty(\eta\gamma u_j x). \tag{3.1}$$

Note that we may restrict to $x \in (\mathfrak{o}_F/\mathfrak{q})^\times$ here, since otherwise $\chi_f(x)$ vanishes.

Changing the order of summation, we get

$$\frac{1}{[\mathfrak{o}_F^\times : \Gamma_{\mathfrak{q}}]} \sum_{\eta \in \Gamma_{\mathfrak{q}}\backslash\mathfrak{o}_F^\times} \chi_\infty(\eta\gamma u_j) \sum_{x \in (\mathfrak{o}_F/\mathfrak{q})^\times} \overline{\chi_f(x)}\psi_\infty(\eta\gamma u_j x).$$

We write $\gamma = b/v_j$ with $b \in \mathfrak{o}_F$ in the inner sum. Note that the inner sum vanishes unless b is invertible mod $\mathfrak{q}$, in which case we may replace x by $(\eta b)^{-1}x$. We arrive at

$$\frac{1}{[\mathfrak{o}_F^\times : \Gamma_{\mathfrak{q}}]} \sum_{\eta \in \Gamma_{\mathfrak{q}}\backslash\mathfrak{o}_F^\times} \chi_\infty(\eta b u_j/v_j)\chi_f(\eta b) \sum_{x \in (\mathfrak{o}_F/\mathfrak{q})^\times} \overline{\chi_f(x)}\psi_\infty(u_j x/v_j).$$

Now, for any character ξ on $(\mathfrak{o}_F/\mathfrak{q})^\times$ and $z \in \mathfrak{q}^{-1}\mathfrak{d}^{-1}$, let $\tau_{\mathfrak{q}}(z, \xi)$ denote the Gauss sum

$$\sum_{x \in (\mathfrak{o}_F/\mathfrak{q})^\times} \xi(x)\psi_\infty(xz).$$

We then recognize the inner sum above as $\tau_{\mathfrak{q}}(u_j/v_j, \overline{\chi}_f)$. Since $((u_j/v_j)\mathfrak{q}\mathfrak{d}, \mathfrak{q}) = 1$, it has modulus $\sqrt{N(\mathfrak{q})}$, by [Neu99, VII, Theorem 6.4]; in particular, it is non-zero. Moreover, the fractional ideal $(v_j)^{-1}\mathfrak{a}_j$ is also prime to $\mathfrak{q}$, and hence we have

$$\chi_\infty(\eta b)\chi_f(\eta b) = \chi((\eta b)) = \chi((b)) = \frac{\chi(\mathfrak{a})}{\chi((v_j)^{-1}\mathfrak{a}_j)}.$$

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Thus, the dependence on η disappears, so we get

$$\frac{\chi_\infty(u_j/v_j)\tau_{\mathfrak{q}}(u_j/v_j,\overline{\chi}_f)}{\chi((v_j)^{-1}\mathfrak{a}_j)}\chi(\mathfrak{a})=\kappa_j\chi(\mathfrak{a}), \tag{3.2}$$

where κ_j is a non-zero number independent of $\mathfrak{a}$. Finally, we have

$$\sum_{\mathfrak{a}}\frac{\lambda(\mathfrak{a})\chi(\mathfrak{a})}{N(\mathfrak{a})^s}=\sum_{j=1}^{h}\kappa_j^{-1}\sum_{\substack{\alpha\in\mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{q}^{-1}/\mathfrak{a}_j\mathfrak{d}^{-1}\\ \text{coset reps. }\alpha\in F^\times}}\overline{\chi((u_j^{-1}\alpha))}\sum_{\mathfrak{a}\sim\mathfrak{a}_j}\lambda(\mathfrak{a})e_{\mathfrak{q}}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)N(\mathfrak{a})^{-s}. \tag{3.3}$$

Thus, every multiplicatively twisted L-function is indeed a linear combination of additively twisted ones.

3.2 Additive to multiplicative twists

Next, we need to express each additive twist appearing in (3.3) in terms of multiplicative twists. To that end, we consider the $\mathbb{C}$-vector space V of Dirichlet series spanned by those of the form $N(\mathfrak{m})^{-s}L(s,\lambda\times\chi)$, where χ is any Größencharakter and $\mathfrak{m}$ is an integral ideal. We will show that this space is closed under additive twists; in particular, any additive twist of $L(s,\lambda)$ may be expressed as a linear combination of multiplicative twists, with coefficients that are Dirichlet polynomials.

Let $\chi=\chi_f\chi_\infty$ be a Größencharakter, as above. A calculation shows that

$$e_{\mathfrak{q}}((z),\chi_\infty)=\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}}]}\sum_{\eta\in\Gamma_{\mathfrak{q}}\backslash\mathfrak{o}_F^\times}\psi_\infty(\eta z)\chi_\infty(\eta z)=\frac{\chi_\infty(z)}{|(\mathfrak{o}_F/\mathfrak{q})^\times|}\sum_{\substack{\xi\in\widehat{(\mathfrak{o}_F/\mathfrak{q})^\times}\\ \chi_f\xi|_{\mathfrak{o}_F^\times}=1}}\tau_{\mathfrak{q}}(z,\xi), \tag{3.4}$$

where, in the latter sum, ξ runs over all characters of $(\mathfrak{o}_F/\mathfrak{q})^\times$ that are inverse to χ_f on the image of $\mathfrak{o}_F^\times$.

Now, suppose that $\mathfrak{q}=\mathfrak{q}_1\mathfrak{q}_2$ is composite, with $(\mathfrak{q}_1,\mathfrak{q}_2)=1$. Then we also have a factorization of the characters ξ as $\xi_1\xi_2$, where ξ_i is a character on $(\mathfrak{o}_F/\mathfrak{q}_i)^\times$; the corresponding Gauss sums satisfy the identity $\tau_{\mathfrak{q}}(z,\xi)=\tau_{\mathfrak{q}_1}(z_1,\xi_1)\tau_{\mathfrak{q}_2}(z_2,\xi_2)$, where $z_i\in\mathfrak{q}_i^{-1}\mathfrak{d}^{-1}$ satisfy $z\equiv z_i \pmod{\mathfrak{q}_i\mathfrak{q}^{-1}\mathfrak{d}^{-1}}$. Then we have

$$\tau_{\mathfrak{q}_i}(z_i,\xi_i)=\sum_{x\in(\mathfrak{o}_F/\mathfrak{q}_i)^\times}\xi_i(x)\psi_\infty(xz_i)=\sum_{y\in(\mathfrak{o}_F/\mathfrak{q}_i)^\times/(\mathfrak{o}_F^\times/\Gamma_{\mathfrak{q}_i})}\xi_i(y)\sum_{\eta\in\mathfrak{o}_F^\times/\Gamma_{\mathfrak{q}_i}}\xi_i(\eta)\psi_\infty(\eta y z_i).$$

Now, since $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are relatively prime, we may assume that the coset representatives $y\in\mathfrak{o}_F$ are chosen to lie in $\mathfrak{q}\mathfrak{q}_i^{-1}$. It follows that $\psi_\infty(\eta y z_i)=\psi_\infty(\eta y z)$. Moreover, there are characters $\chi_{i,\infty}:F_\infty^\times\to S^1$ which complete ξ_i^{-1} to Größencharakters, i.e. such that $\xi_i^{-1}(\epsilon)\chi_{i,\infty}(\epsilon)=1$ for all $\epsilon\in\mathfrak{o}_F^\times$, and we may choose these to satisfy $\chi_{1,\infty}\chi_{2,\infty}=\chi_\infty$. Thus, we get

$$\tau_{\mathfrak{q}_i}(z_i,\xi_i)=\sum_{\substack{y\in(\mathfrak{o}_F/\mathfrak{q}_i)^\times/(\mathfrak{o}_F^\times/\Gamma_{\mathfrak{q}_i})\\ \text{coset reps. }y\in\mathfrak{q}\mathfrak{q}_i^{-1}}}\frac{\xi_i(y)}{\chi_{i,\infty}(yz)}\sum_{\eta\in\mathfrak{o}_F^\times/\Gamma_{\mathfrak{q}_i}}\psi_\infty(\eta y z)\chi_{i,\infty}(\eta y z).$$

Substituting this into (3.4), we have

$$e_{\mathfrak{q}}((z),\chi_\infty)=\frac{1}{|(\mathfrak{o}_F/\mathfrak{q})^\times|}\sum_{\substack{\xi\in\widehat{(\mathfrak{o}_F/\mathfrak{q})^\times}\\ \chi_f\xi|_{\mathfrak{o}_F^\times}=1}}\ \sum_{\substack{y_1\in(\mathfrak{o}_F/\mathfrak{q}_1)^\times/(\mathfrak{o}_F^\times/\Gamma_{\mathfrak{q}_1})\\ \text{coset reps. } y_1\in\mathfrak{q}_2}}\ \sum_{\substack{y_2\in(\mathfrak{o}_F/\mathfrak{q}_2)^\times/(\mathfrak{o}_F^\times/\Gamma_{\mathfrak{q}_2})\\ \text{coset reps. } y_2\in\mathfrak{q}_1}}$$
$$\times\ \frac{\xi_1(y_1)}{\chi_{1,\infty}(y_1)}\frac{\xi_2(y_2)}{\chi_{2,\infty}(y_2)}e_{\mathfrak{q}_1}((y_1z),\chi_{1,\infty})e_{\mathfrak{q}_2}((y_2z),\chi_{2,\infty}).$$

Thus, if $\sum_{\mathfrak{a}}\nu(\mathfrak{a})N(\mathfrak{a})^{-s}$ is any Dirichlet series, then the additive twist

$$\sum_{\mathfrak{a}\sim\mathfrak{a}_j}\nu(\mathfrak{a})e_{\mathfrak{q}}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)N(\mathfrak{a})^{-s}$$

is a linear combination of series of the form

$$\sum_{\mathfrak{a}\sim\mathfrak{a}_j}\nu(\mathfrak{a})e_{\mathfrak{q}_1}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha y_1),\chi_{1,\infty})e_{\mathfrak{q}_2}(\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha y_2),\chi_{2,\infty})N(\mathfrak{a})^{-s}.$$

In other words, the additive twist by $e_{\mathfrak{q}}$ may be built out of twists by $e_{\mathfrak{q}_1}$ and $e_{\mathfrak{q}_2}$.

Hence, it suffices to show that V is closed under twists by $e_{\mathfrak{q}}$ with $\mathfrak{q}=\mathfrak{p}^r$ a prime power. Suppose that $N(\mathfrak{m})^{-s}\sum_{\mathfrak{a}}\nu(\mathfrak{a})N(\mathfrak{a})^{-s}$ is a typical basis element of V, so that ν is a multiplicative function of integral ideals. Then the corresponding additive twist is the series

$$\sum_{\mathfrak{a}\sim\mathfrak{m}^{-1}\mathfrak{a}_j}\nu(\mathfrak{a})e_{\mathfrak{p}^r}(\mathfrak{m}\mathfrak{a}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)N(\mathfrak{m}\mathfrak{a})^{-s}$$
$$=\sum_{k=0}^{\infty}\frac{\nu(\mathfrak{p}^k)}{N(\mathfrak{p})^{ks}}\sum_{\substack{(\mathfrak{b},\mathfrak{p})=1\\ \mathfrak{b}\sim\mathfrak{p}^{-k}\mathfrak{m}^{-1}\mathfrak{a}_j}}\nu(\mathfrak{b})e_{\mathfrak{p}^r}(\mathfrak{p}^k\mathfrak{m}\mathfrak{b}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)N(\mathfrak{m}\mathfrak{b})^{-s},\tag{3.5}$$

where $\alpha\in\mathfrak{a}_j\mathfrak{d}^{-1}\mathfrak{p}^{-r}$. We handle first the terms for $k<r-\operatorname{ord}_{\mathfrak{p}}(\mathfrak{m})$. Without loss of generality, we may assume that $\mathfrak{m}$ is relatively prime to $\mathfrak{p}$, for otherwise we may simply replace k by $k+\operatorname{ord}_{\mathfrak{p}}(\mathfrak{m})$ and $\mathfrak{m}$ by $\mathfrak{p}^{-\operatorname{ord}_{\mathfrak{p}}(\mathfrak{m})}\mathfrak{m}$ in what follows. For a fixed k, we have $\mathfrak{p}^{-k}\mathfrak{m}^{-1}\mathfrak{a}_j=(\beta)\mathfrak{a}_{j'}$ for some $\beta\in F^\times$ and $1\leqslant j'\leqslant h$. Further,

$$e_{\mathfrak{p}^r}(\mathfrak{p}^k\mathfrak{m}\mathfrak{b}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)=\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{p}^r}]}\sum_{\eta\in\Gamma_{\mathfrak{p}^r}\backslash\mathfrak{o}_F^\times}\chi_\infty(\eta z)\psi_\infty(\eta z),\tag{3.6}$$

where $(z)=\mathfrak{p}^k\mathfrak{m}\mathfrak{b}\mathfrak{a}_j^{-1}(\alpha)\subset\mathfrak{p}^{k-r}\mathfrak{m}\mathfrak{b}\mathfrak{d}^{-1}\subset\mathfrak{p}^{k-r}\mathfrak{d}^{-1}$, so that $z\in\mathfrak{p}^{k-r}\mathfrak{d}^{-1}$. A calculation shows that the sum vanishes unless χ_∞ is trivial on $\Gamma_{\mathfrak{p}^{r-k}}$, in which case we have

$$e_{\mathfrak{p}^r}(\mathfrak{p}^k\mathfrak{m}\mathfrak{b}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)=e_{\mathfrak{p}^{r-k}}(\mathfrak{b}\mathfrak{a}_{j'}^{-1}(\alpha/\beta),\chi_\infty).$$

Now, in the previous section, we derived the identity

$$\sum_{x\in(\mathfrak{o}_F/\mathfrak{q})^\times}\overline{\chi_\infty(x)\chi_f(x)}e_{\mathfrak{q}}((\gamma u_jx),\chi_\infty)=\frac{\chi_\infty(u_j/v_j)\tau_{\mathfrak{q}}(u_j/v_j,\overline{\chi_f})}{\chi((v_j)^{-1}\mathfrak{a}_j)}\chi(\mathfrak{a}).$$

It is not hard to see from the proof that this holds even if χ_f is an imprimitive character mod $\mathfrak{q}$. Hence, we may sum over all $\chi_f:(\mathfrak{o}_F/\mathfrak{q})^\times\to S^1$ such that $\chi_f(\epsilon)\chi_\infty(\epsilon)=1$ for all $\epsilon\in\mathfrak{o}_F^\times$, to obtain

$$e_{\mathfrak{q}}(\mathfrak{a}\mathfrak{a}_j^{-1}(u_j),\chi_\infty)=e_{\mathfrak{q}}((\gamma u_j),\chi_\infty)=\frac{1}{|(\mathfrak{o}_F/\mathfrak{q})^\times|}\sum_{\substack{\chi_f\in\widehat{(\mathfrak{o}_F/\mathfrak{q})^\times}\\ \chi_f\chi_\infty|_{\mathfrak{o}_F^\times}=1}}\frac{\chi_\infty(u_j/v_j)\tau_{\mathfrak{q}}(u_j/v_j,\overline{\chi}_f)}{\chi((v_j)^{-1}\mathfrak{a}_j)}\chi(\mathfrak{a})$$

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for any integral ideal $\mathfrak{a}$ with $(\mathfrak{a},\mathfrak{q})=1$. Applying this in the present situation with $\mathfrak{q}=\mathfrak{p}^{r-k}$, j replaced by j' and $u_j=\alpha/\beta$ (note that $\alpha/\beta\in\mathfrak{a}_{j'}\mathfrak{d}^{-1}\mathfrak{p}^{k-r}-\mathfrak{a}_{j'}\mathfrak{d}^{-1}\mathfrak{p}^{k-r+1}$, since $(\mathfrak{m},\mathfrak{p})=1$), we see that $e_{\mathfrak{p}^{r-k}}(\mathfrak{b}\mathfrak{a}_{j'}^{-1}(\alpha/\beta),\chi_\infty)$ is a linear combination of $\chi(\mathfrak{b})$ for Größencharakters χ of conductor dividing $\mathfrak{p}^{r-k}$. Thus, the inner sum in (3.5) is a linear combination of multiplicatively twisted series $N(\mathfrak{m})^{-s}\sum_{\substack{\mathfrak{b}\sim\mathfrak{a}_{j'}\\(\mathfrak{b},\mathfrak{p})=1}}\nu(\mathfrak{b})\chi(\mathfrak{b})N(\mathfrak{b})^{-s}$. This in turn is equal to

$$\frac{1}{h}\sum_{\xi\in\widehat{\mathrm{Cl}_F}}\xi(\mathfrak{a}_{j'})^{-1}N(\mathfrak{m})^{-s}\sum_{(\mathfrak{b},\mathfrak{p})=1}\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})N(\mathfrak{b})^{-s},$$

where the outer sum ranges over all characters ξ of Cl_F. Finally, the inner sum is simply the full series $\sum_{\mathfrak{b}}\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})N(\mathfrak{b})^{-s}$ with its Euler factor at $\mathfrak{p}$ removed; from the hypotheses, that factor amounts to a polynomial in $N(\mathfrak{p})^{-s}$.

It remains only to treat the terms with $k\geqslant r-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})$ from (3.5). For those, we see from (3.6) that $e_{\mathfrak{p}^r}(\mathfrak{p}^k\mathfrak{m}\mathfrak{b}\mathfrak{a}_j^{-1}(\alpha),\chi_\infty)$ is identically zero unless χ_∞ is trivial on $\mathfrak{o}_F^\times$, i.e. χ_∞ is the infinite part of a Größencharakter χ of conductor $\mathfrak{o}_F$, in which case it equals $\chi_\infty(z)$. Thus, we have either zero or the sum

$$\sum_{k\geqslant r-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})}\sum_{\substack{(\mathfrak{b},\mathfrak{p})=1\\ \mathfrak{b}\sim\mathfrak{p}^{-k}\mathfrak{m}^{-1}\mathfrak{a}_j}}\nu(\mathfrak{p}^k\mathfrak{b})\chi(\mathfrak{p}^k\mathfrak{m}\mathfrak{b}\mathfrak{a}_j^{-1}(\alpha))N(\mathfrak{p}^k\mathfrak{m}\mathfrak{b})^{-s}.$$

Again, by summing over ideal class characters ξ, we arrive at

$$\frac{1}{h}\sum_{\xi\in\widehat{\mathrm{Cl}_F}}\xi(\mathfrak{a}_j)^{-1}\chi(\mathfrak{m}\mathfrak{a}_j^{-1}(\alpha))N(\mathfrak{m})^{-s}\sum_{k\geqslant r-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})}\frac{\nu(\mathfrak{p}^k)\chi(\mathfrak{p}^k)\xi(\mathfrak{p}^k)}{N(\mathfrak{p})^{ks}}\sum_{(\mathfrak{b},\mathfrak{p})=1}\frac{\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})}{N(\mathfrak{b})^s}.$$

Now, the sum over k is simply the Euler factor at $\mathfrak{p}$ of $\sum_{\mathfrak{b}}\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})N(\mathfrak{b})^{-s}$, with the first $r-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})$ terms removed. Hence, we have

$$\begin{aligned}\sum_{k\geqslant r-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})}&\frac{\nu(\mathfrak{p}^k)\chi(\mathfrak{p}^k)\xi(\mathfrak{p}^k)}{N(\mathfrak{p})^{ks}}\sum_{(\mathfrak{b},\mathfrak{p})=1}\frac{\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})}{N(\mathfrak{b})^s}\\&=\sum_{\mathfrak{b}}\frac{\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})}{N(\mathfrak{b})^s}-\bigg(\sum_{k<r-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})}\frac{\nu(\mathfrak{p}^k)\chi(\mathfrak{p}^k)\xi(\mathfrak{p}^k)}{N(\mathfrak{p})^{ks}}\bigg)\sum_{(\mathfrak{b},\mathfrak{p})=1}\frac{\nu(\mathfrak{b})\chi(\mathfrak{b})\xi(\mathfrak{b})}{N(\mathfrak{b})^s}.\end{aligned}$$

As before, the restriction to $(\mathfrak{b},\mathfrak{p})=1$ and the remaining sum over k amount to polynomials in $N(\mathfrak{p})^{-s}$.

4. Some generalities of the GL_2 theory

Let π be as in the statement of Theorem 1.1. We write V_π to denote the space of π, $W(\pi_v,\psi_v)$ the ψ_v-Whittaker model of π_v for each v and $W(\pi,\psi)=\bigotimes_v W(\pi_v,\psi_v)$ the global Whittaker model of π. Let $\mathfrak{N}$ be the conductor of π; we have $\mathfrak{N}=\prod_{v<\infty}\mathfrak{p}_v^{f(\pi_v)}$, where $f(\pi_v)=0$ whenever π_v is unramified and $f(\pi_v)>0$ otherwise. Let

$$K_{1,v}(\mathfrak{p}_v^{f(\pi_v)})=\left\{g\in\mathrm{GL}_2(\mathfrak{o}_v):g\equiv\begin{pmatrix}*&*\\0&1\end{pmatrix}\bmod\mathfrak{p}_v^{f(\pi_v)}\right\}$$

and

$$K_{0,v}(\mathfrak{p}_v^{f(\pi_v)})=\left\{g\in\mathrm{GL}_2(\mathfrak{o}_v):g\equiv\begin{pmatrix}*&*\\0&*\end{pmatrix}\bmod\mathfrak{p}_v^{f(\pi_v)}\right\}.$$

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Now, set $K_1(\mathfrak{N}) = \prod_{v<\infty} K_{1,v}(\mathfrak{p}_v^{f(\pi_v)})$ and $K_0(\mathfrak{N}) = \prod_{v<\infty} K_{0,v}(\mathfrak{p}_v^{f(\pi_v)})$. The definition of $\mathfrak{N}$ is such that the dimension of the space of $K_{1,v}(\mathfrak{p}_v^{f(\pi_v)})$-fixed vectors in π_v is 1 for each finite v. Moreover, $K_{0,v}(\mathfrak{p}_v^{f(\pi_v)})$ acts on this one-dimensional space by the central character ω_{π_v}, where ω_{π_v} determines a character of $K_{0,v}(\mathfrak{p}_v^{f(\pi_v)})$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \omega_{\pi_v}(d)$ if $f(\pi_v) > 0$, and 1 otherwise.

Note that for an archimedean place v, V_{π_v} is by definition a Harish-Chandra module, which is not equipped with an action of $\mathrm{GL}_2(F_v)$. Since we consider only unitary representations at archimedean places, V_{π_v} can be realized as the space of K_v-finite vectors (where K_v is the standard maximal compact, as defined below) in some unitary representation of $\mathrm{GL}_2(F_v)$ on a Hilbert space $\mathfrak{h}_v$, which is unique up to isomorphism. Further, the space of smooth vectors $\mathfrak{h}_v^\infty \subset \mathfrak{h}_v$ is the canonical Casselman–Wallach completion of V_{π_v}. Let

$$V_\pi^\infty = \bigotimes_{v|\infty} \mathfrak{h}_v^\infty \otimes \bigotimes_{v<\infty} V_{\pi_v}$$

be the smooth extension of V_π.

There is an intertwining map, unique up to scalar multiplication, from V_π^∞ onto $W(\pi,\psi)$, which we will denote by $\xi \mapsto W_\xi$. Let $\Phi(\pi)$ denote the set of all functions of the form

$$\phi_\xi(g) = \sum_{\gamma\in F^\times} W_\xi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \quad \text{with } \xi \in V_\pi^\infty, g \in \mathrm{GL}_2(\mathbb{A}_F).$$

Then $\Phi(\pi)$ is $\mathrm{GL}_2(\mathbb{A}_F)$-invariant under right translation.

4.1 New vectors at the finite places

For each finite place v there is a unique choice of $\xi_v \in V_{\pi_v}$, called the *new vector*, such that W_{ξ_v} transforms via the central character ω_{π_v} for the action of $K_{0,v}(\mathfrak{p}_v^{f(\pi_v)})$ and $W_{\xi_v}(\left(\begin{smallmatrix} d_v^{-1} & \\ & 1 \end{smallmatrix}\right)) = 1$, with d as in § 1.1. It satisfies

$$\int_{F_v^\times} W_{\xi_v}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \|y\|_v^{s-\frac{1}{2}}\, d^\times y = \|d_v\|_v^{\frac{1}{2}-s} L(s,\pi_v).$$

4.2 Archimedean Whittaker functions

In this section we review some facts about the representation theory of $\mathrm{GL}_2(F_v)$ at archimedean places v, and describe a choice of test vector ξ_v such that the associated Whittaker function has Mellin transform $L(s,\pi_v\otimes\omega_v)$ for a given local character ω_v.

Set $K_v = \mathrm{O}_2(F_v)$ for $v \in S_\mathbb{R}$, and $K_v = \mathrm{U}_2(F_v)$ for $v \in S_\mathbb{C}$; then K_v is a maximal compact subgroup of $\mathrm{GL}_2(F_v)$. The irreducible representations of K_v are indexed by non-negative integers. (See [Wei71, ch. VIII], for example.) Let ρ_N, $N \in \mathbb{Z}_{\geqslant 0}$, denote all such representations. Then the restriction of π_v to K_v decomposes into a direct sum of irreducibles. The smallest non-negative integer N such that $0 \neq \rho_N \subset \pi_v|_{K_v}$ is called the weight of π_v, and we will denote it by $k(\pi_v)$. Moreover, if ρ_{N_1} and ρ_{N_2} are non-trivial constituents of $\pi_v|_{K_v}$, then $N_1 \equiv N_2 \pmod 2$ (cf. [Wei71, ch. VIII]).

4.2.1 *Real places.* Suppose that $v \in S_\mathbb{R}$. For any two quasi-characters μ_1 and μ_2 of $F_v^\times = \mathbb{R}^\times$, form the induced representation $\mathcal{B}(\mu_1,\mu_2)$; it is an admissible $(\mathfrak{gl}_2(\mathbb{R}), K_v)$-module. (See [JL70, ch. 1, § 5] or [God70, § 2]; the latter provides a concise exposition of such representations.) Let us write $\mu_i = \|\cdot\|_v^{\nu_i}\,\mathrm{sgn}_v^{\epsilon_i}$, $\nu_i \in \mathbb{C}$, $\epsilon_i \in \{0,1\}$, $i = 1, 2$. Then $\mathcal{B}(\mu_1,\mu_2)$ is irreducible unless $\nu_1 - \nu_2$ is a non-zero integer and $\epsilon_1 \neq \epsilon_2$. When $\mathcal{B}(\mu_1,\mu_2)$ is reducible, its composition series is of length two

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with a unique infinite-dimensional factor; if we take $\nu_1 - \nu_2$ to be a positive integer (as we may by swapping μ_1 and μ_2 if necessary), then this factor is the unique invariant subspace of $\mathcal{B}(\mu_1, \mu_2)$. We shall denote by $\pi(\mu_1, \mu_2)$ the representation $\mathcal{B}(\mu_1, \mu_2)$ if it is irreducible, and by $\sigma(\mu_1, \mu_2)$ the infinite-dimensional factor in the composition series otherwise. The representation $\sigma(\mu_1, \mu_2)$ is called a discrete series, while the representation $\pi(\mu_1, \mu_2)$ is called a principal series unless $\nu_1 = \nu_2$ and $\epsilon_1 \neq \epsilon_2$, in which case it is a limit of discrete series.

Moreover, these types exhaust the infinite-dimensional representation theory of $\mathrm{GL}_2(F_v)$, i.e. any infinite-dimensional irreducible π_v is isomorphic either to a principal series representation with $\epsilon_1 \leqslant \epsilon_2$ or to a discrete series or limit of discrete series representation with $\nu_1 - \nu_2 \in \mathbb{Z}_{\geqslant 0}$ and $(\epsilon_1, \epsilon_2) = (0, 1)$. In these two cases we have, respectively, $k(\pi_v) = \epsilon_2 - \epsilon_1$ and $k(\pi_v) = \nu_1 - \nu_2 + 1$. We set $\nu(\pi_v) = (\nu_1 - \nu_2)/2$ and $\epsilon(\pi_v) = \epsilon_1$. Further, we write $S_{\mathbb{R}} = S_d \cup S_0 \cup S_1$, corresponding to those places v for which π_v is a discrete series or limit of discrete series, weight 0 principal series and weight 1 principal series, respectively.

For any $\xi_v \in V_{\pi_v}$, let W_{ξ_v} denote the associated vector in the Whittaker model $W(\pi_v, \psi_v)$. We have a decomposition of V_{π_v} into weight spaces for the action of $\mathrm{SO}_2(F_v) \subset K_v$, i.e.

$$V_{\pi_v} = \bigoplus_{k \in k(\pi_v) + 2\mathbb{Z}} V_{\pi_v}^{(k)},$$

where

$$\xi_v \in V_{\pi_v}^{(k)} \implies W_{\xi_v}\left(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{ik\theta} W_{\xi_v}(g) \quad \text{for all } g \in \mathrm{GL}_2(F_v), \theta \in F_v.$$

It is known that each $V_{\pi_v}^{(k)}$ has dimension at most one; precisely, it is one dimensional for every $k \equiv k(\pi_v) \pmod 2$ unless π_v is a discrete series representation and $|k| < k(\pi_v)$, in which case $V_{\pi_v}^{(k)} = \{0\}$.

For each k, we choose a basis vector $\xi_v^{(k)}$ for $V_{\pi_v}^{(k)}$ (or 0 if this space is trivial), and set $W_k = W_{\xi_v^{(k)}}$ for shorthand. Further, let

$$f_k(y) = \omega_{\pi_v}(|y|^{-1/2}) W_k\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right), \quad y \in F_v^{\times}.$$

We can choose the basis vectors in such a way that the f_k are related by raising and lowering operators (see [God70, p. 2.19, Equation (77)] or [JL70, Proof of Lemma 5.13.1]):

$$\begin{aligned} (2\nu(\pi_v) + 1 + k) f_{k+2}(y) &= 2y f_k'(y) - (4\pi y - k) f_k(y), \\ (2\nu(\pi_v) + 1 - k) f_{k-2}(y) &= 2y f_k'(y) + (4\pi y - k) f_k(y), \end{aligned} \tag{4.1}$$

from which one concludes that [God70, p. 2.20, Equation (78)][2]

$$f_k''(y) + \left(\frac{1/4 - \nu(\pi_v)^2}{y^2} + \frac{2\pi k}{y} - 4\pi^2\right) f_k(y) = 0.$$

Moreover, by considering the action of $\pi_v\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right)$ on the $\xi_v^{(k)}$, one finds that [God70, p. 2.7, Equation (21)]

$$f_{-k}(y) = (-1)^{\epsilon(\pi_v)} f_k(-y). \tag{4.2}$$

[2] The equation in [God70] is incorrect; the constant 1 in (78) should be replaced by $4\pi^2$. This is a consequence of replacing u by $2\pi u$ as explained in the footnote on [God70, p. 2.19].

These identities, together with the fact that $f_k(y)$ must have moderate growth as $y\to\pm\infty$, uniquely determine the family f_k (and hence also W_k) up to multiplication by a scalar, which can be chosen *a posteriori*. (In fact, this is essentially the proof of the local multiplicity one theorem; see [JL70, Lemma 5.13.1].)

For any $\nu\in\mathbb{C}$, let $\kappa_\nu(y)=4\sqrt{|y|}K_\nu(2\pi|y|)$, where K_ν is the classical K-Bessel function; then κ_ν satisfies the second-order differential equation

$$y^2\kappa_\nu''(y)+(\tfrac14-\nu^2-4\pi^2y^2)\kappa_\nu(y)=0. \tag{4.3}$$

Moreover, $\kappa_\nu(y)\sim\kappa_{\frac12}(y)=2e^{-2\pi|y|}$ as $y\to\pm\infty$, and

$$\int_{\mathbb{R}^\times}\kappa_\nu(y)|y|^{s-\frac12}\,\frac{dy}{2|y|}=\Gamma_{\mathbb{R}}(s+\nu)\Gamma_{\mathbb{R}}(s-\nu). \tag{4.4}$$

For $v\in S_0$, one can check that $f_0(y)=\operatorname{sgn}(y)^{\epsilon(\pi_v)}\kappa_{\nu(\pi_v)}(y)$ satisfies all of the above conditions, and the other f_k are then determined by (4.1); in particular,

$$f_{\pm2}(y)=\frac{y\operatorname{sgn}(y)^{\epsilon(\pi_v)}}{\nu(\pi_v)+\frac12}(\kappa_{\nu(\pi_v)}'(y)\mp2\pi\kappa_{\nu(\pi_v)}(y)).$$

For $v\in S_1$, one checks using (4.3) and the identities

$$y\kappa_{\nu-\frac12}'(y)=\nu\kappa_{\nu-\frac12}(y)-2\pi|y|\kappa_{\nu+\frac12}(y),$$
$$y\kappa_{\nu+\frac12}'(y)=-\nu\kappa_{\nu+\frac12}(y)-2\pi|y|\kappa_{\nu-\frac12}(y)$$

that the choice

$$f_{\pm1}(y)=|y|^{1/2}\kappa_{\nu(\pi_v)-\frac12}(y)\pm\operatorname{sgn}(y)|y|^{1/2}\kappa_{\nu(\pi_v)+\frac12}(y)$$

works. Similarly, for $v\in S_d$, we may take

$$f_{k(\pi_v)}(y)=(1+\operatorname{sgn}(y))|y|^{k(\pi_v)/2}\kappa_{1/2}(y)=\begin{cases}4y^{k(\pi_v)/2}e^{-2\pi y} & \text{if } y>0,\\ 0 & \text{if } y<0.\end{cases}$$

Given a character ω_v of $F_v^\times$, we choose a test vector ξ_v such that $W_{\xi_v}=\frac12(W_k+(-1)^{\epsilon(\pi_v)+\epsilon(\omega_v)}W_{-k})$, where $k=k(\pi_v)$ unless $v\in S_0$ and $\epsilon(\omega_v)\neq\epsilon(\pi_v)$, in which case we take $k=2$. With this choice, one can verify using (4.4) and (4.2) that we can choose the constant of proportionality so that

$$\begin{aligned}\int_{F_v^\times}W_{\xi_v}\left(\begin{pmatrix}y&\\&1\end{pmatrix}\right)\omega_v(y)\|y\|_v^{s-\frac12}\,d^\times y&=L(s,\pi_v\otimes\omega_v)\\ &=\begin{cases}\Gamma_{\mathbb{R}}(t+|\epsilon(\omega_v)-\epsilon(\pi_v)|+\nu(\pi_v))\Gamma_{\mathbb{R}}(t+|\epsilon(\omega_v)-\epsilon(\pi_v)|-\nu(\pi_v)) & \text{if } v\in S_0,\\ \Gamma_{\mathbb{R}}(t+|\epsilon(\omega_v)-\epsilon(\pi_v)|+\nu(\pi_v))\Gamma_{\mathbb{R}}(t+1-|\epsilon(\omega_v)-\epsilon(\pi_v)|-\nu(\pi_v)) & \text{if } v\in S_1,\\ \Gamma_{\mathbb{C}}\left(t+\dfrac{k(\pi_v)-1}{2}\right) & \text{if } v\in S_d,\end{cases}\end{aligned}$$

where $t=s+\nu(\omega_{\pi_v})/2+\nu(\omega_v)$.

4.2.2 *Complex places.* Suppose that $v\in S_{\mathbb{C}}$, i.e. $F_v=\mathbb{C}$. The situation here is similar to the real places except that there are no discrete series representations. To be precise, π_v being an irreducible admissible generic representation of $\mathrm{GL}_2(F_v)$ can be realized as a full induced representation $\mathcal{B}(\mu_1,\mu_2)$ (see [JL70, ch. 1, § 6]). We denote this representation by $\pi(\mu_1,\mu_2)$.

For such a π_v, let $\mu(z)=\mu_1\mu_2^{-1}(z)$. By interchanging μ_1 and μ_2 if necessary, we may suppose that $k(\mu)=k(\mu_1)-k(\mu_2)\geqslant 0$; then in fact $k(\pi_v)=k(\mu)$ [JL70, Lemma 6.1(ii)], and we define $\nu(\pi_v)=\nu(\mu)/2$. In particular,

$$\pi_v|_{\mathrm{SU}_2(F_v)}\cong\rho_{k(\mu)}\oplus\rho_{k(\mu)+2}\oplus\cdots,$$

where ρ_N is the unique $(N+1)$-dimensional irreducible representation of $\mathrm{SU}_2(F_v)$ which acts by right translation on the space V_N of homogeneous polynomials of degree N in two variables. Let us fix the basis

$$\{X^{(N+k)/2}Y^{(N-k)/2}:|k|\leqslant N, k\equiv N \pmod 2\}$$

for V_N. If $\rho_N\subset\pi_v$, there is an inclusion $i_N:V_N\to W(\pi_v,\psi)$ which commutes with the action of $\mathrm{SU}_2(F_v)$. For each $N\geqslant k(\pi_v)$, let $W_{N,k}$ be the image of the monomial $X^{(N+k)/2}Y^{(N-k)/2}$. Using the identity

$$\begin{pmatrix} y & x\\ & 1\end{pmatrix}=u\begin{pmatrix} |y| & x\\ & 1\end{pmatrix}\begin{pmatrix} u & \\ & \overline{u}\end{pmatrix},$$

where $x\in F_v$, $y\in F_v^\times$ and $u^2=y/|y|$, we have

$$W_{N,k}\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right)=\omega_{\pi_v}(u)u^kW_{N,k}\left(\begin{pmatrix} |y| & x\\ & 1\end{pmatrix}\right).$$

Let $f_{N,k}$ be the function on $\mathbb{R}_{>0}$ given by

$$f_{N,k}(t)=\omega_{\pi_v}(t^{-1/2})W_{N,k}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\right),\quad t\in\mathbb{R}_{>0}.$$

It is shown in [JL70] that the functions $f_{N,k}$ satisfy the differential equations[3]

$$\begin{aligned}-4\pi i(N+k)\frac{f_{N,k-2}(t)}{t}&=f''_{N,k}(t)-(1-k)\frac{f'_{N,k}(t)}{t}\\&\quad+\left[\frac{(2-k)^2-(k(\pi_v)+4\nu(\pi_v))^2}{4t^2}-16\pi^2\right]f_{N,k}(t),\\4\pi i(N-k)\frac{f_{N,k+2}(t)}{t}&=f''_{N,k}(t)-(1+k)\frac{f'_{N,k}(t)}{t}\\&\quad+\left[\frac{(2+k)^2-(k(\pi_v)-4\nu(\pi_v))^2}{4t^2}-16\pi^2\right]f_{N,k}(t),\end{aligned}\tag{4.5}$$

analogous to (4.1) in the real case.

For fixed N, these equations, together with the condition of moderate growth, determine the family of $f_{N,k}$ up to scalar multiplication. In particular, in the boundary cases $k=\pm N$ and $N=k(\pi_v)$, one can show that $f_{N,k}(t)$ is proportional to $t^{1+N/2}K_{2\nu(\pi_v)\mp k(\pi_v)/2}(4\pi t)$ and $t^{1+k(\pi_v)/2}K_{2\nu(\pi_v)-k/2}(4\pi t)$, respectively (cf. the proof of Lemma 5.2 below).

We now describe our choice of Whittaker functions at v. Let ω_v be a fixed character of $F_v^\times$, and set $l=-k(\omega_{\pi_v})-2k(\omega_v)$, $N=\max(k(\pi_v),|l|)$. Then we pick ξ_v such that $W_{\xi_v}=W_{N,l}$; note that this is always one of the boundary cases described above. Using the identity

$$\int_0^\infty t^{1+N/2}K_{2\nu}(4\pi t)t^{2s-1}\frac{dt}{t}=\frac{1}{16}\Gamma_{\mathbb{C}}\left(s+\frac{N}{4}+\nu\right)\Gamma_{\mathbb{C}}\left(s+\frac{N}{4}-\nu\right)$$

[3] As pointed out in [Pop08], the equations in [JL70] are not exact; the factor u there should be replaced by $4\pi u$.

and the above formulas for $f_{N,k}$, it is straightforward to check that we can choose the constant of proportionality so that

$$\int_{F_v^\times} W_{\xi_v}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\omega_v(y)\|y\|_v^{s-\frac{1}{2}}\,d^\times y$$
$$= L(s,\pi_v\otimes\omega_v) = \Gamma_{\mathbb{C}}\left(t+\nu(\pi_v)+\frac{|l-k(\pi_v)|}{4}\right)\Gamma_{\mathbb{C}}\left(t-\nu(\pi_v)+\frac{|l+k(\pi_v)|}{4}\right),$$

where $t = s + (\nu(\omega_{\pi_v})/2) + \nu(\omega_v)$.

4.3 Bounds for the archimedean parameters

For $v \in S_\infty$, let π_v be as in § 4.2. In particular, π_v is either a $\pi(\mu_1,\mu_2)$ or a $\sigma(\mu_1,\mu_2)$ for some quasi-characters μ_1, μ_2. Using the classification of irreducible unitary generic representations of $\mathrm{GL}_2(F_v)$, in the notation of § 4.2, one has

$$\Re(\nu_1)+\Re(\nu_2) = 0 \quad \text{if } \pi_v = \sigma(\mu_1,\mu_2),$$
$$|\Re(\nu_1)|, |\Re(\nu_2)| < 1/2 \quad \text{if } \pi_v = \pi(\mu_1,\mu_2).$$

In particular, for $v \in S_0 \cup S_1 \cup S_{\mathbb{C}}$, we have $|\Re\nu(\pi_v)| = |\Re(\nu_1-\nu_2)|/2 < \frac{1}{2}$. As a consequence, note that, for any character ω_v of $F_v^\times$, $L(s,\pi_v\otimes\omega_v)$ is holomorphic in a right half-plane $\Re(s) > \frac{1}{2}-\delta$ for some $\delta > 0$. Also, recall that if $\pi_v = \pi(\mu_1,\mu_2)$ is a local component of a global unitary cuspidal representation, then the Ramanujan conjecture predicts that π_v is tempered, i.e. $\Re(\nu_i) = 0$ for $i = 1, 2$.

4.4 Fourier coefficients and Dirichlet series

For each $v < \infty$, let $\xi_v \in V_{\pi_v}$ be the new vector introduced in § 4.1; for $v|\infty$, let $\xi_v \in V_{\pi_v}$ be as defined in § 4.2, with ω_v equal to the trivial character. Set $\xi = \bigotimes \xi_v$; then by construction we have

$$\Lambda(s,\pi) = N(\mathfrak{d})^{\frac{1}{2}-s}\int_{F^\times\backslash\mathbb{A}_F^\times} \phi_\xi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\|y\|^{s-\frac{1}{2}}\,d^\times y. \tag{4.6}$$

In this section we will write $L(s,\pi)$ as a Dirichlet series and describe the Dirichlet coefficients explicitly in terms of the finite part of the Whittaker function W_ξ. For $y \in \mathbb{A}_F^\times$, $x \in \mathbb{A}_F$, we have

$$\phi_\xi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right) = \sum_{\gamma\in F^\times} W_\xi\left(\begin{pmatrix} \gamma y & \gamma x\\ & 1\end{pmatrix}\right) = \sum_{\gamma\in F^\times}\psi(\gamma x)W_\xi\left(\begin{pmatrix} \gamma y & \\ & 1\end{pmatrix}\right)$$
$$= \sum_{\gamma\in F^\times} a_\xi(y,\gamma)\psi(\gamma x)W_{\xi_\infty}\left(\begin{pmatrix} \gamma y_\infty & \\ & 1\end{pmatrix}\right),$$

where $a_\xi(y,\gamma) = \prod_{v<\infty} W_{\xi_v}\left(\left(\begin{smallmatrix} \gamma y_v & \\ & 1\end{smallmatrix}\right)\right)$ and $W_{\xi_\infty} = \prod_{v|\infty} W_{\xi_v}$. Observe that $a_\xi(y,\gamma)$ depends only on the finite part y_f of y. Moreover, for $v < \infty$ and $z \in \mathfrak{o}_v^\times$, we have

$$W_{\xi_v}\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & z\\ & 1\end{pmatrix}\right) = W_{\xi_v}\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)$$

for all $a \in F_v^\times$. Consequently, $(\psi_v(az)-1)W_{\xi_v}\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\right) = 0$, and hence $W_{\xi_v}\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\right) \neq 0 \implies a \in \mathfrak{d}_v^{-1}$. Therefore, $a_\xi(y,\gamma) = 0$ unless $(\gamma y) \subset \mathfrak{d}^{-1}$, and thus

$$\phi_\xi\left(\begin{pmatrix} y & x\\ & 1\end{pmatrix}\right) = \sum_{\gamma\in(y)^{-1}\mathfrak{d}^{-1}\cap F^\times} a_\xi(y,\gamma)\psi(\gamma x)W_{\xi_\infty}\left(\begin{pmatrix} \gamma y_\infty & \\ & 1\end{pmatrix}\right).$$

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In particular, setting $x=0$ in the above equation and using the fact that $a_\xi(y,\gamma)=a_\xi(y,\epsilon\gamma)$ for all $\epsilon\in\mathfrak{o}_F^\times$, we may break the sum over γ into equivalence classes modulo $\mathfrak{o}_F^\times$ to obtain

$$\phi_\xi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)=\sum_{\gamma\in\mathfrak{o}_F^\times\backslash((y)^{-1}\mathfrak{d}^{-1}\cap F^\times)} a_\xi(y,\gamma)\sum_{\eta\in\mathfrak{o}_F^\times} W_{\xi_\infty}\left(\begin{pmatrix}\eta\gamma y_\infty & \\ & 1\end{pmatrix}\right). \tag{4.7}$$

Next, since the function $y\mapsto\phi_\xi(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))$ is invariant under multiplication by elements of $\prod_{v<\infty}\mathfrak{o}_v^\times$, by strong approximation we may rewrite (4.6) as

$$\begin{aligned}\Lambda(s,\pi)&=N(\mathfrak{d})^{\frac12-s}\int_{F^\times\backslash\mathbb{A}_F^\times/\prod_v\mathfrak{o}_v^\times}\phi_\xi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\|y\|^{s-\frac12}\,d^\times y\\ &=N(\mathfrak{d})^{\frac12-s}\int_{\mathfrak{o}_F^\times\backslash F_\infty^\times}\sum_{j=1}^h\phi_\xi\left(\begin{pmatrix}(y_\infty,t_j) & \\ & 1\end{pmatrix}\right)\|y_\infty\|_\infty^{s-\frac12}\|t_j\|^{s-\frac12}\,d^\times y_\infty,\end{aligned}$$

where the t_j are the finite idèles defined in § 1.1. Using (4.7) and replacing y_∞ by $\gamma_\infty^{-1}y_\infty$ in the above integral, we get

$$\begin{aligned}\Lambda(s,\pi)=N(\mathfrak{d})^{\frac12-s}\sum_{j=1}^h&\sum_{\gamma\in\mathfrak{o}_F^\times\backslash((t_j)^{-1}\mathfrak{d}^{-1}\cap F^\times)} a_\xi(t_j,\gamma)\|t_j\|^{s-\frac12}\|\gamma\|_\infty^{\frac12-s}\\ &\times\int_{F_\infty^\times}W_{\xi_\infty}\left(\begin{pmatrix} y_\infty & \\ & 1\end{pmatrix}\right)\|y_\infty\|_\infty^{s-\frac12}\,d^\times y_\infty.\end{aligned} \tag{4.8}$$

Now, for any non-zero integral ideal $\mathfrak{a}$ in F, set

$$\lambda_\pi(\mathfrak{a})=a_\xi(t_j,\gamma)\sqrt{N(\mathfrak{a})},$$

where j, $1\leqslant j\leqslant h$, is the unique index such that $\mathfrak{a}=(\gamma)(t_j)\mathfrak{d}$ for some $\gamma\in F$. Noting that $\|\gamma\|_\infty=N(\gamma)$, it follows from (4.8) that

$$\Lambda(s,\pi)=\sum_{\mathfrak{a}}\frac{\lambda_\pi(\mathfrak{a})}{N(\mathfrak{a})^s}\prod_{v|\infty}L(s,\pi_v),$$

and thus

$$L(s,\pi)=\sum_{\mathfrak{a}}\frac{\lambda_\pi(\mathfrak{m})}{N(\mathfrak{a})^s}.$$

Moreover, it can be checked that the Dirichlet coefficients $\lambda_\pi(\mathfrak{a})$ are multiplicative, i.e. $\lambda_\pi(\mathfrak{a}\mathfrak{b})=\lambda_\pi(\mathfrak{a})\lambda_\pi(\mathfrak{b})$ if $\mathfrak{a}+\mathfrak{b}=\mathfrak{o}_F$.

Next, we consider the integral

$$\int_{F^\times\backslash\mathbb{A}_F^\times}\phi_\xi\left(w\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\|y\|^{s-\frac12}\,d^\times y,$$

where $w=\left(\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right)$. A simple calculation using the fact that the central character ω_π is trivial on $F^\times$ shows that this is

$$\prod_v\int_{F_v^\times}W_{\xi_v}\left(w\begin{pmatrix} y_v & \\ & 1\end{pmatrix}\right)\|y_v\|_v^{s-\frac12}\,d^\times y_v.$$

Now, using [JL70, Theorems 2.18, 5.15 and 6.4], keeping the notation there, we see that

$$\int_{F_v^\times}W_{\xi_v}\left(w\begin{pmatrix} y_v & \\ & 1\end{pmatrix}\right)\|y_v\|_v^{s-\frac12}\,d^\times y_v=\|d_v\|_v^{\frac12-s}\epsilon(s,\pi_v,\psi_v)L(1-s,\widetilde{\pi}_v). \tag{4.9}$$

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Thus, we obtain

$$\int_{F^\times\backslash\mathbb{A}_F^\times}\phi_\xi\left(w\begin{pmatrix}y&\\&1\end{pmatrix}\right)\|y\|^{s-\frac12}\,d^\times y=N(\mathfrak{d})^{s-\frac12}\epsilon(s,\pi)\Lambda(1-s,\widetilde{\pi}),\tag{4.10}$$

where $\Lambda(s,\widetilde{\pi})=\prod_v L(s,\widetilde{\pi}_v)$ and $\epsilon(s,\pi)=\prod_v\epsilon(s,\pi_v,\psi_v)$. For $v<\infty$, the local epsilon factors $\epsilon(s,\pi_v,\psi_v)$ are given by

$$\epsilon(s,\pi_v,\psi_v)=\epsilon(\pi_v,\psi_v)\omega_{\pi_v}(d_v)\|d_v\|_v^{2s-1}q_v^{f(\pi_v)(\frac12-s)},\tag{4.11}$$

where $\epsilon(\pi_v,\psi_v)$ is a complex number of unit modulus called the *root number* of π_v. (Note that ψ_v has conductor $\mathfrak{d}_v^{-1}=(d_v^{-1})$.) Moreover, they satisfy the functional equation $\epsilon(s,\pi_v,\psi_v)\epsilon(1-s,\widetilde{\pi}_v,\psi_v)=\omega_{\pi_v}(-1)$ for all v.

Note that

$$\int_{F^\times\backslash\mathbb{A}_F^\times}\phi_\xi\left(w\begin{pmatrix}y&\\&1\end{pmatrix}\right)\|y\|^{s-\frac12}\,d^\times y=\int_{F^\times\backslash\mathbb{A}_F^\times}\phi_\xi\left(\begin{pmatrix}y^{-1}&\\&1\end{pmatrix}w\right)\omega_\pi(y)\|y\|^{s-\frac12}\,d^\times y\tag{4.12}$$

and

$$\phi_\xi\left(\begin{pmatrix}y^{-1}&\\&1\end{pmatrix}w\right)\omega_\pi(y)=\sum_{\gamma\in F^\times}\tilde{a}_\xi(y^{-1},\gamma)W_{\xi_\infty}\left(\begin{pmatrix}\gamma y_\infty^{-1}&\\&1\end{pmatrix}w\right)\omega_{\pi_\infty}(\gamma y_\infty^{-1})^{-1},$$

where

$$\tilde{a}_\xi(y^{-1},\gamma)=\prod_{v<\infty}W_{\xi_v}\left(\begin{pmatrix}\gamma y_v^{-1}&\\&1\end{pmatrix}w\right)\omega_{\pi_v}(\gamma y_v^{-1})^{-1}.$$

As above, we see that $\tilde{a}_\xi(y,\gamma)$ is invariant under $y\mapsto uy$ for $u\in\prod_{v<\infty}\mathfrak{o}_v^\times$, and $\tilde{a}_\xi(y,\gamma)=0$ unless $\gamma\in(y)\mathfrak{N}^{-1}\mathfrak{d}^{-1}$. Thus, we have

$$\begin{aligned}\phi_\xi\left(\begin{pmatrix}y^{-1}&\\&1\end{pmatrix}w\right)\omega_\pi(y)=&\sum_{\gamma\in\mathfrak{o}_F^\times\backslash((y)\mathfrak{d}^{-1}\mathfrak{N}^{-1}\cap F^\times)}\tilde{a}_\xi(y^{-1},\gamma)\\&\times\sum_{\eta\in\mathfrak{o}_F^\times}W_{\xi_\infty}\left(\begin{pmatrix}\eta\gamma y_\infty^{-1}&\\&1\end{pmatrix}w\right)\omega_{\pi_\infty}(\eta\gamma y_\infty^{-1})^{-1}.\end{aligned}$$

Substituting this into (4.12) and using (4.10), we obtain

$$\begin{aligned}N(\mathfrak{d})^{s-\frac12}\epsilon(s,\pi)\Lambda(1-s,\tilde{\pi})=&\sum_{j=1}^h\sum_{\gamma\in\mathfrak{o}_F^\times\backslash((t_j)\mathfrak{d}^{-1}\mathfrak{N}^{-1}\cap F^\times)}\tilde{a}_\xi(t_j^{-1},\gamma)\|t_j\|^{s-\frac12}\|\gamma\|_\infty^{s-\frac12}\\&\times\int_{F_\infty^\times}W_{\xi_\infty}\left(\begin{pmatrix}y_\infty&\\&1\end{pmatrix}w\right)\omega_{\pi_\infty}(y_\infty)^{-1}\|y_\infty\|_\infty^{\frac12-s}\,d^\times y_\infty.\end{aligned}\tag{4.13}$$

Now, for any non-zero integral ideal $\mathfrak{a}$, set $\tilde{a}_\pi(\mathfrak{a})=\tilde{a}_\xi(t_j^{-1},\gamma)\sqrt{N(\mathfrak{a})}$, where j is the unique index such that $\mathfrak{a}=(\gamma)\mathfrak{N}\mathfrak{d}(t_j)^{-1}$ for some $\gamma\in F$. Then it follows from (4.9), (4.11) and (4.13) that

$$\omega_\pi(d)\epsilon_f(\pi,\psi)L(1-s,\tilde{\pi})=\sum_{\mathfrak{a}}\frac{\tilde{a}_\pi(\mathfrak{a})}{N(\mathfrak{a})^{1-s}},$$

where $\epsilon_f(\pi,\psi)=\prod_{v<\infty}\epsilon(\pi_v,\psi_v)$. Consequently,

$$\tilde{a}_\pi(\mathfrak{a})=\omega_\pi(d)\epsilon_f(\pi,\psi)\lambda_{\widetilde{\pi}}(\mathfrak{a}).$$

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Recall that $\epsilon_f(\pi,\psi)$ is of unit modulus, and satisfies the identity

$$\epsilon_f(\pi,\psi)\epsilon_f(\tilde{\pi},\psi)=\prod_{v<\infty}\omega_{\pi_v}(-1).$$

5. Proof of Theorem 1.1

With these preliminaries in place, we can now proceed with the proof. Firstly, it is well known that for almost all places v of F the representation π_v is unramified; for such a v, let $\begin{pmatrix}\alpha_v & \\ & \beta_v\end{pmatrix}\in\mathrm{GL}_2(\mathbb{C})$ be the corresponding Satake parameters associated to π_v. Since the products defining $\Lambda(s,\pi\otimes\omega)$ and $\Lambda(s,\widetilde{\pi}\otimes\omega^{-1})$ converge for $\Re(s)>\sigma$, it is not hard to see that the inequality $|\alpha_v|,|\beta_v|\leqslant q_v^{\sigma'}$ must hold for some $\sigma'\geqslant 0$. Moreover, from the functional equation, it follows that all poles are confined to the strip $\{s\in\mathbb{C}:1-\sigma\leqslant\Re(s)\leqslant\sigma\}$.

5.1 Reduction to additively twisted L-functions

Let ω be an idèle class character. We aim to show that $\Lambda(s,\pi\otimes\omega)$ is entire, so that our hypotheses reduce to those of the traditional converse theorem. Let χ_ω be the Größencharakter associated to ω. Suppose that $L(s,\pi)$ has Dirichlet coefficients λ_π, so that

$$L(s,\pi)=\sum_{\mathfrak{a}}\lambda_\pi(\mathfrak{a})N(\mathfrak{a})^{-s}.$$

The relationship between the classical and adelic formulations of the twisted L-functions is such that

$$L(s,\lambda_\pi\times\chi_\omega)=\prod_{\substack{v<\infty\\ \omega_v\text{ unramified}}}L(s,\pi_v\otimes\omega_v),$$

i.e. $L(s,\lambda_\pi\times\chi_\omega)$ is $L(s,\pi\otimes\omega)$ with its Euler factors removed at all places v for which ω_v is ramified. When π_v is unramified for all such v, the relevant Euler factors are 1, so the finite twisted L-functions agree. However, when π_v is also ramified, the local L-factor $L(s,\pi_v\otimes\omega_v)$ may be non-trivial, and removing it amounts to multiplying by a polynomial function of q_v^{-s}. That could cancel a pole of $\Lambda(s,\pi\otimes\omega)$, invalidating our subsequent arguments.

To get around this problem, we apply a version of the converse theorem [CKM04, Lecture 10, Theorem 10.1] that allows one to twist only by those characters ω that are unramified at places v in a given finite set S; in our case, we can choose S to be the set of finite places v for which π_v is ramified.[4] The expense of doing so is that we only conclude quasi-automorphy of π, i.e. there is an automorphic representation Π such that $\pi_v\cong\Pi_v$ for all $v\notin S$. To conclude full automorphy, we apply the standard 'stability of γ-factors' argument [CKM04, Lecture 6, § 5]. Precisely, we fix a place $u\in S$ and a local character ω_u. We then consider idèle class characters ω which have component ω_u at u and are highly ramified at all other places $v\in S$. One knows then that the local functional equations are the same for $\pi_v\otimes\omega_v$ and $\Pi_v\otimes\omega_v$ for all places $v\neq u$. We also know that the global functional equations hold for $\Lambda(s,\pi\otimes\omega)$ and $\Lambda(s,\Pi\otimes\omega)$. Together these imply that the local functional equations for $\pi_u\otimes\omega_u$ and $\Pi_u\otimes\omega_u$ agree, and it follows that $\pi_u\cong\Pi_u$.

Thus, we may assume without loss of generality that ω_v is unramified for all $v\in S$, so we have $L(s,\pi\otimes\omega)=L(s,\lambda_\pi\times\chi_\omega)$. By Proposition 3.1, this L-series is a $\mathbb{C}$-linear combination

[4] This is essentially Weil's version of the converse theorem [Wei71, Theorem 7], though he stated his results in somewhat different language.

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of additive twists $L_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})$. We define a complete version of these additive twists by attaching the same Γ-factors as in $\Lambda(s,\pi\otimes\omega)$; precisely, we set

$$\Lambda_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})=N((\alpha)^{-1}\mathfrak{a}_j)^{s-\frac{1}{2}}L_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})\prod_{v\mid\infty}L(s,\pi_v\otimes\omega_v).$$

(The exponential factor $N((\alpha)^{-1}\mathfrak{a}_j)^{s-\frac{1}{2}}$ turns out to be a convenient normalization; see (5.6) below.) Then $\Lambda(s,\pi\otimes\omega)$ is a linear combination of terms of the form $N((\alpha)^{-1}\mathfrak{a}_j)^{\frac{1}{2}-s}\Lambda_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})$.

On the other hand, by the converse direction of Proposition 3.1, $L_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})$ is a linear combination (with coefficients that are finite Dirichlet series) of multiplicative twists $L(s,\lambda_\pi\times\chi_{\omega'})$. Again using the fact that passing from adelic twists to classical twists involves multiplying by a Dirichlet polynomial (which in this direction is harmless), we see that $L_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})$ is also a combination of adelic twists $L(s,\pi\otimes\omega')$. Moreover, the ω' that occur in this expansion satisfy $\omega'_\infty=\omega_\infty$, and it follows that the corresponding L-functions have the same Γ-factors. Hence, $\Lambda_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})$ is a linear combination of complete twisted L-functions $\Lambda(s,\pi\otimes\omega')$, with coefficients that are entire. Our hypotheses thus imply that $\Lambda_{\mathfrak{a}_j}(s,\lambda_\pi,\alpha,\omega_\infty^{-1})$ is meromorphic, with poles confined to the strip $\{s\in\mathbb{C}:1-\sigma\leqslant\Re(s)\leqslant\sigma\}$.

Note that the above remarks apply equally well with π, ω replaced by $\widetilde{\pi}$, ω^{-1}, respectively. Because of the structure of the argument, it will be convenient to show first that $\Lambda_{\mathfrak{a}_j}(s,\lambda_{\widetilde{\pi}},\alpha,\chi_\infty)$ has no poles for $\Re(s)\leqslant 1/2$ for arbitrary α,χ_∞; thus, $\Lambda(s,\widetilde{\pi}\otimes\omega^{-1})$ is holomorphic for $\Re(s)\leqslant 1/2$ as well, for any ω as above. Reversing the roles of π and $\widetilde{\pi}$ in the argument, it then follows from the functional equation that $\Lambda(s,\pi\otimes\omega)$ and $\Lambda(s,\widetilde{\pi}\otimes\omega^{-1})$ are entire, as required.

5.2 Automorphy relations from unramified twists

We begin by showing that for any $\xi\in V_\pi$ that is $K_1(\mathfrak{N})$-invariant, ϕ_ξ satisfies an automorphy relation akin to (2.2) from the classical argument. We say that a representation π satisfies condition A_ω, where ω is a character of $F^\times\backslash\mathbb{A}_F^\times$, if $\Lambda(s,\pi\otimes\omega)$ and $\Lambda(s,\widetilde{\pi}\otimes\omega^{-1})$ are entire functions bounded in each vertical strip of finite width, and satisfy the functional equation

$$\Lambda(s,\pi\otimes\omega)=\epsilon(s,\pi\otimes\omega)\Lambda(1-s,\widetilde{\pi}\otimes\omega^{-1}).$$

For our particular π, note that the hypotheses of Theorem 1.1 and the Phragmén–Lindelöf principle imply A_ω for all ω that are unramified at every finite place. We recall a lemma due to Piatetski-Shapiro, which generalizes a result proved by Jacquet and Langlands [JL70, Proof of Theorem 11.3].

LEMMA 5.1 [Pja75, Lemma 4]. *Let $X\subset\mathbb{A}_{F,f}^\times$ be a compact subgroup. Assume that M is a linear subspace of $\Phi(\pi)$ and*

$$\phi\left(\begin{pmatrix}ax&\\&1\end{pmatrix}\right)=\phi\left(\begin{pmatrix}a&\\&1\end{pmatrix}\right)\quad\text{for all }\phi\in M,\ a\in\mathbb{A}_F^\times,\ x\in X.$$

If A_ω is satisfied for all characters of $F^\times\backslash\mathbb{A}_F^\times$ which are trivial on X, then, for any $\phi\in M$,

$$\phi\left(\begin{pmatrix}a&\\&1\end{pmatrix}\right)=\phi\left(w\begin{pmatrix}a&\\&1\end{pmatrix}\right)\quad\text{for all }a\in\mathbb{A}_F^\times,$$

where $w=\left(\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right)$.

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We take $X=\prod_{v<\infty}\mathfrak{o}_v^\times$ and M to be the linear span of $\{R_{\left(\begin{smallmatrix}1 & (x_\infty,0)\\ & 1\end{smallmatrix}\right)}\phi_\xi : x_\infty\in F_\infty\}$, where R_g denotes the right regular action by $g\in \mathrm{GL}_2(\mathbb{A}_F)$. Applying the conclusion of the lemma to $R_{\left(\begin{smallmatrix}1 & (x_\infty,0)\\ & 1\end{smallmatrix}\right)}\phi_\xi$, we have

$$\phi_\xi\left(\begin{pmatrix}a & \\ & 1\end{pmatrix}\begin{pmatrix}1 & (x_\infty,0)\\ & 1\end{pmatrix}\right)=\phi_\xi\left(w\begin{pmatrix}a & \\ & 1\end{pmatrix}\begin{pmatrix}1 & (x_\infty,0)\\ & 1\end{pmatrix}\right)$$

for all $x_\infty\in F_\infty, a\in\mathbb{A}_F^\times$. Choosing $x_\infty=a_\infty^{-1}\beta_\infty$ for some $\beta\in F^\times$, we have $\left(\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & (x_\infty,0)\\ & 1\end{smallmatrix}\right)=\left(\begin{smallmatrix}a & (\beta_\infty,0)\\ & 1\end{smallmatrix}\right)$; finally, replacing a by $(\beta_\infty a_\infty, a_f)$, we get

$$\phi_\xi\left(\begin{pmatrix}(\beta_\infty a_\infty, a_f) & (\beta_\infty,0)\\ & 1\end{pmatrix}\right)=\phi_\xi\left(w\begin{pmatrix}(\beta_\infty a_\infty, a_f) & (\beta_\infty,0)\\ & 1\end{pmatrix}\right) \tag{5.1}$$

for all $a\in\mathbb{A}_F^\times$ and $\beta\in F^\times$.

5.3 Producing additive twists

Define class group representatives $\mathfrak{a}_j=(t_j^{-1})\mathfrak{N}\mathfrak{d}$ and $\mathfrak{a}_j'=(t_j)\mathfrak{d}$. Suppose that we are given a j, $1\leqslant j\leqslant h$, a non-zero integral ideal $\mathfrak{q}$ and a continuous character $\chi_\infty:\Gamma_\mathfrak{q}\backslash F_\infty^\times\to S^1$; we regard these data as fixed from now on, and will not explicitly indicate their dependence in various expressions. Set $\chi_\infty'=\omega_{\pi_\infty}\chi_\infty$ and choose $\xi\in V_\pi$ as described in § 4 so that ξ_v is the new vector at all finite v and

$$\int_{F_v^\times} W_{\xi_v}\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}\right)\chi_v'(y)^{-1}\|y\|_v^{s-\frac12}\,d^\times y=L(s,\pi_v\otimes\chi_v'^{-1})$$

for $v|\infty$. With this choice fixed, we define, for $\alpha\in\mathfrak{a}_j\mathfrak{d}\mathfrak{q}^{-1}\cap F^\times$ and $y\in F_\infty^\times$,

$$\Phi(y,\alpha)=\chi_\infty'(y)^{-1}\phi_\xi\left(\begin{pmatrix}(\beta_\infty y, t_j) & (\beta_\infty,0)\\ & 1\end{pmatrix}\right),$$

where $\beta=-\alpha^{-1}$. Further, let T be a finite subset of $(1+\mathfrak{q})\cap F^\times$ containing 1, to be chosen later, and set

$$\mathfrak{q}'=\mathfrak{q}\cap\mathfrak{N}\cap\bigcap_{\lambda\in T}(\lambda\beta^{-1})\mathfrak{a}_j'\mathfrak{d}.$$

Then $\lambda^{-1}\beta\in\mathfrak{a}_j'\mathfrak{d}\mathfrak{q}'^{-1}\cap F^\times$ for every $\lambda\in T$, and χ_∞' factors through $\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times$. We have

$$\begin{aligned}\Phi(y,\alpha)&=\chi_\infty'(y)^{-1}\sum_{\gamma\in F^\times}W_\xi\left(\begin{pmatrix}\gamma(\beta_\infty y,t_j) & \gamma(\beta_\infty,0)\\ & 1\end{pmatrix}\right)\\ &=\chi_\infty'(y)^{-1}\sum_{\gamma\in F^\times}\psi_\infty(\gamma\beta)a_\xi(t_j,\gamma)W_{\xi_\infty}\left(\begin{pmatrix}\gamma\beta y & \\ & 1\end{pmatrix}\right),\end{aligned} \tag{5.2}$$

where $a_\xi(a,\gamma)=\prod_{v<\infty}W_{\xi_v}(\left(\begin{smallmatrix}\gamma_v a_v & \\ & 1\end{smallmatrix}\right))$ and $W_{\xi_\infty}=\prod_{v|\infty}W_{\xi_v}$. For $v<\infty$, we saw in § 4.4 that $W_{\xi_v}(\left(\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right))=0$ for $a\notin\mathfrak{d}_v^{-1}$, where $\mathfrak{d}_v^{-1}$ is the conductor of ψ_v. Further, we break the sum over γ into equivalence classes modulo $\mathfrak{o}_F^\times$, to get

$$\Phi(y,\alpha)=\chi_\infty'(y)^{-1}\sum_{\gamma\in\mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)}a_\xi(t_j,\gamma)\sum_{\eta\in\mathfrak{o}_F^\times}\psi_\infty(\eta\gamma\beta)W_{\xi_\infty}\left(\begin{pmatrix}\eta\gamma\beta y & \\ & 1\end{pmatrix}\right).$$

Since $\gamma\beta \in \mathfrak{d}\mathfrak{q}'^{-1}$, the map $\eta \mapsto \psi_\infty(\eta\gamma\beta)$ factors through $\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times$, and thus

$$\Phi(y,\alpha) = \chi'_\infty(y)^{-1} \sum_{\gamma\in \mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)} a_\xi(t_j,\gamma) \sum_{\eta\in\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times} \psi_\infty(\eta\gamma\beta) \sum_{\epsilon\in\Gamma_{\mathfrak{q}'}} W_{\xi_\infty}\left(\begin{pmatrix} \epsilon\eta\gamma\beta y & \\ & 1\end{pmatrix}\right). \quad (5.3)$$

We consider the integral

$$\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times} \Phi(y,\alpha)\|y\|_\infty^{s-\frac{1}{2}}\, d^\times y. \quad (5.4)$$

In view of the bound on the Satake parameters of π, it will be clear from the arguments below that this converges for $\Re(s)$ sufficiently large. Substituting (5.3) into this expression, we have

$$\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times} \sum_{\gamma\in \mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)} a_\xi(t_j,\gamma) \sum_{\eta\in\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times} \psi_\infty(\eta\gamma\beta)$$
$$\times \sum_{\epsilon\in\Gamma_{\mathfrak{q}'}} W_{\xi_\infty}\left(\begin{pmatrix} \epsilon\eta\gamma\beta y & \\ & 1\end{pmatrix}\right)\chi'_\infty(y)^{-1}\|y\|_\infty^{s-\frac{1}{2}}\, d^\times y.$$

Combining the integral and the sum over $\Gamma_{\mathfrak{q}'}$, and making the change of variables $y\mapsto(\gamma\eta\beta)^{-1}y$, we obtain

$$\|\beta\|_\infty^{\frac{1}{2}-s} \sum_{\gamma\in \mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)} \frac{\|\gamma\|_\infty^{\frac{1}{2}-s} a_\xi(t_j,\gamma)}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]} \sum_{\eta\in\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times} \psi_\infty(\eta\gamma\beta)\chi'_\infty(\eta\gamma\beta)$$
$$\times \int_{F_\infty^\times} W_{\xi_\infty}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\chi'_\infty(y)^{-1}\|y\|_\infty^{s-\frac{1}{2}}\, d^\times y.$$

Note that the inner sum $1/[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}}]\sum_{\eta\in\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times}\psi_\infty(\eta\gamma\beta)\chi'_\infty(\eta\gamma\beta)$ is precisely $e_{\mathfrak{q}'}((\gamma\beta),\chi'_\infty)$. Thus, we get

$$\|\beta\|_\infty^{\frac{1}{2}-s} \sum_{\gamma\in \mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)} \frac{a_\xi(t_j,\gamma)e_{\mathfrak{q}'}((\gamma\beta),\chi'_\infty)}{\|\gamma\|_\infty^{s-\frac{1}{2}}} \int_{F_\infty^\times} W_{\xi_\infty}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\chi'_\infty(y)^{-1}\|y\|_\infty^{s-\frac{1}{2}}\, d^\times y. \quad (5.5)$$

As shown in § 4.4,

$$a_\xi(t_j,\gamma) = \frac{\lambda_\pi(\mathfrak{a})}{\sqrt{N(\mathfrak{a})}},$$

where $\mathfrak{a} = (\gamma)\mathfrak{a}_j'$, so that

$$\sum_{\gamma\in \mathfrak{o}_F^\times\backslash(\mathfrak{a}_j'^{-1}\cap F^\times)} \frac{a_\xi(t_j,\gamma)e_{\mathfrak{q}'}((\gamma\beta),\chi'_\infty)}{\|\gamma\|_\infty^{s-\frac{1}{2}}} = N(\mathfrak{a}_j')^{s-\frac{1}{2}} \sum_{\mathfrak{a}\sim\mathfrak{a}_j'} \frac{\lambda_\pi(\mathfrak{a})e_{\mathfrak{q}'}(\mathfrak{a}\mathfrak{a}_j'^{-1}(\beta),\chi'_\infty)}{N(\mathfrak{a})^s}.$$

This is precisely the additive twist $N(\mathfrak{a}_j')^{s-\frac{1}{2}}L_{\mathfrak{a}_j'}(s,\lambda_\pi,\beta,\chi'_\infty)$. Moreover, the integral in (5.5) equals $\prod_{v|\infty}L(s,\pi_v\otimes\chi_v'^{-1})$. Putting everything together, we have

$$\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times} \Phi(y,\alpha)\|y\|_\infty^{s-\frac{1}{2}}\, d^\times y$$
$$= N((\beta)^{-1}\mathfrak{a}_j')^{s-\frac{1}{2}}L_{\mathfrak{a}_j'}(s,\lambda_\pi,\beta,\chi'_\infty)\prod_{v|\infty}L(s,\pi_v\otimes\chi_v'^{-1}) = \Lambda_{\mathfrak{a}_j'}(s,\lambda_\pi,\beta,\chi'_\infty). \quad (5.6)$$

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For future reference, we note that if we replace (y,α) in this analysis by $(\lambda^{-1}y,\lambda\alpha)$ for some $\lambda\in T$ (and hence also replace β by $\lambda^{-1}\beta$), we arrive at

$$\frac{1}{[\mathfrak{o}_F^\times:\Gamma_{\mathfrak{q}'}]}\int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times}\Phi(\lambda^{-1}y,\lambda\alpha)\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y=N((\lambda))^{s-\frac{1}{2}}\Lambda_{\mathfrak{a}'_j}(s,\lambda_\pi,\lambda^{-1}\beta,\chi'_\infty).$$

5.4 Taylor expansions

We now recompute (5.4) using (5.1). For any $y\in\mathbb{A}_F^\times$, the Iwasawa decomposition yields

$$w\begin{pmatrix}(\beta_\infty y,t_j) & (\beta_\infty,0)\\ & 1\end{pmatrix}=\begin{pmatrix}z & \\ & z\end{pmatrix}\begin{pmatrix}y' & x\\ & 1\end{pmatrix}k,$$

where $x\in\mathbb{A}_F$, $y',z\in\mathbb{A}_F^\times$ and k is an element of $\prod_{v|\infty}K_v\cdot\prod_{v<\infty}\mathrm{GL}_2(\mathfrak{o}_v)$. A calculation shows that we may take

$$y'_v=\begin{cases}t_j^{-1} & \text{if } v<\infty,\\ -\dfrac{\alpha_v y_v}{1+y_v\overline{y_v}} & \text{if } v|\infty,\end{cases}\qquad x_v=\begin{cases}0 & \text{if } v<\infty,\\ \dfrac{\alpha_v}{1+y_v\overline{y_v}} & \text{if } v|\infty,\end{cases}$$

$$z_v=\begin{cases}t_j & \text{if } v<\infty,\\ \alpha_v^{-1}\sqrt{1+y_v\overline{y_v}} & \text{if } v|\infty,\end{cases}\qquad k_v=\begin{cases}w & \text{if } v<\infty,\\ \dfrac{1}{\sqrt{1+y_v\overline{y_v}}}\begin{pmatrix}1 & -\overline{y_v}\\ y_v & 1\end{pmatrix} & \text{if } v|\infty.\end{cases}$$

Then

$$\begin{aligned}\Phi(y,\alpha)&=\chi'_\infty(y)^{-1}\phi_\xi\left(\begin{pmatrix}z & \\ & z\end{pmatrix}\begin{pmatrix}y' & x\\ & 1\end{pmatrix}k\right)\\&=\sum_{\gamma\in F^\times}\chi'_\infty(y)^{-1}W_\xi\left(\begin{pmatrix}z & \\ & z\end{pmatrix}\begin{pmatrix}\gamma y' & \gamma x\\ & 1\end{pmatrix}k\right)\\&=\sum_{\gamma\in F^\times}\chi'_\infty(y)^{-1}\omega_\pi(\gamma^{-1}z)\psi(\gamma x)W_\xi\left(\begin{pmatrix}\gamma y' & \\ & 1\end{pmatrix}k\right),\end{aligned}$$

where we have used our assumption that ω_π factors through $F^\times\backslash\mathbb{A}_F^\times$. Thus,

$$\Phi(y,\alpha)=\sum_{\gamma\in F^\times}\tilde{a}_\xi(y',\gamma)\chi'_\infty(y)^{-1}\omega_{\pi_\infty}(\gamma^{-1}z)\psi_\infty(\gamma x)W_{\xi_\infty}\left(\begin{pmatrix}\gamma y' & \\ & 1\end{pmatrix}k\right),\tag{5.7}$$

where

$$\tilde{a}_\xi(y',\gamma)=\prod_{v<\infty}\omega_{\pi_v}(\gamma_v y'_v)^{-1}W_{\xi_v}\left(\begin{pmatrix}\gamma_v y'_v & \\ & 1\end{pmatrix}w\right).$$

Recall that the support of the function $W_{\xi_v}\left(\left(\begin{smallmatrix}\cdot & \\ & 1\end{smallmatrix}\right)w\right)$ is $\mathfrak{d}_v^{-1}\mathfrak{N}_v^{-1}$. Since $y'_v=t_j^{-1}$ for $v<\infty$, we may rewrite (5.7) as

$$\Phi(y,\alpha)=\sum_{\gamma\in\mathfrak{a}_j^{-1}\cap F^\times}\tilde{a}_\xi(t_j^{-1},\gamma)\chi'_\infty(y)^{-1}\omega_{\pi_\infty}(\gamma^{-1}z)\psi_\infty(\gamma x)W_{\xi_\infty}\left(\begin{pmatrix}\gamma y' & \\ & 1\end{pmatrix}k\right).$$

Note that for $v|\infty$ we have

$$\omega_{\pi_v}(\gamma^{-1}z)=\omega_{\pi_v}(\gamma\alpha)^{-1}\omega_{\pi_v}(\sqrt{1+y_v\overline{y_v}})$$

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and

$$\psi_v(\gamma x) = \psi_v(\alpha\gamma)\psi_v\biggl(-\frac{\alpha_v\gamma_v y_v\overline{y_v}}{1+y_v\overline{y_v}}\biggr).$$

Recalling that $\omega_{\pi_\infty}^{-1}\chi'_\infty = \chi_\infty$, we therefore have

$$\Phi(y,\alpha) = \chi'_\infty(-1)\sum_{\gamma\in\mathfrak{a}_j^{-1}\cap F^\times}\tilde{a}_\xi(t_j^{-1},\gamma)\chi_\infty(\gamma\alpha)\psi_\infty(\gamma\alpha)\prod_{v\mid\infty}\Phi_v(y_v,\gamma_v\alpha_v), \tag{5.8}$$

where

$$\Phi_v(y,A) = \chi'_v(-Ay)^{-1}\omega_{\pi_v}(\sqrt{1+y\overline{y}})\psi_v\biggl(-\frac{Ay\overline{y}}{1+y\overline{y}}\biggr)W_{\xi_v}\left(\begin{pmatrix}-\frac{Ay}{1+y\overline{y}} & \\ & 1\end{pmatrix}\frac{\begin{pmatrix}1 & -\overline{y}\\ y & 1\end{pmatrix}}{\sqrt{1+y\overline{y}}}\right)$$

for any $y, A\in F_v^\times$.

Although $\Phi(y,\alpha)$ is invariant under $y\mapsto\epsilon y$ for any $\epsilon\in\Gamma_{\mathfrak{q}'}$, that is not apparent from the right-hand side of (5.8). This will be remedied in the process of taking a linear combination over different choices of α. However, in so doing we must carry out the arguments in a different order from the classical case, taking the linear combination before we take the Mellin transform. The key results concerning the Taylor expansion of (5.8) that we need in order to complete the proof are contained in the following lemma; we postpone its rather technical proof until the next section.

LEMMA 5.2. (i) *For $v\in S_\mathbb{C}$, there are polynomials $P_v(s;m,n)$ such that*

$$\begin{aligned}\Phi_v(y,A) = O_{M,K}(|y|^M\|Ay\|_v^{-K}) &+ \sum_{m=0}^{M-1}\sum_{\substack{-m\leqslant n\leqslant m\\ n\equiv m \ (\mathrm{mod}\ 2)}}\frac{\|A\|_v^{-m/2}\theta_v(A)^{-n}}{2\pi i}\\ &\times\int_{\Re(s)=\frac12}P_v(s;m,n)L(s,\widetilde{\pi}_v\otimes\chi_v^{-1}\otimes\theta_v^{-n})\|Ay\|_v^{(m/2)+\frac12-s}\,ds\end{aligned}$$

for all $M\in\mathbb{Z}_{\geqslant0}$, $K\in\mathbb{R}_{\geqslant0}$ and $y,A\in F_v^\times$ with $|y|<\frac12$. In particular, for m even,

$$P_v(s;m,0) = \biggl(\frac{(2\pi i)^{m/2}}{(m/2)!}\biggr)^2\frac{L((m/2)+1-s,\pi_v\otimes\chi_v)}{L(1-s,\pi_v\otimes\chi_v)}.$$

(ii) *For $v\in S_\mathbb{R}$, there are polynomials $P_v(s;m)$ such that*

$$\begin{aligned}\Phi_v(y,A) &= O_{M,K}(|y|^M\|Ay\|_v^{-K})\\ &\quad+\sum_{\substack{m\in2\mathbb{Z}\\ 0\leqslant m/2<M}}\frac{A^{-m/2}}{2\pi i}\int_{\Re(s)=\frac12}P_v(s;m)L(s,\widetilde{\pi}_v\otimes\chi_v^{-1}\otimes\mathrm{sgn}_v^{m/2})\|Ay\|_v^{(m/2)+\frac12-s}\,ds\end{aligned}$$

for all $M\in\mathbb{Z}_{\geqslant0}$, $K\in\mathbb{R}_{\geqslant0}$ and $y,A\in F_v^\times$ with $|y|<\frac12$. In particular, for $m\equiv0\ (\mathrm{mod}\ 4)$,

$$P_v(s;m) = \frac{(2\pi i)^{m/2}}{(m/2)!}\frac{L((m/2)+1-s,\pi_v\otimes\chi_v)}{L(1-s,\pi_v\otimes\chi_v)}.$$

We express the conclusion of the lemma in more compact notation as follows. For each $v\in S_\mathbb{R}$, let $\mu_v: F_v^\times\to\mathbb{C}^\times$ be a quasi-character of the form $\mu_v(y) = y^{m_v/2}$ for some $m_v\in2\mathbb{Z}_{\geqslant0}$, and put

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$\deg(\mu_v) = m_v/2$ and

$$I_v(y,\mu_v) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} P_v(s;m_v)L(s,\widetilde{\pi}_v\otimes\chi_v^{-1}\otimes\mathrm{sgn}_v^{m_v/2})\|y\|_v^{(m_v/2)+\frac{1}{2}-s}\,ds,$$

where P_v is as given by the lemma. Similarly, for $v\in S_{\mathbb{C}}$, let $\mu_v(y)=\|y\|_v^{m_v/2}\theta_v(y)^{n_v}=y^{(m_v+n_v)/2}\overline{y}^{(m_v-n_v)/2}$ for some $m_v\in\mathbb{Z}_{\geqslant 0}$, $n_v\in m_v+2\mathbb{Z}$ with $|n_v|\leqslant m_v$, and put $\deg(\mu_v)=m_v$ and

$$I_v(y,\mu_v) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} P_v(s;m_v,n_v)L(s,\widetilde{\pi}_v\otimes\chi_v^{-1}\otimes\theta_v^{-n_v})\|y\|_v^{(m_v/2)+\frac{1}{2}-s}\,ds.$$

Then, for each $v|\infty$, we have

$$\Phi_v(y,A) = O_{M,K}(|y|^M\|Ay\|_v^{-K}) + \sum_{\mu_v:\deg(\mu_v)<M}\mu_v(A)^{-1}I_v(Ay,\mu_v).$$

Next, define $\mu: F_\infty^\times\to\mathbb{C}^\times$ by $\mu(y)=\prod_{v|\infty}\mu_v(y_v)$, and put $\deg(\mu)=\sum_{v|\infty}\deg(\mu_v)$, $I(y,\mu)=\prod_{v|\infty}I_v(y_v,\mu_v)$. Note that μ is a monomial of total degree $\deg(\mu)$ in the $[F:\mathbb{Q}]$ variables y_v for $v\in S_{\mathbb{R}}$ and $y_v,\overline{y_v}$ for $v\in S_{\mathbb{C}}$. Let R_M denote the set of all such μ with $\deg(\mu)<M$. Multiplying the series given by Lemma 5.2 over all $v|\infty$, we have

$$\prod_{v|\infty}\Phi_v(y_v,A_v) = O_{M,K}\left(\|Ay\|_\infty^{-K}\max_{v|\infty}|y_v|^M\right) + \sum_{\mu\in R_M}\mu(A)^{-1}I(Ay,\mu)$$

for any $y, A\in F_\infty^\times$ with $\max_{v|\infty}|y_v|<\frac{1}{2}$. Moreover, replacing (y,A) by $(\lambda^{-1}y,\lambda A)$ for some $\lambda\in T$, we have

$$\prod_{v|\infty}\Phi_v(\lambda_v^{-1}y_v,\lambda_vA_v) = O_{M,K,T}\left(\|Ay\|_\infty^{-K}\max_{v|\infty}|y_v|^M\right) + \sum_{\mu\in R_M}\mu(\lambda A)^{-1}I(Ay,\mu), \tag{5.9}$$

provided that y satisfies

$$\max_{v|\infty}|y_v| < \delta_T = \frac{1}{2}\min_{\lambda\in T}\min_{v|\infty}|\lambda_v|. \tag{5.10}$$

Note also that the implied constant in (5.9) now depends on the set T.

Let $c_\lambda\in\mathbb{C}$ for each $\lambda\in T$. Substituting (5.9) into (5.8), we have

$$\begin{aligned}\sum_{\lambda\in T}c_\lambda\chi_\infty(\lambda)^{-1}\Phi(\lambda^{-1}y,\lambda\alpha) &= O_{M,K,T}\left(\max_{v|\infty}|y_v|^M\sum_{\gamma\in\mathfrak{a}_j^{-1}\cap F^\times}\sum_{\lambda\in T}|c_\lambda\tilde{a}_\xi(t_j^{-1},\gamma)|\cdot\|\gamma\alpha y\|_\infty^{-K}\right)\\ &+\ \chi'_\infty(-1)\sum_{\gamma\in\mathfrak{a}_j^{-1}\cap F^\times}\tilde{a}_\xi(t_j^{-1},\gamma)\chi_\infty(\gamma\alpha)\psi_\infty(\gamma\alpha)\sum_{\mu\in R_M}\sum_{\lambda\in T}c_\lambda\mu(\lambda\gamma\alpha)^{-1}I(\gamma\alpha y,\mu).\end{aligned} \tag{5.11}$$

Lemma 5.3. *Let $M\in\mathbb{Z}_{\geqslant 0}$ and $\mu_0\in R_M$. Then there is a finite subset $T\subset(1+\mathfrak{q})\cap F^\times$ and coefficients $c_\lambda\in\mathbb{C}$ for each $\lambda\in T$ such that*

$$\sum_{\lambda\in T}c_\lambda\mu(\lambda)^{-1} = \begin{cases}1 & \text{if } \mu=\mu_0,\\ 0 & \text{if } \mu\neq\mu_0\end{cases}$$

for all $\mu\in R_M$.

Proof. Let $\tau_i: F\to\mathbb{C}$, $1\leqslant i\leqslant[F:\mathbb{Q}]$, be the distinct field embeddings of F into $\mathbb{C}$ and $\tau: F\to\mathbb{C}^{[F:\mathbb{Q}]}$ the 'canonical embedding' $\tau=(\tau_1,\ldots,\tau_{[F:\mathbb{Q}]})$. It is well known that the image of any non-zero integral ideal under τ is a lattice of full rank, and it follows easily that the

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set $S = \{\tau(\lambda^{-1}) : \lambda \in (1+\mathfrak{q}) \cap F^\times\}$ is Zariski dense in $\mathbb{C}^{[F:\mathbb{Q}]}$. Moreover, as noted above, $\mu(\lambda^{-1})$ is a monomial function of $\tau(\lambda^{-1})$ of total degree $\deg(\mu)$.

Next, consider the vector space V of functions $f : R_M \to \mathbb{C}$. To each $\lambda \in (1+\mathfrak{q}) \cap F^\times$ we associate the function $f_\lambda \in V$ defined by $f_\lambda(\mu) = \mu(\lambda^{-1})$, and define $V' \subset V$ to be the span of the f_λ. If $V' \neq V$, then there is a non-zero linear functional $L : V \to \mathbb{C}$ which vanishes on V'. Note that L evaluated at f_λ is a polynomial function of $\tau(\lambda^{-1})$, and hence its existence contradicts the Zariski density of S. Therefore, $V' = V$, and the conclusion follows by choosing T such that $\{f_\lambda : \lambda \in T\}$ is a basis. □

We now fix $m_0 \in 4\mathbb{Z}$ with $0 \leqslant m_0 < M$, and take $\mu_0(y) = \|y\|_\infty^{m_0/2}$ in Lemma 5.3. Then (5.11) simplifies to

$$\begin{aligned}\sum_{\lambda\in T} & c_\lambda \chi_\infty(\lambda)^{-1}\Phi(\lambda^{-1}y,\lambda\alpha)\\ &= E_{M,m_0,\alpha}(y) + \chi'_\infty(-1)\sum_{\gamma\in\mathfrak{a}_j^{-1}\cap F^\times}\tilde{a}_\xi(t_j^{-1},\gamma)\chi_\infty(\gamma\alpha)\psi_\infty(\gamma\alpha)\|\gamma\alpha\|_\infty^{-m_0/2} I(\gamma\alpha y,\mu_0),\end{aligned} \tag{5.12}$$

where $E_{M,m_0,\alpha}(y)$ is defined by this equation and satisfies

$$E_{M,m_0,\alpha}(y) \ll_{M,K,m_0} \max_{v|\infty}|y_v|^M \sum_{\gamma\in\mathfrak{a}_j^{-1}\cap F^\times} |\tilde{a}_\xi(t_j^{-1},\gamma)|\cdot\|\gamma\alpha y\|_\infty^{-K}.$$

Applying the same folding analysis as before, the sum over γ in (5.12) can be factored as

$$\chi'_\infty(-1)\sum_{\gamma\in\,\mathfrak{o}_F^\times\backslash(\mathfrak{a}_j^{-1}\cap F^\times)}\tilde{a}_\xi(t_j^{-1},\gamma)\sum_{\eta\in\Gamma_{\mathfrak{q}'}\backslash\mathfrak{o}_F^\times}\psi_\infty(\eta\gamma\alpha)\chi_\infty(\eta\gamma\alpha)\|\gamma\alpha\|_\infty^{-m_0/2}\sum_{\epsilon\in\Gamma_{\mathfrak{q}'}} I(\epsilon\eta\gamma\alpha y,\mu_0).$$

Multiplying this part by $\|y\|_\infty^{s-\frac{1}{2}}/[\mathfrak{o}_F^\times : \Gamma_{\mathfrak{q}'}]$, integrating over $\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times$ and unfolding the sum over ϵ, we get

$$\chi'_\infty(-1)\sum_{\gamma\in\,\mathfrak{o}_F^\times\backslash(\mathfrak{a}_j^{-1}\cap F^\times)}\tilde{a}_\xi(t_j^{-1},\gamma)e_{\mathfrak{q}'}((\gamma\alpha),\chi_\infty)\|\gamma\alpha\|_\infty^{\frac{1}{2}-(m_0/2)-s}\int_{F_\infty^\times} I(y,\mu_0)\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y. \tag{5.13}$$

Note that

$$\int_{F_\infty^\times} I(y,\mu_0)\|y\|_\infty^{s-\frac{1}{2}}\,d^\times y = P\Big(s+\frac{m_0}{2};m_0\Big)L\Big(s+\frac{m_0}{2},\widetilde{\pi}_\infty\otimes\chi_\infty^{-1}\Big),$$

where

$$\begin{aligned}P(s;m_0) &= \prod_{v\in S_\mathbb{R}} P_v(s;m_0)\prod_{v\in S_\mathbb{C}} P_v(s;m_0,0)\\ &= \bigg(\frac{(2\pi i)^{m_0/2}}{(m_0/2)!}\bigg)^{[F:\mathbb{Q}]}\frac{L(1-s+(m_0/2),\pi_\infty\otimes\chi_\infty)}{L(1-s,\pi_\infty\otimes\chi_\infty)}.\end{aligned} \tag{5.14}$$

Recall also that the Fourier coefficients $\tilde{a}_\xi(t_j^{-1},\gamma)$ are related to the Dirichlet coefficients $\lambda_{\widetilde{\pi}}$ of $L(s,\widetilde{\pi})$ via

$$\tilde{a}_\xi(t_j^{-1},\gamma) = \omega_\pi(d)\epsilon_f(\pi,\psi)\frac{\lambda_{\widetilde{\pi}}(\mathfrak{a})}{\sqrt{N(\mathfrak{a})}},$$

where $\mathfrak{a} = (\gamma)\mathfrak{a}_j$. Hence, (5.13) is precisely $\kappa P(s+(m_0/2);m_0)\Lambda_{\mathfrak{a}_j}(s+(m_0/2),\lambda_{\widetilde{\pi}},\alpha,\chi_\infty)$, where $\kappa = \chi'_\infty(-1)\omega_\pi(d)\epsilon_f(\pi,\psi)$.

Moreover, choosing $K=\sigma+1$, the error term converges and gives the estimate

$$E_{M,m_0,\alpha}(y) \ll_{M,m_0,\alpha} \max_{v|\infty} |y_v|^M \|y\|_\infty^{-K} \tag{5.15}$$

for any y satisfying (5.10). It is now clear from (5.12) that $E_{M,m_0,\alpha}$ factors through $\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times$ and decays rapidly as $\|y\|_\infty \to \infty$. Hence, the integral

$$H_{M,m_0,\alpha}(s) = \frac{1}{[\mathfrak{o}_F^\times : \Gamma_{\mathfrak{q}'}]} \int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^\times} E_{M,m_0,\alpha}(y) \|y\|_\infty^{s-\frac{1}{2}} \, d^\times y$$

converges for $\Re(s)$ sufficiently large. Let $F_\infty^1 = \{y \in F_\infty : \|y\|_\infty = 1\}$. Then any $y \in F_\infty^\times$ can be expressed as $t^{1/[F:\mathbb{Q}]}y'$, where $t = \|y\|_\infty$ and $y' \in F_\infty^1$. Thus,

$$H_{M,m_0,\alpha}(s) = \frac{[F:\mathbb{Q}]}{[\mathfrak{o}_F^\times : \Gamma_{\mathfrak{q}'}]} \int_0^\infty t^{s-\frac{1}{2}} \int_{\Gamma_{\mathfrak{q}'}\backslash F_\infty^1} E_{M,m_0,\alpha}(t^{1/[F:\mathbb{Q}]}y) \, d^\times y \, \frac{dt}{t}.$$

Fix a compact fundamental domain $\mathcal{F}_{\mathfrak{q}'}$ for $\Gamma_{\mathfrak{q}'}\backslash F_\infty^1$ and set $\delta_T' = \inf_{y\in\mathcal{F}_{\mathfrak{q}'}} \min_{v|\infty} (\delta_T/|y_v|)^{[F:\mathbb{Q}]}$. Then, by (5.15), the inner integral is $O_{M,m_0,\alpha}(t^{M/[F:\mathbb{Q}]-K})$ for $t < \delta_T'$, and it follows that $H_{M,m_0,\alpha}$ is holomorphic for $s > K + \frac{1}{2} - (M/[F:\mathbb{Q}])$.

Putting this together with the analysis of the left-hand side, we have

$$\begin{aligned} &\kappa P\left(s+\frac{m_0}{2}; m_0\right) \Lambda_{\mathfrak{a}_j}\left(s+\frac{m_0}{2}, \lambda_{\widetilde{\pi}}, \alpha, \chi_\infty\right) \\ &\quad = \sum_{\lambda\in T} c_\lambda \chi_\infty(\lambda)^{-1} N((\lambda))^{s-\frac{1}{2}} \Lambda_{\mathfrak{a}_j'}(s, \lambda_\pi, \lambda^{-1}\beta, \chi_\infty') - H_{M,m_0,\alpha}(s). \end{aligned}$$

Choosing $m_0 > 4\sigma - 2$ and $M > \max(m_0, [F:\mathbb{Q}](m_0 + 2\sigma + 2K - 1)/2)$, we see that $P(s;m_0)\Lambda_{\mathfrak{a}_j}(s, \lambda_{\widetilde{\pi}}, \alpha, \chi_\infty)$ is entire. Finally, in view of (5.14), note that the zeros of $P(s;m_0)$ are among the poles of $L(1-s, \pi_\infty \otimes \chi_\infty)$. As observed in § 4.3, since π_v is unitary for every $v|\infty$, there are no such poles for $\Re(s) \leqslant \frac{1}{2}$, and it follows that $\Lambda_{\mathfrak{a}_j}(s, \lambda_{\widetilde{\pi}}, \alpha, \chi_\infty)$ is holomorphic for $\Re(s) \leqslant \frac{1}{2}$, as required.

5.5 Proof of Lemma 5.2

Note first that we may express the result in terms of π_v rather than $\widetilde{\pi}_v$, thanks to the isomorphism $\widetilde{\pi}_v \otimes \chi_v^{-1} \cong \pi_v \otimes \chi_v'^{-1}$; this is more convenient since our test functions are chosen from the Whittaker model for π_v. Moreover, it is easy to see that if the conclusions of Lemma 5.2 hold for some choice of χ_v then they hold for $|\cdot|^\nu \chi_v$ as well, for any $\nu \in i\mathbb{R}$. Hence, we may assume without loss of generality that $\nu(\omega_{\pi_v}) + 2\nu(\chi_v) = 0$. With this normalization, a short computation shows that the polynomials that we expect are

$$\begin{aligned} P_v(s;m,0) &= (-1)^{m/2} \binom{s-1-\frac{|k(\pi_v)+k(\omega_{\pi_v})-2k(\chi_v')|}{4}+\nu(\pi_v)}{m/2} \\ &\quad \times \binom{s-1-\frac{|k(\pi_v)-k(\omega_{\pi_v})+2k(\chi_v')|}{4}-\nu(\pi_v)}{m/2}, \end{aligned} \tag{5.16}$$

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for $v \in S_{\mathbb{C}}$ and m even, and

$$P_v(s;m) = \begin{cases} \frac{(-4)^{m/4}}{\binom{m/2}{m/4}} \binom{\frac{s-1-|\epsilon(\chi_v')-\epsilon(\pi_v)|+\nu(\pi_v)}{2}}{m/4} \binom{\frac{s-1-|\epsilon(\chi_v')-\epsilon(\pi_v)|-\nu(\pi_v)}{2}}{m/4} & \text{if } v \in S_0, \\ \frac{(-4)^{m/4}}{\binom{m/2}{m/4}} \binom{\frac{s+\chi_v'(-1)\nu(\pi_v)-1}{2}}{m/4} \binom{\frac{s-\chi_v'(-1)\nu(\pi_v)-2}{2}}{m/4} & \text{if } v \in S_1, \\ (-1)^{m/4} \binom{s-\frac{k(\pi_v)+1}{2}}{m/2} & \text{if } v \in S_d \end{cases} \tag{5.17}$$

for $v \in S_{\mathbb{R}}$ and $m \equiv 0 \pmod 4$.

5.5.1 *Complex places.* Let notation be as in § 4.2. Our first task is to get a better understanding of all of the Whittaker functions $W_{N,k}$, since it is not only the boundary ones that play a role. It is easiest to do this via their Mellin transforms; precisely, we define

$$\widetilde{f}_{N,k}(s) = \int_{F_v^\times} f_{N,k}(|z|)\|z\|_v^s \, d^\times z,$$

so that

$$f_{N,k}(t) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma_0} \widetilde{f}_{N,k}(s) t^{1-2s} \, ds \tag{5.18}$$

for $t > 0$ and any $\sigma_0 > 0$ sufficiently large. Then the differential equations (4.5) become

$$4\pi i(N+k)\widetilde{f}_{N,k-2}(s) = 16\pi^2 \widetilde{f}_{N,k}\left(s+\frac{1}{2}\right) - \left[\left(2s-1-\frac{k}{2}\right)^2 - p^2\right]\widetilde{f}_{N,k}\left(s-\frac{1}{2}\right) \tag{5.19}$$

and

$$-4\pi i(N-k)\widetilde{f}_{N,k+2}(s) = 16\pi^2 \widetilde{f}_{N,k}\left(s+\frac{1}{2}\right) - \left[\left(2s-1+\frac{k}{2}\right)^2 - q^2\right]\widetilde{f}_{N,k}\left(s-\frac{1}{2}\right), \tag{5.20}$$

where we write $p = 2\nu(\pi_v) + k(\pi_v)/2$, $q = 2\nu(\pi_v) - (k(\pi_v)/2)$.

Next, let

$$\gamma_k(s) = \Gamma_{\mathbb{C}}\left(s + \nu(\pi_v) + \frac{|k-k(\pi_v)|}{4}\right)\Gamma_{\mathbb{C}}\left(s - \nu(\pi_v) + \frac{|k+k(\pi_v)|}{4}\right)$$

and

$$F_{N,k}(s) = \frac{\widetilde{f}_{N,k}(s)}{\gamma_k(s)}.$$

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Up to a shift, $\gamma_k(s)$ is the L-factor of π_v twisted by any character that is not orthogonal to $W_{N,k}$. In terms of $F_{N,k}$, the differential equations become

$$-\frac{N+k}{4\pi i}F_{N,k-2}(s)=\begin{cases} F_{N,k}(s+1/2)-F_{N,k}(s-1/2) & \text{if } k\leqslant -k(\pi_v),\\ \dfrac{2s-1+k/2+q}{4\pi}F_{N,k}(s+1/2) & \\ \qquad -\dfrac{2s-1-k/2+p}{4\pi}F_{N,k}(s-1/2) & \text{if } -k(\pi_v)<k\leqslant k(\pi_v),\\ \dfrac{(2s-1+k/2)^2-q^2}{16\pi^2}F_{N,k}(s+1/2) & \\ \qquad -\dfrac{(2s-1-k/2)^2-p^2}{16\pi^2}F_{N,k}(s-1/2) & \text{if } k>k(\pi_v)\end{cases}$$

and

$$\frac{N-k}{4\pi i}F_{N,k+2}(s)=\begin{cases} \dfrac{(2s-1-k/2)^2-p^2}{16\pi^2}F_{N,k}(s+1/2) & \\ \qquad -\dfrac{(2s-1+k/2)^2-q^2}{16\pi^2}F_{N,k}(s-1/2) & \text{if } k<-k(\pi_v),\\ \dfrac{2s-1-k/2+p}{4\pi}F_{N,k}(s+1/2) & \\ \qquad -\dfrac{2s-1+k/2+q}{4\pi}F_{N,k}(s-1/2) & \text{if } -k(\pi_v)\leqslant k<k(\pi_v),\\ F_{N,k}(s+1/2)-F_{N,k}(s-1/2) & \text{if } k\geqslant k(\pi_v).\end{cases}$$

In particular, we have $F_{N,\pm N}(s+1/2)-F_{N,\pm N}(s-1/2)=0$, i.e. $F_{N,\pm N}$ is periodic with period 1. From Stirling's formula and the fact that $\widetilde{f}_{N,k}$ must decay in vertical strips (since $W_{N,k}$ is smooth), it follows that $F_{N,\pm N}$ is constant. A simple inductive argument then shows that $F_{N,k}$ is a polynomial of degree $(N-\max(k(\pi_v),|k|))/2$ and, if we normalize matters so that $F_{N,l}$ is monic for some $l\in[-N,N]$, then $F_{N,k}$ has leading coefficient $(2\pi)^{(\max(k(\pi_v),|k|)-\max(k(\pi_v),|l|))/2}i^{(k-l)/2}$. From these facts, it is easy to verify the formulas for $f_{N,k}$ given in § 4.2.

Since π_v is unitary, $\widetilde{f}_{N,k}(s)=F_{N,k}(s)\gamma_k(s)$ is holomorphic in a right half-plane $\Re(s)>\frac{1}{2}-\delta$ for some $\delta>0$, as noted in § 4.3. It follows from Stirling's formula that we have

$$\widetilde{f}_{N,k}(s)\ll_{\pi_v,N,k,\Re(s),\varepsilon}e^{-(\pi-\varepsilon)|s|}$$

for fixed $\varepsilon>0$ and any $s\in\mathbb{C}$ with fixed real part $\Re(s)\geqslant\frac{1}{2}$. We shift the contour of (5.18) to $\Re(s)=K+\frac{1}{2}$ for some $K\in\mathbb{R}_{\geqslant 0}$ and differentiate to obtain

$$\begin{aligned}\frac{t^a f^{(a)}_{N,k}(t)}{a!}&=\frac{1}{2\pi i}\int_{\Re(s)=K+\frac{1}{2}}\binom{1-2s}{a}\widetilde{f}_{N,k}(s)t^{1-2s}\,ds\\ &\ll_{\pi_v,N,k,K,\varepsilon}t^{-2K}\int_{\Re(s)=K+\frac{1}{2}}\left|\binom{1-2s}{a}\right|e^{-(\pi-\varepsilon)|s|}\,|ds|.\end{aligned}$$

Note that

$$\left|\binom{1-2s}{a}\right|=\left|\binom{2s+a-2}{a}\right|\leqslant\binom{|2s-1|+a-1}{a}\leqslant 2^{|2s|+a}.$$

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Since $4 < e^{\pi}$, we may choose ε small enough that the integral converges, giving the estimate

$$\frac{t^a f_{N,k}^{(a)}(t)}{a!} \ll_{\pi_v,N,k,K} 2^a t^{-2K} \tag{5.21}$$

for all $a \in \mathbb{Z}_{\geqslant 0}$, $K \in \mathbb{R}_{\geqslant 0}$.

Next, we compute the right action of $1/\sqrt{1+y\overline{y}}\left(\begin{smallmatrix} 1 & -\overline{y} \\ y & 1 \end{smallmatrix}\right)$ on $W_{\xi_v} = W_{N,l}$, where $l = -k(\omega_{\pi_v}) + 2k(\chi'_v) = k(\omega_{\pi_v}) + 2k(\chi_v)$. Note that

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} 1 & -\overline{y} \\ y & 1 \end{pmatrix} = \begin{pmatrix} X + yY & Y - \overline{y}X \end{pmatrix},$$

so that $X^{(N+l)/2}Y^{(N-l)/2}$ is mapped to

$$\begin{aligned} &\frac{(X+yY)^{(N+l)/2}(Y-\overline{y}X)^{(N-l)/2}}{(1+y\overline{y})^{N/2}} \\ &\quad = (1+y\overline{y})^{-N/2} \sum_{j_1=0}^{(N+l)/2} \sum_{j_2=0}^{(N-l)/2} \binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2} y^{j_1}(-\overline{y})^{j_2} X^{((N+l)/2)+j_2-j_1} Y^{((N-l)/2)+j_1-j_2} \\ &\quad = (1+y\overline{y})^{-N/2} \sum_{j_1=0}^{(N+l)/2} \sum_{j_2=0}^{(N-l)/2} (-1)^{j_2} \binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2} \\ &\qquad \times |y|^{j_1+j_2} \theta_v(y)^{j_1-j_2} X^{((N+l)/2)+j_2-j_1} Y^{((N-l)/2)+j_1-j_2}. \end{aligned}$$

Thus, after a short computation using our assumption that $\nu(\omega_{\pi_v}) + 2\nu(\chi_v) = 0$, we have

$$\begin{aligned} &\chi'_v(-Ay)^{-1}\omega_{\pi_v}(\sqrt{1+y\overline{y}})e\left(-\operatorname{tr}_{F_v/\mathbb{R}} \frac{Ay\overline{y}}{1+y\overline{y}}\right) W_{\xi_v}\left(\begin{pmatrix} -\frac{Ay}{1+y\overline{y}} & \\ & 1 \end{pmatrix} \frac{\left(\begin{smallmatrix} 1 & -\overline{y} \\ y & 1 \end{smallmatrix}\right)}{\sqrt{1+y\overline{y}}}\right) \\ &\quad = \sum_{j_1,j_2 \geqslant 0} \binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2} (-1)^{j_2} |y|^{j_1+j_2} \theta_v(-A)^{j_2-j_1} \\ &\qquad \times (1+y\overline{y})^{-N/2} e\left(-\frac{(A+\overline{A})y\overline{y}}{1+y\overline{y}}\right) f_{N,l+2j_1-2j_1}\left(\frac{|Ay|}{1+y\overline{y}}\right). \end{aligned} \tag{5.22}$$

Next, using (5.21), we have

$$\begin{aligned} f_{N,k}\left(\frac{|Ay|}{1+y\overline{y}}\right) &= \sum_{a=0}^{\infty} \frac{f_{N,k}^{(a)}(|Ay|)}{a!}\left(-\frac{|Ay|y\overline{y}}{1+y\overline{y}}\right)^a \\ &= \sum_{\substack{a \in \mathbb{Z}_{\geqslant 0} \\ 2a < a_0}} \frac{f_{N,k}^{(a)}(|Ay|)}{a!}\left(-\frac{|Ay|y\overline{y}}{1+y\overline{y}}\right)^a + O_{\pi_v,N,k,K}\left(|Ay|^{-2K} \sum_{\substack{a \in \mathbb{Z}_{\geqslant 0} \\ 2a \geqslant a_0}} \left(\frac{2y\overline{y}}{1+y\overline{y}}\right)^a\right) \\ &= \sum_{\substack{a \in \mathbb{Z}_{\geqslant 0} \\ 2a < a_0}} \frac{f_{N,k}^{(a)}(|Ay|)}{a!}\left(-\frac{|Ay|y\overline{y}}{1+y\overline{y}}\right)^a + O_{\pi_v,N,k,K,a_0}(\|Ay\|_v^{-K}|y|^{a_0}) \end{aligned}$$

for any $a_0 \in \mathbb{Z}_{\geqslant 0}$ and y such that $|y| < \frac{1}{2}$, say. Multiplying by $e(-(A+\overline{A})y\overline{y}/(1+y\overline{y}))$, which has modulus 1, we have

$$f_{N,k}\left(\frac{|Ay|}{1+y\overline{y}}\right)e\left(-\frac{(A+\overline{A})y\overline{y}}{1+y\overline{y}}\right) = \sum_{\substack{a\in\mathbb{Z}_{\geqslant 0}\\ 2a<a_0}} \frac{f_{N,k}^{(a)}(|Ay|)}{a!}\left(-\frac{|Ay|y\overline{y}}{1+y\overline{y}}\right)^a e\left(-\frac{(A+\overline{A})y\overline{y}}{1+y\overline{y}}\right) + O_{\pi_v,N,k,K,a_0}(\|Ay\|_v^{-K}|y|^{a_0}). \tag{5.23}$$

Since $e(x)$ is a bounded function of $x \in \mathbb{R}$, Taylor's theorem yields

$$e(x) = \sum_{b=0}^{b_0-1} \frac{(2\pi i x)^b}{b!} + O_{b_0}(|x|^{b_0})$$

for any $b_0 \in \mathbb{Z}_{\geqslant 0}$. Taking $b_0 = a_0 - 2a$ and substituting into (5.23), we have

$$\begin{aligned} &f_{N,k}\left(\frac{|Ay|}{1+y\overline{y}}\right)e\left(-\frac{(A+\overline{A})y\overline{y}}{1+y\overline{y}}\right) \\ &= \sum_{\substack{a\in\mathbb{Z}_{\geqslant 0}\\ 2a<a_0}} \frac{f_{N,k}^{(a)}(|Ay|)}{a!}\left(-\frac{|Ay|y\overline{y}}{1+y\overline{y}}\right)^a \sum_{b=0}^{a_0-2a-1} \frac{1}{b!}\left(-\frac{2\pi i(A+\overline{A})y\overline{y}}{1+y\overline{y}}\right)^b \\ &\quad + O_{\pi_v,N,k,K,a_0}\left(\sum_{\substack{a\in\mathbb{Z}_{\geqslant 0}\\ 2a<a_0}} \left|\frac{(Ay)^a f_{N,k}^{(a)}(|Ay|)}{a!}\right| (y\overline{y})^a |Ay\overline{y}|^{a_0-2a}\right) \\ &\quad + O_{\pi_v,N,k,K,a_0}(\|Ay\|_v^{-K}|y|^{a_0}). \end{aligned}$$

Applying (5.21) with K replaced by $K + a_0/2 - a$ (and taking the maximum of the implied constants over all choices of a), we see that the first error term can be absorbed into the second. Expanding the bth power, we have

$$\begin{aligned} &f_{N,k}\left(\frac{|Ay|}{1+y\overline{y}}\right)e\left(-\frac{(A+\overline{A})y\overline{y}}{1+y\overline{y}}\right) \\ &= \sum_{\substack{a,b\in\mathbb{Z}_{\geqslant 0}\\ 2a+b<a_0}} \sum_{t=0}^{b} (-1)^a \frac{|Ay|^{a+b} f_{N,k}^{(a)}(|Ay|)}{a!} \frac{(y\overline{y})^{a+(b/2)}}{(1+y\overline{y})^{a+b}} \frac{(2\pi i)^b}{b!} \binom{b}{t} \theta_v(-A)^{2t-b} \\ &\quad + O_{\pi_v,N,k,K,a_0}(\|Ay\|_v^{-K}|y|^{a_0}). \end{aligned} \tag{5.24}$$

Next, using the inequality

$$\left|\binom{-a-b-\frac{N}{2}}{r}\right| = \left|\binom{r+a+b+\frac{N}{2}-1}{r}\right| \leqslant 2^{r+a+b+N/2},$$

we have

$$(y\overline{y})^{a+(b/2)}(1+y\overline{y})^{-a-b-(N/2)} = \sum_{r=0}^{\infty}\binom{-a-b-\frac{N}{2}}{r}\|y\|_v^{(2a+2r+b)/2}$$
$$= \sum_{\substack{r\in\mathbb{Z}_{\geqslant 0}\\ 2a+2r+b<a_0}}\binom{-a-b-\frac{N}{2}}{r}\|y\|_v^{(2a+2r+b)/2} + O_{N,M}\Bigg(\sum_{\substack{r\in\mathbb{Z}_{\geqslant 0}\\ 2a+2r+b\geqslant a_0}}(2\|y\|_v)^{(2a+2r+b)/2}\Bigg)$$
$$= \sum_{\substack{r\in\mathbb{Z}_{\geqslant 0}\\ 2a+2r+b<a_0}}\binom{-a-b-\frac{N}{2}}{r}\|y\|_v^{(2a+2r+b)/2} + O_{N,M}(|y|^{a_0})$$

for y with $|y|<\frac{1}{2}$. Substituting into (5.24) and applying (5.21) with K replaced by $K+b/2$, we get

$$(1+y\overline{y})^{-N/2}f_{N,k}\bigg(\frac{|Ay|}{1+y\overline{y}}\bigg)e\bigg(-\frac{(A+\overline{A})y\overline{y}}{1+y\overline{y}}\bigg)$$
$$= \sum_{\substack{a,b,r\in\mathbb{Z}_{\geqslant 0}\\ 2a+2r+b<a_0}}\sum_{t=0}^{b}(-1)^a\frac{|Ay|^{a+b}f_{N,k}^{(a)}(|Ay|)}{a!}\frac{(2\pi i)^b}{b!}\binom{b}{t}\binom{-a-b-\frac{N}{2}}{r}\|y\|_v^{(2a+2r+b)/2}$$
$$\times\ \theta_v(-A)^{2t-b} + O_{\pi_v,N,k,K,a_0}(\|Ay\|_v^{-K}|y|^{a_0}). \tag{5.25}$$

Now, any term of (5.22) with $j_1+j_2\geqslant M$ becomes part of the final error term by substituting (5.25) with $a_0=0$. For $j_1+j_2<M$, we set $a_0=M-j_1-j_2$. Putting everything together, (5.22) becomes

$$\sum_{\substack{j_1,j_2,a,b,r,t\in\mathbb{Z}_{\geqslant 0}\\ 2a+2r+b+j_1+j_2<M\\ t\leqslant b}}(-1)^a\frac{f_{N,l+2j_2-2j_1}^{(a)}(|Ay|)}{a!}\frac{(2\pi i)^b}{b!}\binom{b}{t}\binom{-a-b-\frac{N}{2}}{r}$$
$$\times\ (-1)^{j_2}\binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2}\|y\|_v^{(2a+2r+b+j_1+j_2)/2}|Ay|^{a+b}\theta_v(-A)^{2t-b+j_2-j_1}$$
$$+\ O_{\pi_v,N,k,K,M}(\|Ay\|_v^{-K}|y|^M). \tag{5.26}$$

Now, writing $m=2a+2r+b+j_1+j_2$ and $n=b-2t+j_1-j_2$ (thus replacing the variable t), this is

$$\sum_{m=0}^{M-1}\sum_{\substack{n\in\mathbb{Z}\\ |n|\leqslant m,n\equiv m\ (\mathrm{mod}\ 2)}}\|y\|_v^{m/2}\theta_v(-A)^{-n}\sum_{\substack{j_1,j_2,a,b,r\in\mathbb{Z}_{\geqslant 0}\\ 2a+2r+b+j_1+j_2=m\\ |n+j_2-j_1|\leqslant b}}$$
$$\times\ (-1)^{a+j_2}|Ay|^{a+b}\frac{f_{N,l+2j_2-2j_1}^{(a)}(|Ay|)}{a!}\frac{(2\pi i)^b}{b!}\binom{b}{\frac{b+j_1-j_2-n}{2}}\binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2}\binom{-a-b-\frac{N}{2}}{r}$$
$$+\ O_{\pi_v,N,k,K,M}(\|Ay\|_v^{-K}|y|^M).$$

By Mellin inversion, we have

$$|Ay|^{a+b}\frac{f_{N,l+2j_2-2j_1}^{(a)}(|Ay|)}{a!} = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}}\binom{1-b-2s}{a}\widetilde{f}_{N,l+2j_2-2j_1}(s+b/2)\|Ay\|_v^{\frac{1}{2}-s}\,ds.$$

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Further, we replace a by $((m-b-j_1-j_2)/2)-r$ to get

$$\sum_{m=0}^{M-1} \|y\|_v^{m/2} \sum_{\substack{n\in\mathbb{Z}\\ |n|\leqslant m, n\equiv m \pmod 2}} \theta_v(-A)^{-n} \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \sum_{j_1,j_2\geqslant 0} \binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2}$$
$$\times \sum_{\substack{b\in\mathbb{Z}\\ |n+j_2-j_1|\leqslant b\leqslant m-j_1-j_2\\ b\equiv n+j_2-j_1 \pmod 2}} (-1)^{j_2}\frac{(2\pi i)^b}{b!}\binom{b}{\frac{b+j_1-j_2-n}{2}}$$
$$\times \sum_{r=0}^{(m-b-j_1-j_2)/2} (-1)^{((m-b-j_2-j_1)/2)-r}\binom{r-\frac{m+b+N-j_1-j_2}{2}}{r}$$
$$\times \binom{1-b-2s}{\frac{m-b-j_1-j_2}{2}-r}\widetilde{f}_{N,l+2j_2-2j_1}(s+b/2)\|Ay\|_v^{\frac{1}{2}-s}\,ds + O_{\pi_v,N,k,K,M}(\|Ay\|_v^{-K}|y|^M).$$

Now,

$$\sum_{r=0}^{m-b-j_1-j_2/2} (-1)^{((m-b-j_2-j_1)/2)-r}\binom{r-\frac{m+b+N-j_1-j_2}{2}}{r}\binom{1-b-2s}{\frac{m-b-j_1-j_2}{2}-r}$$
$$= \sum_{r=0}^{(m-b-j_1-j_2)/2} (-1)^{(m-b-j_2-j_1)/2}\binom{\frac{m+b+N-j_1-j_2}{2}-1}{r}\binom{1-b-2s}{\frac{m-b-j_1-j_2}{2}-r}$$
$$= (-1)^{(m-b-j_2-j_1)/2}\binom{\frac{m+N-b-j_1-j_2}{2}-2s}{\frac{m-b-j_1-j_2}{2}} = \binom{2s-\frac{N}{2}-1}{\frac{m-b-j_1-j_2}{2}}. \tag{5.27}$$

Substituting this yields

$$\sum_{m=0}^{M-1} \|y\|_v^{m/2} \sum_{\substack{n\in\mathbb{Z}\\ |n|\leqslant m, n\equiv m \pmod 2}} \theta_v(-A)^{-n} \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \sum_{j_1,j_2\geqslant 0} \binom{\frac{N+l}{2}}{j_1}\binom{\frac{N-l}{2}}{j_2}$$
$$\times \sum_{\substack{b\in\mathbb{Z}\\ |n+j_2-j_1|\leqslant b\leqslant m-j_1-j_2\\ b\equiv n+j_2-j_1 \pmod 2}} (-1)^{j_2}\frac{(2\pi i)^b}{b!}\binom{b}{\frac{b+j_1-j_2-n}{2}}\binom{2s-\frac{N}{2}-1}{\frac{m-b-j_1-j_2}{2}}$$
$$\times\ \widetilde{f}_{N,l+2j_2-2j_1}(s+b/2)\|Ay\|_v^{\frac{1}{2}-s}\,ds + O_{\pi_v,N,k,K,M}(\|Ay\|_v^{-K}|y|^M).$$

It is easy to verify that

$$\frac{\widetilde{f}_{N,l+2j_2-2j_1}(s+t/2)}{\gamma_{l-2n}(s)} = F_{N,l+2j_2-2j_1}(s+b/2)\frac{\gamma_{l+2j_2-2j_1}(s+b/2)}{\gamma_{l-2n}(s)}$$

is a polynomial function of s, so the whole expression takes the form

$$\sum_{m=0}^{M-1} \|y\|_v^{m/2} \sum_{\substack{n\in\mathbb{Z}\\ |n|\leqslant m, n\equiv m \pmod 2}} \theta_v(A)^{-n} \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} P_v(s;m,n)\gamma_{l-2n}(s)\|Ay\|_v^{\frac{1}{2}-s}\,ds$$
$$+ O_{\pi_v,N,k,K,M}(\|Ay\|_v^{-K}|y|^M),$$

where $P_v(s;m,n)$ is a polynomial. Moreover, since we have assumed that $\nu(\omega_{\pi_v})+2\nu(\chi_v)=0$,

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we have

$$\gamma_{l-2n}(s) = L(s, \pi_v \otimes \chi_v'^{-1} \otimes \theta_v^{-n}) = L(s, \widetilde{\pi}_v \otimes \chi_v^{-1} \otimes \theta_v^{-n}).$$

It remains to evaluate this polynomial when $n = 0$ and m is an even integer. In this case, we have

$$P_v(s; m, 0) = \sum_{j_1, j_2 \geqslant 0} (-1)^{j_2} \binom{\frac{N+l}{2}}{j_1} \binom{\frac{N-l}{2}}{j_2} \times \sum_{\substack{b \in \mathbb{Z} \\ |j_2 - j_1| \leqslant b \leqslant m - j_1 - j_2 \\ b \equiv j_1 + j_2 \ (\mathrm{mod}\ 2)}} \frac{(2\pi i)^b}{b!} \binom{b}{\frac{b+j_1-j_2}{2}} \binom{2s - \frac{N}{2} - 1}{\frac{m-b-j_1-j_2}{2}} \frac{\widetilde{f}_{N,l+2j_2-2j_1}(s + b/2)}{\gamma_l(s)}.$$

We make the change of variables $b = 2r - j_1 - j_2$:

$$\sum_{\substack{j_1, j_2, r \in \mathbb{Z} \\ 0 \leqslant j_1, j_2 \leqslant r}} (-1)^{j_2} \binom{\frac{N+l}{2}}{j_1} \binom{\frac{N-l}{2}}{j_2} \frac{(2\pi i)^{2r-j_1-j_2}}{(r-j_1)(r-j_2)!} \binom{2s - \frac{N}{2} - 1}{\frac{m}{2} - r} \frac{\widetilde{f}_{N,l+2j_2-2j_1}(s + r + (j_1+j_2)/2)}{\gamma_l(s)}.$$

We evaluate this first in the case when $N = k(\pi_v)$. Then $F_{N,k}(s)$ is the constant $i^{(k-l)/2}$, so that

$$\begin{aligned}
&\frac{(2\pi i)^{2r-j_1-j_2}}{(r-j_1)(r-j_2)!} \frac{\widetilde{f}_{N,l+2j_2-2j_1}(s + r + (j_1+j_2)/2)}{\gamma_l(s)} \\
&\quad = (-1)^{r+j_1} \frac{(2\pi)^{2r-j_1-j_2}}{(r-j_1)(r-j_2)!} \frac{\gamma_{l+2j_2-2j_1}(s + r + (j_1+j_2)/2)}{\gamma_l(s)} \\
&\quad = (-1)^{r+j_1} \binom{s - \nu(\pi_v) + \frac{N+l}{4} + r - j_1 - 1}{r - j_1} \binom{s + \nu(\pi_v) + \frac{N-l}{4} + r - j_2 - 1}{r - j_2} \\
&\quad = (-1)^{r+j_2} \binom{-s + \nu(\pi_v) - \frac{N+l}{4}}{r - j_1} \binom{-s - \nu(\pi_v) - \frac{N-l}{4}}{r - j_2}.
\end{aligned}$$

This gives

$$\sum_{\substack{j_1, j_2, r \in \mathbb{Z} \\ 0 \leqslant j_1, j_2 \leqslant r}} (-1)^{r} \binom{\frac{N+l}{2}}{j_1} \binom{\frac{N-l}{2}}{j_2} \binom{2s - \frac{N}{2} - 1}{\frac{m}{2} - r} \binom{-s + \nu(\pi_v) - \frac{N+l}{4}}{r - j_1} \binom{-s - \nu(\pi_v) - \frac{N-l}{4}}{r - j_2}.$$

Applying the Chu–Vandermonde identity to the j_1 and j_2 sums, this simplifies to

$$\sum_{r \geqslant 0} (-1)^r \binom{2s - \frac{N}{2} - 1}{\frac{m}{2} - r} \binom{-s - \nu(\pi_v) + \frac{N-l}{4}}{r} \binom{-s + \nu(\pi_v) + \frac{N+l}{4}}{r}.$$

Finally, by Lemma A.1(i), this is

$$(-1)^{m/2} \binom{s + \nu(\pi_v) - \frac{N-l}{4} - 1}{m/2} \binom{s - \nu(\pi_v) - \frac{N+l}{4} - 1}{m/2},$$

in agreement with (5.16).

Next, we handle the case $l = -N$. Then the only non-zero term in the sum over j_1 is the one with $j_1 = 0$. Writing j in place of j_2, we thus have

$$\sum_{r \geqslant 0} \binom{2s - \frac{N}{2} - 1}{\frac{m}{2} - r} \sum_{j=0}^{\min(r,N)} \frac{(-2\pi i)^{2r-j}}{r!(r-j)!} \binom{N}{j} \frac{\widetilde{f}_{N,2j-N}(s + r - j/2)}{\gamma_{-N}(s)}. \tag{5.28}$$

Set

$$G_{N,k}(s) = \frac{\widetilde{f}_{N,k}(s-k/4)}{\gamma_N(s-N/4)},$$

so that

$$\frac{\widetilde{f}_{N,2j-N}(s+r-j/2)}{\gamma_{-N}(s)} = \frac{G_{N,2j-N}(s+r-N/4)\gamma_N(s+r-N/2)}{\gamma_{-N}(s)} \tag{5.29}$$

and

$$G_{N,-N}(s) = (2\pi)^{-N}\left(\frac{N+k(\pi_v)}{2}\right)!\left(\frac{N-k(\pi_v)}{2}\right)! \\ \times \binom{s+\nu(\pi_v)+\frac{2N+k(\pi_v)}{4}-1}{\frac{N+k(\pi_v)}{2}}\binom{s-\nu(\pi_v)+\frac{2N-k(\pi_v)}{4}-1}{\frac{N-k(\pi_v)}{2}}.$$

Moreover, substituting into (5.20), we have

$$G_{N,k}(s) = G_{N,k}(s-1) - \frac{i(N-k)}{4\pi}G_{N,k+2}(s). \tag{5.30}$$

It follows that $G_{N,k}(s)$ is a polynomial of degree $(N-k)/2$.

We now consider the sum

$$S_r(s) = \sum_{j=0}^{N} \frac{j!}{(-2\pi i)^j}\binom{r}{j}\binom{N}{j}G_{N,2j-N}(s).$$

Substituting into (5.30), we have

$$\begin{aligned} S_r(s) &= S_r(s-1) - \sum_{j=0}^{N-1}\frac{(j+1)!}{(-2\pi i)^{j+1}}\binom{r}{j}\binom{N}{j+1}G_{N,2j+2-N}(s) \\ &= S_r(s-1) - \sum_{j=0}^{N}\frac{j!}{(-2\pi i)^j}\binom{r}{j-1}\binom{N}{j}G_{N,2j-N}(s) \\ &= S_r(s-1) - S_{r+1}(s) + S_r(s), \end{aligned}$$

since $\binom{r}{j-1} = \binom{r+1}{j} - \binom{r}{j}$. Hence, $S_{r+1}(s) = S_r(s-1)$, so by induction we have $S_r(s) = S_0(s-r) = G_{N,-N}(s-r)$.

By (5.29), we have

$$\begin{aligned} &\sum_{j=0}^{\min(r,N)} \frac{(-2\pi i)^{2r-j}}{r!(r-j)!}\binom{N}{j}\frac{\widetilde{f}_{N,2j-N}(s+r-j/2)}{\gamma_{-N}(s)} = \frac{(-4\pi^2)^r}{r!^2}S_r\left(s+r-\frac{N}{4}\right)\frac{\gamma_N(s+r-N/2)}{\gamma_{-N}(s)} \\ &= \frac{(-4\pi^2)^r}{r!^2}G_{N,-N}\left(s-\frac{N}{4}\right)\frac{\gamma_N(s+r-N/2)}{\gamma_{-N}(s)} = \frac{(-4\pi^2)^r}{r!^2}\frac{\gamma_N(s+r-N/2)}{\gamma_N(s-N/2)} \\ &= (-1)^r\binom{s+\nu(\pi_v)-\frac{N+k(\pi_v)}{4}+r-1}{r}\binom{s-\nu(\pi_v)-\frac{N-k(\pi_v)}{4}+r-1}{r} \\ &= (-1)^r\binom{-s-\nu(\pi_v)+\frac{N+k(\pi_v)}{4}}{r}\binom{-s+\nu(\pi_v)+\frac{N-k(\pi_v)}{4}}{r}. \end{aligned}$$

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Finally, applying Lemma A.1(i), we have that $P_v(s;m,0)$ is

$$\sum_{r\geqslant 0}(-1)^r\binom{2s-\frac{N}{2}-1}{\frac{m}{2}-r}\binom{-s-\nu(\pi_v)+\frac{N+k(\pi_v)}{4}}{r}\binom{-s+\nu(\pi_v)+\frac{N-k(\pi_v)}{4}}{r}$$
$$=(-1)^{m/2}\binom{s+\nu(\pi_v)-\frac{N+k(\pi_v)}{4}-1}{m/2}\binom{s-\nu(\pi_v)-\frac{N-k(\pi_v)}{4}-1}{m/2},$$

as required.

The case $l=N$ is handled by an entirely similar argument. Alternatively, using the isomorphism $\pi_v\otimes\chi_v'^{-1}\cong\widetilde{\pi}_v\otimes\chi_v^{-1}$ and working instead with the Whittaker model for $\widetilde{\pi}_v$ effectively replaces l and $\nu(\pi_v)$ by their negatives, and thus the result for $l=N$ follows formally from that for $l=-N$.

5.5.2 *Real places.* Again let notation be as in § 4.2. The function that we must analyze is

$$\chi_v'(-Ay)^{-1}\omega_{\pi_v}(\sqrt{1+y^2})e\left(-\frac{Ay^2}{1+y^2}\right)W_{\xi_v}\left(\begin{pmatrix}-\frac{Ay}{1+y^2}&\\&1\end{pmatrix}\frac{\begin{pmatrix}1&-y\\y&1\end{pmatrix}}{\sqrt{1+y^2}}\right).$$

We start by considering W_k in place of W_{ξ_v}:

$$\begin{aligned}\chi_v'(-Ay)^{-1}&\omega_{\pi_v}(\sqrt{1+y^2})e\left(-\frac{Ay^2}{1+y^2}\right)W_k\left(\begin{pmatrix}-\frac{Ay}{1+y^2}&\\&1\end{pmatrix}\frac{\begin{pmatrix}1&-y\\y&1\end{pmatrix}}{\sqrt{1+y^2}}\right)\\&=\chi_v'(-Ay)^{-1}\omega_{\pi_v}(\sqrt{1+y^2})e\left(-\frac{Ay^2}{1+y^2}\right)\frac{(1-iy)^k}{(1+y^2)^{k/2}}W_k\left(\begin{pmatrix}-\frac{Ay}{1+y^2}&\\&1\end{pmatrix}\right)\\&=\chi_v'(-Ay)^{-1}\omega_{\pi_v}(\sqrt{|Ay|})\frac{(1-i\,\mathrm{sgn}(k)y)^{|k|}}{(1+y^2)^{|k|/2}}e\left(-\frac{Ay^2}{1+y^2}\right)f_k\left(-\frac{Ay}{1+y^2}\right).\end{aligned}\tag{5.31}$$

We apply a similar argument to that leading up to (5.26) in the complex case to derive the Taylor expansion

$$\begin{aligned}\frac{(1-i\,\mathrm{sgn}(k)y)^{|k|}}{(1+y^2)^{|k|/2}}&e\left(-\frac{Ay^2}{1+y^2}\right)f_k\left(-\frac{Ay}{1+y^2}\right)\\&=O_{\pi_v,k,K}(\|Ay\|_v^{-K}|y|^M)+\sum_{\substack{a,b,r,j\in\mathbb{Z}_{\geqslant 0}\\2a+2r+b+j<M}}(-1)^a i^{-j\,\mathrm{sgn}(k)}\frac{(2\pi i)^b}{b!}\\&\quad\times\frac{(-Ay)^{a+b}f_k^{(a)}(-Ay)}{a!}\binom{-a-b-\frac{|k|}{2}}{r}\binom{|k|}{j}y^{2a+2r+b+j}\end{aligned}$$

for all y with $|y|<\frac{1}{2}$. Moreover, assuming that $\nu(\omega_{\pi_v})+2\nu(\chi_v)=0$, we have $\chi_v'(-Ay)^{-1}\omega_{\pi_v}(\sqrt{|Ay|})=\chi_v'(-1)=(-1)^{\epsilon(\chi_v')}$. Substituting

$$\frac{t^{a+b}f_k^{(a)}(t)}{a!}=\sum_{\epsilon\in\{0,1\}}\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}}\widetilde{f}_k(s+b,\epsilon)\binom{\frac{1}{2}-b-s}{a}\,\mathrm{sgn}(t)^{b+\epsilon}\|t\|_v^{\frac{1}{2}-s}\,ds,$$

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(5.31) thus becomes

$$O_{\pi_v,k,K}(\|Ay\|_v^{-K}|y|^M) + \sum_{\substack{a,b,r,j\geqslant 0\\ 2a+2r+b+j<M}} \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \sum_{\epsilon\in\{0,1\}} \operatorname{sgn}(-Ay)^{\epsilon(\chi_v')+b+\epsilon}\|Ay\|_v^{\frac{1}{2}-s}$$
$$\times (-1)^a i^{-j\operatorname{sgn}(k)}\frac{(2\pi i)^b}{b!}\binom{\frac{1}{2}-b-s}{a}\binom{-a-b-\frac{|k|}{2}}{r}\binom{|k|}{j}\widetilde{f}_k(s+b,\epsilon)y^{2a+2r+b+j}\,ds.$$

Next, we write $2a+2r+b+j=m/2$, where $m\in 2\mathbb{Z}_{\geqslant 0}$, to get

$$\begin{aligned}
&O_{\pi_v,k,K}(\|Ay\|_v^{-K}|y|^M)\\
&\quad+\sum_{\substack{m\in 2\mathbb{Z}_{\geqslant 0}\\ m/2<M}}(-A)^{-m/2}\sum_{\substack{a,b,r,j\geqslant 0\\ 2a+2r+b+j=m/2}}\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}}\sum_{\epsilon\in\{0,1\}}\operatorname{sgn}(-Ay)^{\epsilon(\chi_v')+(m/2)+b+\epsilon}\\
&\quad\times\|Ay\|_v^{(m/2)+\frac{1}{2}-s}(-1)^a i^{-j\operatorname{sgn}(k)}\frac{(2\pi i)^b}{b!}\binom{\frac{1}{2}-b-s}{a}\binom{-a-b-\frac{|k|}{2}}{r}\binom{|k|}{j}\widetilde{f}_k(s+b,\epsilon)\,ds\\
&=O_{\pi_v,k,K}(\|Ay\|_v^{-K}|y|^M)\\
&\quad+\sum_{\substack{m\in 2\mathbb{Z}_{\geqslant 0}\\ m/2<M}}(-A)^{-m/2}\sum_{\substack{b,r,j\geqslant 0\\ 2r+b+j\leqslant m/2\\ b+j\equiv m/2\ (\mathrm{mod}\ 2)}}\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}}\sum_{\epsilon\in\{0,1\}}\operatorname{sgn}(-Ay)^{\epsilon(\chi_v')+j+\epsilon}\\
&\quad\times\|Ay\|_v^{(m/2)+\frac{1}{2}-s}(-1)^{m/2-b-j/2-r}i^{-j\operatorname{sgn}(k)}\frac{(2\pi i)^b}{b!}\\
&\quad\times\binom{\frac{1}{2}-b-s}{\frac{m/2-b-j}{2}-r}\binom{\frac{-m/2-b+j-|k|}{2}+r}{r}\binom{|k|}{j}\widetilde{f}_k(s+b,\epsilon)\,ds.
\end{aligned}$$

Note that

$$\sum_{r\geqslant 0}(-1)^{((m/2-b-j)/2)-r}\binom{\frac{1}{2}-b-s}{\frac{m/2-b-j}{2}-r}\binom{\frac{-m/2-b+j-|k|}{2}+r}{r}=\binom{s-\frac{|k|+1}{2}}{\frac{m/2-b-j}{2}},$$

with the same proof as (5.27). Also, using (4.2), we see that

$$\widetilde{f}_k(s,\epsilon)=i^{(\epsilon-\epsilon(\pi_v))(1-\operatorname{sgn}(k))}\widetilde{f}_{|k|}(s,\epsilon).$$

Substituting these, we have

$$\begin{aligned}
&O_{\pi_v,k,K}(\|Ay\|_v^{-K}|y|^M)\\
&\quad+\sum_{\substack{m\in 2\mathbb{Z}_{\geqslant 0}\\ m/2<M}}(-A)^{-m/2}\sum_{\substack{b,j\geqslant 0\\ b+j\leqslant m/2\\ b+j\equiv m/2\ (\mathrm{mod}\ 2)}}\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}}\sum_{\epsilon\in\{0,1\}}\operatorname{sgn}(-Ay)^{\epsilon(\chi_v')+j+\epsilon}\\
&\quad\times\|Ay\|_v^{(m/2)+(1/2)-s}i^{b-j+(\epsilon-\epsilon(\pi_v)+j)(1-\operatorname{sgn}(k))}\frac{(2\pi)^b}{b!}\binom{s-\frac{|k|+1}{2}}{\frac{m/2-b-j}{2}}\binom{|k|}{j}\widetilde{f}_{|k|}(s+b,\epsilon)\,ds.
\end{aligned}$$

Now, considering $W_{\xi_v}=(W_k+(-1)^{\epsilon(\pi_v)+\epsilon(\chi_v')}W_{-k})/2$ with $k\geqslant 0$ in place of W_k selects the term with $\epsilon\equiv j+\epsilon(\chi_v')\pmod 2$. (This holds even when $k=0$, since $\epsilon(\chi_v')=\epsilon(\pi_v)$ and $j=0$

in that case.) Thus, we have

$$O_{\pi_v,k,K}(\|Ay\|_v^{-K}|y|^M) + \sum_{\substack{m\in 2\mathbb{Z}_{\geqslant 0}\\ m/2<M}} (-A)^{-m/2}\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \|Ay\|_v^{(m/2)+\frac{1}{2}-s}$$
$$\times \sum_{\substack{b,j\geqslant 0\\ b+j\leqslant m/2\\ b+j\equiv m/2 \ (\mathrm{mod}\ 2)}} i^{b-j}\frac{(2\pi)^b}{b!}\binom{s-\frac{k+1}{2}}{\frac{m/2-b-j}{2}}\binom{k}{j}\widetilde{f}_k(s+b, m/2+b+\epsilon(\chi_v'))\, ds.$$

As in the complex case, we see that $\widetilde{f}_k(s+b, m/2+b+\epsilon(\chi_v'))$ is a polynomial function times $L(s,\pi_v\otimes\chi_v'^{-1}\otimes \mathrm{sgn}_v^{m/2}) = L(s,\widetilde{\pi}_v\otimes\chi_v^{-1}\otimes\mathrm{sgn}_v^{m/2})$, so the whole expression can be written in the form

$$O_{M,K}(|y|^M\|Ay\|_v^{-K})$$
$$+ \sum_{\substack{m\in 2\mathbb{Z}\\ 0\leqslant m/2<M}} \frac{A^{-m/2}}{2\pi i}\int_{\Re(s)=\frac{1}{2}} P_v(s;m)L(s,\widetilde{\pi}_v\otimes\chi_v^{-1}\otimes\mathrm{sgn}_v^{m/2})\|Ay\|_v^{(m/2)+(1/2)-s}\, ds,$$

as required.

To compute $P_v(s;m)$ for m divisible by 4, we write $b=2r+\delta$, $j=2t+\delta$ with $\delta\in\{0,1\}$, to get

$$P_v(s;m) = \sum_{\delta\in\{0,1\}}\sum_{r,t\geqslant 0}(-1)^{r+t}\frac{(2\pi)^{2r+\delta}}{(2r+\delta)!}\binom{s-\frac{k+1}{2}}{\frac{m}{4}-\delta-r-t}\binom{k}{2t+\delta}\frac{\widetilde{f}_k(s+2r+\delta,\delta+\epsilon(\chi_v'))}{L(s,\pi_v\otimes\chi_v'^{-1})}. \tag{5.32}$$

We now break into cases according to whether $v\in S_0$, $v\in S_1$ or $v\in S_d$.

Weight 1 principal series. In this case we have $k=1$,

$$\widetilde{f}_k(s,\delta+\epsilon(\chi_v')) = \Gamma_{\mathbb{R}}(s+\delta+\chi_v'(-1)\nu(\pi_v))\Gamma_{\mathbb{R}}(s+1-\delta-\chi_v'(-1)\nu(\pi_v))$$

and

$$L(s,\pi_v\otimes\chi_v'^{-1}) = \Gamma_{\mathbb{R}}(s+\chi_v'(-1)\nu(\pi_v))\Gamma_{\mathbb{R}}(s+1-\chi_v'(-1)\nu(\pi_v)),$$

so that

$$\frac{\widetilde{f}_k(s+2r+\delta,\delta+\epsilon(\chi_v'))}{L(s,\pi_v\otimes\chi_v'^{-1})}$$
$$= \pi^{-2r-\delta}r!(r+\delta)!\binom{\frac{s+2r+2\delta+\chi_v'(-1)\nu(\pi_v)}{2}-1}{r+\delta}\binom{\frac{s+2r+1-\chi_v'(-1)\nu(\pi_v)}{2}-1}{r}$$
$$= (-\pi)^{-2r-\delta}r!(r+\delta)!\binom{-\frac{s+\chi_v'(-1)\nu(\pi_v)}{2}}{r+\delta}\binom{-\frac{s+1-\chi_v'(-1)\nu(\pi_v)}{2}}{r}.$$

Moreover, only the $t=0$ term contributes to (5.32), so we have

$$\sum_{\delta\in\{0,1\}}\sum_{r\geqslant 0}(-1)^{r+\delta}\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{s-1}{\frac{m}{4}-\delta-r}\binom{-\frac{s+\chi_v'(-1)\nu(\pi_v)}{2}}{r+\delta}\binom{-\frac{s+1-\chi_v'(-1)\nu(\pi_v)}{2}}{r}.$$

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Replacing r by $r-\delta$ in the inner sum and using the equality $2^{2r-1}/\binom{2r-1}{r-1} = 2^{2r}/\binom{2r}{r}$ for $r \geqslant 1$, this is

$$\sum_{r\geqslant 0}(-1)^r \frac{2^{2r}}{\binom{2r}{r}}\binom{s-1}{\frac{m}{4}-r}\binom{-\frac{s+\chi_v'(-1)\nu(\pi_v)}{2}}{r}\sum_{\delta\in\{0,1\}}\binom{-\frac{s+1-\chi_v'(-1)\nu(\pi_v)}{2}}{r-\delta}$$
$$=\sum_{r\geqslant 0}(-1)^r \frac{2^{2r}}{\binom{2r}{r}}\binom{s-1}{\frac{m}{4}-r}\binom{-\frac{s+\chi_v'(-1)\nu(\pi_v)}{2}}{r}\binom{-\frac{s-1-\chi_v'(-1)\nu(\pi_v)}{2}}{r}.$$

Applying Lemma A.1(ii), we have

$$P_v(s;m) = \frac{(-4)^{m/4}}{\binom{m/2}{m/4}}\binom{\frac{s-\chi_v'(-1)\nu(\pi_v)-2}{2}}{m/4}\binom{\frac{s+\chi_v'(-1)\nu(\pi_v)-1}{2}}{m/4},$$

in agreement with (5.17).

Weight 0 principal series. If $k=0$ (so that $\epsilon(\chi_v')=\epsilon(\pi_v)$), then we have $t=\delta=0$:

$$\sum_{r\geqslant 0}(-1)^r\frac{(2\pi)^{2r}}{(2r)!}\binom{s-\frac{1}{2}}{\frac{m}{4}-r}\frac{\widetilde{f}_0(s+2r,\epsilon(\chi_v'))}{L(s,\pi_v\otimes\chi_v'^{-1})}.$$

Note that

$$\frac{\widetilde{f}_0(s+2r,\epsilon(\chi_v'))}{L(s,\pi_v\otimes\chi_v'^{-1})} = \frac{\Gamma_{\mathbb{R}}(s+2r+\nu(\pi_v))\Gamma_{\mathbb{R}}(s+2r-\nu(\pi_v))}{\Gamma_{\mathbb{R}}(s+\nu(\pi_v))\Gamma_{\mathbb{R}}(s-\nu(\pi_v))}$$
$$=\pi^{-2r}r!^2\binom{\frac{s+2r+\nu(\pi_v)}{2}-1}{r}\binom{\frac{s+2r-\nu(\pi_v)}{2}-1}{r}$$
$$=\pi^{-2r}r!^2\binom{-\frac{s+\nu(\pi_v)}{2}}{r}\binom{-\frac{s-\nu(\pi_v)}{2}}{r}.$$

Therefore, by Lemma A.1(ii),

$$P_v(s;m) = \sum_{r\geqslant 0}(-1)^r\frac{2^{2r}}{\binom{2r}{r}}\binom{s-\frac{1}{2}}{\frac{m}{4}-r}\binom{-\frac{s+\nu(\pi_v)}{2}}{r}\binom{-\frac{s-\nu(\pi_v)}{2}}{r}$$
$$=\frac{(-4)^{m/4}}{\binom{m/2}{m/4}}\binom{\frac{s-\nu(\pi_v)-1}{2}}{m/4}\binom{\frac{s+\nu(\pi_v)-1}{2}}{m/4},$$

as required.

If $k=2$ (so that $\epsilon(\chi_v')=1-\epsilon(\pi_v)$), we have

$$\sum_{\delta\in\{0,1\}}\sum_{r\geqslant 0}(-1)^r\frac{(2\pi)^{2r+\delta}}{(2r+\delta)!}\binom{s-\frac{3}{2}}{\frac{m}{4}-\delta-r}2^{\delta}\frac{\widetilde{f}_2(s+2r+\delta,\delta+\epsilon(\chi_v'))}{L(s,\pi_v\otimes\chi_v'^{-1})}$$
$$-\sum_{r\geqslant 0}(-1)^r\frac{(2\pi)^{2r}}{(2r)!}\binom{s-\frac{3}{2}}{\frac{m}{4}-1-r}\frac{\widetilde{f}_2(s+2r,\epsilon(\chi_v'))}{L(s,\pi_v\otimes\chi_v'^{-1})}.$$

Note that

$$\widetilde{f}_2(s,\epsilon(\chi_v')+\delta) = \left(\frac{s-\frac{1}{2}}{2\pi}\right)^{\delta}\Gamma_{\mathbb{R}}(s+1-\delta+\nu(\pi_v))\Gamma_{\mathbb{R}}(s+1-\delta-\nu(\pi_v))$$

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and

$$L(s, \pi_v \otimes \chi_v'^{-1}) = \Gamma_{\mathbb{R}}(s + 1 + \nu(\pi_v))\Gamma_{\mathbb{R}}(s + 1 - \nu(\pi_v)),$$

so that

$$\begin{aligned}
&\frac{\widetilde{f}_2(s + 2r + \delta, \epsilon(\chi_v') + \delta)}{L(s, \pi_v \otimes \chi_v'^{-1})} \\
&\quad = \pi^{-2r} r!^2 \left(\frac{s + 2r + \frac{1}{2}}{2\pi}\right)^{\delta} \binom{\frac{s+2r+1+\nu(\pi_v)}{2} - 1}{r} \binom{\frac{s+2r+1-\nu(\pi_v)}{2} - 1}{r} \\
&\quad = \pi^{-2r} r!^2 \left(\frac{s + 2r + \frac{1}{2}}{2\pi}\right)^{\delta} \binom{-\frac{s+1+\nu(\pi_v)}{2}}{r} \binom{-\frac{s+1-\nu(\pi_v)}{2}}{r}.
\end{aligned}$$

Moreover, we calculate that

$$\begin{aligned}
&\sum_{\delta \in \{0,1\}} \frac{2^{2r+\delta} r!^2}{(2r + \delta)!} \binom{s - \frac{3}{2}}{\frac{m}{4} - \delta - r} \left(s + 2r + \frac{1}{2}\right)^{\delta} - \frac{2^{2r} r!^2}{(2r)!} \binom{s - \frac{3}{2}}{\frac{m}{4} - 1 - r} \\
&\quad = \frac{2^{2r}}{\binom{2r}{r}} \frac{m/2 + 1}{2r + 1} \binom{s - \frac{1}{2}}{\frac{m}{4} - r}.
\end{aligned}$$

Therefore, applying Lemma A.1(iii), we get

$$\begin{aligned}
P_v(s; m) &= \sum_{r \geqslant 0} \frac{(-4)^r}{\binom{2r}{r}} \frac{m/2 + 1}{2r + 1} \binom{s - \frac{1}{2}}{\frac{m}{4} - r} \binom{-\frac{s+1+\nu(\pi_v)}{2}}{r} \binom{-\frac{s+1-\nu(\pi_v)}{2}}{r} \\
&= \frac{(-4)^{m/4}}{\binom{m/2}{m/4}} \binom{\frac{s-\nu(\pi_v)-2}{2}}{m/4} \binom{\frac{s+\nu(\pi_v)-2}{2}}{m/4},
\end{aligned}$$

as required.

Discrete series. In this case we have $k = k(\pi_v) \geqslant 1$ and

$$\begin{aligned}
\widetilde{f}_k(s, \delta + \epsilon(\chi_v')) &= L(s, \pi_v \otimes \chi_v'^{-1}) \\
&= \Gamma_{\mathbb{C}}\left(s + \frac{k(\pi_v) - 1}{2}\right),
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\widetilde{f}_k(s + 2r + \delta, \delta + \epsilon(\chi_v'))}{L(s, \pi_v \otimes \chi_v'^{-1})} &= (2\pi)^{-2r-\delta}(2r + \delta)! \binom{s + \frac{k(\pi_v)-1}{2} + 2r + \delta - 1}{2r + \delta} \\
&= (-2\pi)^{-2r-\delta}(2r + \delta)! \binom{-s - \frac{k(\pi_v)-1}{2}}{2r + \delta}.
\end{aligned}$$

We make the change of variables $2r + \delta = j_2$, $2t + \delta = 2j_1 - j_2$ with $0 \leqslant j_2 \leqslant 2j_1$, to get

$$\begin{aligned}
P_v(s; m) &= \sum_{j_1, j_2 \geqslant 0} (-1)^{j_1} \binom{s - \frac{k(\pi_v)+1}{2}}{\frac{m}{4} - j_1} \binom{k(\pi_v)}{2j_1 - j_2} \binom{-s - \frac{k(\pi_v)-1}{2}}{j_2} \\
&= \sum_{j_1 \geqslant 0} (-1)^{j_1} \binom{s - \frac{k(\pi_v)+1}{2}}{\frac{m}{4} - j_1} \binom{-s + \frac{k(\pi_v)+1}{2}}{2j_1}.
\end{aligned}$$

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Next, as a special case of (A5), we have

$$\binom{-s+\frac{k(\pi_v)+1}{2}}{2j_1} = \frac{2^{2j_1}}{\binom{2j_1}{j_1}}\binom{-\frac{s}{2}+\frac{k(\pi_v)+1}{4}}{j_1}\binom{-\frac{s}{2}+\frac{k(\pi_v)-1}{4}}{j_1},$$

so, by Lemma A.1(ii), we arrive at

$$P_v(s;m) = (-1)^{m/4}\frac{2^{m/2}}{\binom{m/2}{m/4}}\binom{\frac{s}{2}-\frac{k(\pi_v)+1}{4}}{m/4}\binom{\frac{s}{2}-\frac{k(\pi_v)+3}{4}}{m/4} = (-1)^{m/4}\binom{s-\frac{k(\pi_v)+1}{2}}{m/2},$$

as required.

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Appendix A. Binomial coefficients

The binomial coefficient polynomials $\binom{x}{n}$ are defined for $x \in \mathbb{C}$, $n \in \mathbb{Z}$ by

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} = \frac{\Gamma(x+1)}{n!\Gamma(x+1-n)} \tag{A1}$$

for $n \geqslant 0$ and $\binom{x}{n} = 0$ for $n < 0$. These directly generalize the usual binomial coefficients $\binom{m}{n}$, and it easy to see that many of the familiar properties hold in full generality, e.g.

$$\binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1}, \quad \binom{x}{n}\binom{n}{m} = \binom{x}{m}\binom{x-m}{n-m}$$

and

$$\binom{x+y}{n} = \sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}.$$

(The last one of these is known as the 'Chu–Vandermonde identity'.) For some, one must exercise caution, e.g. $\binom{m}{n} = \binom{m}{m-n}$ holds when $m \in \mathbb{Z}_{\geqslant 0}$, but not in general otherwise. On the other hand, we have $\binom{x}{n} = (-1)^n\binom{n-1-x}{n}$, which is new to the general setting.

We will also need the following less obvious identities.

Lemma A.1. *For $x, y \in \mathbb{C}$ and $n \in \mathbb{Z}$, we have:*

(i) $$\sum_{r=0}^{n}(-1)^r\binom{x+y+1}{n-r}\binom{-x-1}{r}\binom{-y-1}{r} = (-1)^n\binom{x}{n}\binom{y}{n};$$

(ii) $$\sum_{r\geqslant 0}\frac{(-4)^r}{\binom{2r}{r}}\binom{x+y}{n-r}\binom{-x}{r}\binom{-y-\frac{1}{2}}{r} = \frac{(-4)^n}{\binom{2n}{n}}\binom{x-\frac{1}{2}}{n}\binom{y}{n};$$

(iii) $$\sum_{r\geqslant 0}\frac{(-4)^r}{\binom{2r}{r}}\frac{2n+1}{2r+1}\binom{x+y}{n-r}\binom{-x-\frac{1}{2}}{r}\binom{-y-1}{r} = \frac{(-4)^n}{\binom{2n}{n}}\binom{x-1}{n}\binom{y-\frac{1}{2}}{n}.$$

Proof. For (i), we begin with the identity

$$\binom{x}{m}\binom{x}{n} = \sum_{k=\max(m,n)}^{m+n} \frac{k!}{(m+n-k)!(k-m)!(k-n)!}\binom{x}{k} \tag{A2}$$

for $x, m, n \in \mathbb{Z}_{\geqslant 0}$, which has the following combinatorial interpretation. Let X be a set of cardinality x, and choose subsets of M, N of cardinality m, n, respectively. The number of ways of making such a choice is the left-hand side, $\binom{x}{m}\binom{x}{n}$. On the right-hand side, we consider all possibilities for the cardinality k of the union $K = M \cup N$. Given such a k, there are $\binom{x}{k}$ ways of choosing the elements of K among those of X; this is then multiplied by the number of ways of dividing those elements among the three sets $M \cap N$, $N \backslash M$ and $M \backslash N$, which is the trinomial coefficient $k!/(m+n-k)!(k-m)!(k-n)!$.

Thus, (A2) holds for all $x \in \mathbb{Z}_{\geqslant 0}$. Since both sides are clearly polynomial functions of x and $\mathbb{Z}_{\geqslant 0}$ is Zariski dense in $\mathbb{C}$, it in fact holds for all $x \in \mathbb{C}$. Further, by the Chu–Vandermonde identity we have

$$\begin{aligned} \frac{k!}{(m+n-k)!(k-m)!(k-n)!} &= \binom{k}{m}\binom{m}{k-n} = \sum_r \binom{k-n}{m-r}\binom{n}{r}\binom{m}{k-n} \\ &= \sum_r \binom{m}{r}\binom{n}{r}\binom{r}{m+n-k}. \end{aligned}$$

Thus,

$$\binom{x}{m}\binom{x}{n} = \sum_{k,r} \binom{m}{r}\binom{n}{r}\binom{r}{m+n-k}\binom{x}{k} = \sum_r \binom{m}{r}\binom{n}{r}\binom{x+r}{m+n} \tag{A3}$$

for all $x \in \mathbb{C}$. Next, from (A1), we see that

$$\binom{x+r}{m+n}\binom{m+n}{n}\binom{n}{r} = \binom{x+r}{r}\binom{x}{m}\binom{x-m}{n-r}.$$

Multiplying (A3) by $\binom{m+n}{n}/\binom{x}{m}$, we thus have

$$\binom{m+n}{n}\binom{x}{n} = \sum_r \binom{m}{r}\binom{x+r}{r}\binom{x-m}{n-r}$$

or

$$(-1)^n\binom{-m-1}{n}\binom{x}{n} = \sum_r (-1)^r \binom{-m-1+r}{r}\binom{x+r}{r}\binom{x-m}{n-r},$$

which is the required identity with $y = -m-1$. Since the negative integers are Zariski dense in $\mathbb{C}$, the identity holds for all x, y.

To prove (ii) and (iii), we first derive the identity

$$\frac{(-1)^t x}{x-t}(z-1)^{2x-t}\frac{d^t}{dz^t}(\sqrt{z} \pm 1)^{2t-2x} = \frac{d^t}{dz^t}(\sqrt{z} \mp 1)^{2x} \tag{A4}$$

for $z > 1$ and $t \in \mathbb{Z}_{\geqslant 0}$. This clearly holds for $t = 0$, since $z - 1 = (\sqrt{z}+1)(\sqrt{z}-1)$. Hence, by induction, it suffices to show that replacing t by $t+1$ in the left-hand side has the same effect

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as differentiating with respect to z. To that end, we have

$$\begin{aligned}&\frac{d}{dz}\frac{(-1)^t x}{x-t}(z-1)^{2x-t}\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2t-2x}\\&\quad=\frac{(-1)^t x}{x-t}(z-1)^{2x-t-1}\biggl((2x-t)\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2t-2x}+(z-1)\frac{d^{t+1}}{dz^{t+1}}(\sqrt{z}\pm 1)^{2t-2x}\biggr).\end{aligned}$$

By the Leibniz rule, we see that

$$(z-1)\frac{d^{t+1}}{dz^{t+1}}(\sqrt{z}\pm 1)^{2t-2x}=\frac{d^{t+1}}{dz^{t+1}}((z-1)(\sqrt{z}\pm 1)^{2t+2-2x})-(t+1)\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2t-2x},$$

so that

$$\begin{aligned}&\frac{d}{dz}\frac{(-1)^t x}{x-t}(z-1)^{2x-t}\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2t-2x}\\&\quad=\frac{(-1)^t x}{x-t}(z-1)^{2x-t-1}\biggl((2x-2t-1)\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2t-2x}+\frac{d^{t+1}}{dz^{t+1}}((z-1)(\sqrt{z}\pm 1)^{2t-2x})\biggr).\end{aligned}$$

On the other hand, replacing t by $t+1$ on the left-hand side of (A4) yields

$$(z-1)^{2x-t-1}\frac{(-1)^{t+1}x}{x-t-1}\frac{d^{t+1}}{dz^{t+1}}(\sqrt{z}\pm 1)^{2t+2-2x}.$$

Thus, we must show the equality

$$\begin{aligned}&(2x-2t-1)\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2t-2x}+\frac{d^{t+1}}{dz^{t+1}}((z-1)(\sqrt{z}\pm 1)^{2t-2x})\\&\quad=-\frac{x-t}{x-t-1}\frac{d^{t+1}}{dz^{t+1}}(\sqrt{z}\pm 1)^{2t+2-2x}.\end{aligned}$$

This follows from the directly verifiable identity

$$(2x-1)(\sqrt{z}\pm 1)^{-2x}+\frac{d}{dz}((z-1)(\sqrt{z}\pm 1)^{-2x})=-\frac{x}{x-1}\frac{d}{dz}(\sqrt{z}\pm 1)^{2-2x}$$

by changing x to $x-t$ and applying d^t/dz^t.

Next, we show how (ii) and (iii) are deduced from (A4). First note that, by the identities

$$\frac{(-4)^k}{(2k+1)\binom{2k}{k}}=\frac{(-1)^k}{2}\frac{2^{2k+1}}{(k+1)\binom{2k+1}{k}}\quad\text{and}\quad\frac{1}{k+1}\binom{x-1}{k}=\frac{1}{x}\binom{x}{k+1},$$

(iii) is equivalent to

$$\sum_{r\geqslant 0}(-1)^{n-r}\frac{2^{2r+1}}{\binom{2r+1}{r}}\binom{x+y}{n-r}\binom{-y}{r+1}\binom{-x-\frac{1}{2}}{r}=-\frac{y}{x}\frac{2^{2n+1}}{\binom{2n+1}{n}}\binom{x}{n+1}\binom{y-\frac{1}{2}}{n}.$$

Thus, (ii) and (iii) can be written in the common form

$$\begin{aligned}&\sum_{r\geqslant 0}(-1)^{n-r}\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{x+y}{n-r}\binom{-y+\frac{\delta-1}{2}}{r+\delta}\binom{-x-\frac{\delta}{2}}{r}\\&\quad=\Bigl(-\frac{y}{x}\Bigr)^{\delta}\frac{2^{2n+\delta}}{\binom{2n+\delta}{n}}\binom{x+\frac{\delta-1}{2}}{n+\delta}\binom{y-\frac{\delta}{2}}{n}\end{aligned}$$

for $\delta\in\{0,1\}$. Again, by a density argument, it is enough to show this when $y=x-t$ for $t\in\mathbb{Z}_{\geqslant 0}$,

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in which case the desired identity is

$$\sum_{r\geqslant 0}(-1)^{n-r}\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{2x-t}{n-r}\binom{t-x+\frac{\delta-1}{2}}{r+\delta}\binom{-x-\frac{\delta}{2}}{r}$$
$$=\left(\frac{t-x}{x}\right)^{\delta}\frac{2^{2n+\delta}}{\binom{2n+\delta}{n}}\binom{x+\frac{\delta-1}{2}}{n+\delta}\binom{x-\frac{2t+\delta}{2}}{n}.$$

Working from (A1), we derive

$$\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{x+\frac{\delta-1}{2}}{r+\delta}\binom{x-\frac{2t+\delta}{2}}{r}\binom{x-\frac{\delta}{2}}{t}=\binom{2x}{2r+\delta}\binom{x-\frac{2r+\delta}{2}}{t}. \tag{A5}$$

From the binomial theorem we have, for $z>1$,

$$\sum_{n=0}^{\infty}\binom{2x}{n}\binom{x-\frac{n}{2}}{t}(\pm\sqrt{z})^{-n}=z^{t-x}\frac{d^t}{dz^t}\sum_{n=0}^{\infty}(\pm 1)^n\binom{2x}{n}z^{x-(n/2)}=z^{t-x}\frac{d^t}{dz^t}(\sqrt{z}\pm 1)^{2x}.$$

Taking the sum and difference over both choices of the sign, we have

$$\sum_{r=0}^{\infty}\binom{2x}{2r+\delta}\binom{x-\frac{2r+\delta}{2}}{t}z^{-r}=z^{t-x+(\delta/2)}\frac{d^t}{dz^t}\frac{(\sqrt{z}+1)^{2x}+(-1)^{\delta}(\sqrt{z}-1)^{2x}}{2}.$$

Substituting (A5), we get

$$\binom{x-\frac{\delta}{2}}{t}\sum_{r=0}^{\infty}\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{x+\frac{\delta-1}{2}}{r+\delta}\binom{x-\frac{2t+\delta}{2}}{r}z^{-r}=z^{t-x+(\delta/2)}\frac{d^t}{dz^t}\frac{(\sqrt{z}+1)^{2x}+(-1)^{\delta}(\sqrt{z}-1)^{2x}}{2}.$$

Replacing x by $t-x$, we get

$$(-1)^t\binom{x+\frac{\delta}{2}-1}{t}\sum_{r=0}^{\infty}\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{-x-\frac{\delta}{2}}{r}\binom{t-x+\frac{\delta-1}{2}}{r+\delta}z^{-r}$$
$$=z^{x+(\delta/2)}\frac{d^t}{dz^t}\frac{(\sqrt{z}+1)^{2t-2x}+(-1)^{\delta}(\sqrt{z}-1)^{2t-2x}}{2}.$$

Thus,

$$(-1)^t\binom{x+\frac{\delta}{2}-1}{t}\sum_{n=0}^{\infty}z^{-n}\sum_{r=0}^{\infty}(-1)^{n-r}\frac{2^{2r+\delta}}{\binom{2r+\delta}{r}}\binom{2x-t}{n-r}\binom{-x-\frac{\delta}{2}}{r}\binom{t-x+\frac{\delta-1}{2}}{r+\delta}$$
$$=(1-z^{-1})^{2x-t}z^{x+(\delta/2)}\frac{d^t}{dz^t}\frac{(\sqrt{z}+1)^{2t-2x}+(-1)^{\delta}(\sqrt{z}-1)^{2t-2x}}{2}.$$

Therefore, (ii) and (iii) are equivalent to the equality of generating functions

$$\frac{(1-z^{-1})^{2x-t}z^{x+(\delta/2)}}{(-1)^t\binom{x+\delta/2-1}{t}}\frac{d^t}{dz^t}\frac{(\sqrt{z}+1)^{2t-2x}+(-1)^{\delta}(\sqrt{z}-1)^{2t-2x}}{2}$$
$$=\left(\frac{t-x}{x}\right)^{\delta}\frac{z^{t-x+(\delta/2)}}{\binom{x-\delta/2}{t}}\frac{d^t}{dz^t}\frac{(\sqrt{z}+1)^{2x}+(-1)^{\delta}(\sqrt{z}-1)^{2x}}{2}$$

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or

$$\frac{(-1)^{t+\delta}x}{x-t}(z-1)^{2x-t}\frac{d^t}{dz^t}[(\sqrt{z}+1)^{2t-2x}+(-1)^{\delta}(\sqrt{z}-1)^{2t-2x}]$$
$$=\frac{d^t}{dz^t}[(\sqrt{z}+1)^{2x}+(-1)^{\delta}(\sqrt{z}-1)^{2x}],$$

which follows immediately from (A4). □

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A STRENGTHENING OF THE GL(2) CONVERSE THEOREM

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