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AN ANALYSIS OF RIEMANN SOLUTIONS FOR THE NONHOMOGENEOUS AW–RASCLE MODEL OF TRAFFIC FLOW WITH THE BORN–INFELD EQUATION OF STATE

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Abstract

This paper focuses on the Aw–Rascle model of traffic flow for the Born–Infeld equation of state with Coulomb-like friction, whose Riemann problem is solved with the variable substitution method. Four kinds of nonself-similar solutions are derived. The delta shock occurs in the solutions, although the system is strictly hyperbolic with a genuinely nonlinear characteristic field and a linearly degenerate characteristic field. The generalized Rankine–Hugoniot relation and entropy condition for the delta shock are clarified. The delta shock can be used to describe the serious traffic jam. Under the impact of the friction term, the rarefaction wave (R), shock wave (S), contact discontinuity (J) and delta shock (δ) are bent into parabolic curves. Furthermore, it is proved that the $S + J$ solution and δ solution of the nonhomogeneous Aw–Rascle model tend to be the δ solution of the zero-pressure Euler system with friction; the $R + J$ solution and $R + \text{Vac} + J$ solution tend to be the vacuum solution of the zero-pressure Euler system with friction.

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1. Introduction

Consider the Aw–Rascle (AR) model [1] of traffic flow with Coulomb-like friction,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho(u + p(\rho)))_t + (\rho u(u + p(\rho)))_x = \beta\rho, \end{cases} \tag{1.1}$$

where ρ is the traffic density to indicate the space occupied by vehicles, u denotes the actual speed of vehicles and $\beta\rho$ with constant β denotes the Coulomb-like friction term. From the unidirectional feature of traffic flow, it is natural to demand the

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requirements $\rho \geq 0$ and $u \geq 0$. Additionally, $p(\rho)$ denotes the speed deviation, which is commonly known as the traffic pressure obtained from the analogy of gas dynamics. It is important to note that the traffic pressure $p(\rho)$ is used to indicate the difference between the expected speed $v = u + p(\rho)$ and the actual one u, which reflects the fact that drivers usually need to consider the traffic status ahead. Since different traffic pressures $p(\rho)$ can be used to reflect different road surface and traffic circumstances, the traffic pressure $p(\rho)$ plays a crucial role in the study of the AR model of traffic flow (1.1).

In 2000, the homogeneous AR model of traffic flow, that is, the system (1.1) with $\beta = 0$ describing the formation and dynamics of traffic jams, was proposed by Aw and Rascle [1]. In [1], the Riemann solutions for the homogeneous AR model with the classical pressure $p(\rho) = k\rho^{\gamma}$ $(k, \gamma > 0)$ were obtained. The interactions of elementary waves were investigated by Sun [38]. Shen and Sun [33] analysed the limiting behaviour of Riemann solutions for the homogeneous AR model with the classical pressure. Pan and Han [23] solved the Riemann problem for the homogeneous AR model with the Chaplygin gas pressure law $p(\rho) = -B/\rho$ $(B > 0)$ [2, 4, 41], and obtained two kinds of solutions involving delta shocks. Sheng and Zeng [36] discussed the Riemann problem with delta initial data for the Chaplygin pressure AR model. Cheng and Yang [7] analysed the limit of Riemann solutions for the homogeneous AR model with the modified Chaplygin pressure law $p(\rho) = k\rho - B/\rho$ as the pressure tends to the Chaplygin gas pressure. Wang [42] solved the Riemann problem for the homogeneous AR model with the generalized Chaplygin gas law $p(\rho) = -B/\rho^{\alpha}$ $(0 < \alpha < 1)$ and constructed three kinds of solutions involving delta shocks. Li et al. [21] carried out an analysis of Riemann solutions of the homogeneous AR model with general pressure. Shao [30] discussed the Riemann problem for the homogeneous AR model with the pressure function $p(\rho) = -\epsilon/\rho^2$ and proved that the limits of the Riemann solutions are exactly those of the pressureless Euler system as the traffic pressure vanishes. Li and Wang [20] studied the phenomena of concentration and cavitation and the formation of delta shocks and vacuums in Riemann solutions of the homogeneous Aw–Rascle model with Umami Chaplygin gas. Li and Shen [22] studied the asymptotic behaviour of Riemann solutions for the homogeneous AR model of traffic flow with the polytropic and logarithmic combined pressure term. For the nonhomogeneous AR model (1.1), Yin and Chen [46] studied the Riemann problem and stability of the solutions for (1.1) with classical pressure. Li [19] discussed the Riemann problem for (1.1) with anti-Chaplygin pressure $p(\rho) = B/\rho$. Zhang [48] considered the Riemann problem for (1.1) with Chaplygin gas pressure and analysed the limiting behaviour of the solutions.

In regards to the delta shock, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function [37, 39, 40] with the discontinuity as its support. Physically, it represents the process of overcompression in car traffic flow, which is able to explain the serious traffic jam. Interested readers can refer to [5, 8, 10, 11, 15–18, 24, 25, 32, 37, 39, 40, 47] for related research of delta shocks.

From the above analysis, it is found that the delta shocks [16, 40] only appear in the Chaplygin gas traffic flow model [23] and the generalized Chaplygin gas traffic flow model [42]. Nevertheless, it is worth noting that the Chaplygin gas with equation of state $p(\rho) = -B/\rho$ and the generalized Chaplygin gas with equation of state $p(\rho) = -B/\rho^{\alpha}(0 < \alpha < 1)$, which can effectively play the role of dark energy, generate the negative pressure. The present study focuses on a Born–Infeld type fluid whose equation of state is of the form [6, 44, 45]

$$p(\rho) = \frac{A\rho}{A+\rho} = A - \frac{A^2}{A+\rho} \geq 0, \tag{1.2}$$

where A stays in the interval $0 < A < +\infty$. In the context of k-essence cosmology, the classical Born–Infeld model gives rise to a big-bang scenario for the evolution of the universe. The matter pressure in the Born–Infeld model remains finite at the initial moment despite of the fact that the matter density is infinite. More importantly, there is a close relation between the Born–Infeld type fluid model and the Chaplygin gas model. As described in [6], the two Lagrangians differ by a constant which adds a cosmological or dark energy term. If $\bar{\rho} = \rho + A$ and $\bar{p} = p - A$, then one has $\bar{p} = -A^2/\bar{\rho}$. Nevertheless, unlike the Chaplygin gas, the Born–Infeld equation of state, which satisfies both the dominant and strong energy conditions for all values of the energy density, is a nonnegative function of density. Therefore, it is interesting and meaningful to conduct the theoretical analysis for the AR model of traffic flow (1.1) by introducing the Born–Infeld type fluid model from the mathematical point of view.

Moreover, it should be noted that the specially designed AR model of traffic flow (1.1)–(1.2) could be invoked to explain some complicated traffic phenomena in the realistic situation of traffic flow resulting from different road surface and traffic circumstances by properly adjusting the perturbed parameter A and even being zero. For instance, a very light traffic situation with a few slow drivers can be approximately described by the state near vacuum as well as the congestion traffic situation, and the traffic jam and even the traffic accident can be illustrated by the state with high traffic density and even the singular δ-mass concentration. Since the Riemann problem reflects many practical situations in traffic flow, it makes sense to investigate the related Riemann problem for traffic flow models in hyperbolic form. For example, accelerating will produce a rarefaction wave and braking will produce a shock wave, events that are well treated in the study of related Riemann problems. More importantly, serious traffic congestion can be explained by the delta shock. Indeed, the decisive property of a traffic flow model is its ability to offer simple and physically correct solutions for traffic flows, and accordingly, the Riemann problem is one of the most classical benchmarks for a traffic flow model at the level of hyperbolic conservation laws.

The main purpose of this paper is to derive the exact solutions of the Riemann problem for the model (1.1) subject to the equation of state (1.2) with initial data

$$(\rho, u)(x, t = 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \tag{1.3}$$

and explore whether the Riemann solutions tend to these of zero-pressure Euler system with friction

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \beta\rho, \end{cases} \tag{1.4}$$

as the pressure approaches to zero, that is, $A \to 0$, where $\rho_\pm > 0$ and $u_\pm \geq 0$ are given constants. The homogeneous zero-pressure Euler system, that is, (1.4) with $\beta = 0$, can describe the physical phenomenon of an aggregate of sticky free particles: when these particles collide with each other, they fuse into a single one which combines their masses and moves with a velocity that conserves the total momentum [3]. It can also explain the formation of large-scale structures in the universe [29, 43]. Additionally, in [26], it has been shown that under suitable scaling, the pressureless flow is actually a hypersonic limiting flow. If the speed of the piston is very large compared with the flow, in the high Mach number limit, polytropic gas may also be considered as pressureless flow. Taking the friction effect into consideration, Shen [31] solved the Riemann problem for the system (1.4), and constructed two kinds of nonself-similar solutions involving the delta shock and vacuum. Interestingly enough, the contact discontinuity and delta shock are bent into parabolic curves under the influence of the friction term. Because of the interesting finding, the system (1.4) has received much attention recently (see [9, 12, 13, 28, 34, 35, 49]).

The basic idea is to use variable substitution. First, by introducing a transformation $u(x, t) = v(x, t) + \beta t$, the system (1.1)–(1.2) is transformed into an homogeneous conservative system. The classical elementary waves, which contain rarefaction wave (R), shock wave (S) and contact discontinuity (J), are obtained. By means of these curves starting from (ρ_-, u_-), the phase plane (ρ, v) is divided into four domains $I, II, III, IV(\rho_-, u_-)$. We obtain the solutions with four kinds of different structures with the aid of the analysis method in the phase plane. To be exact, the Riemann solutions are

$$\begin{array}{ll} R + \text{Vac} + J & \text{when } (\rho_+, u_+) \in I(\rho_-, u_-), \\ R + J & \text{when } (\rho_+, u_+) \in II(\rho_-, u_-), \\ S + J & \text{when } (\rho_+, u_+) \in III(\rho_-, u_-), \\ \delta & \text{when } (\rho_+, u_+) \in IV(\rho_-, u_-). \end{array}$$

For the delta shock (δ), the generalized Rankine–Hugoniot relation and entropy condition are clarified. Moreover, the position, strength and propagation speed of the delta shock are derived explicitly. Second, by virtue of $(\rho, u)(x, t) = (\rho, v + \beta t)(x, t)$, four forms of Riemann solutions to the original system (1.1)–(1.2) with initial data (1.3) are constructed. Due to the emergence of the Coulomb-like friction term, the constructed Riemann solutions are no longer self-similar. The velocity u changes linearly at the rate β with respect to t. Additionally, the rarefaction wave, shock wave and contact discontinuity as well as delta shock are bent into parabolic curves. Finally, it is shown that as $A \to 0$, the $S + J$ solution and δ solution of the nonhomogeneous AR model tend to be the δ solution of the zero-pressure Euler system with friction; the solutions $R + J$ and $R + \text{Vac} + J$ tend to be the solution containing vacuum to the zero-pressure Euler system with friction.

The article is divided into five sections. Section 2 is devoted to the study of the Riemann solutions to the modified homogeneous system (2.1) with the same Riemann initial data (2.2). Section 3 constructs the solutions for the Riemann problem (1.1)–(1.3). Section 4 analyses the limiting behaviour of the solutions of (1.1)–(1.3) as $A \to 0$. Concluding remarks are given in Section 5.

2. Solutions of Riemann problem for a modified system

In this section, we discuss the Riemann problem for the modified homogeneous system (2.1) which is derived by using transformation $u(x,t) = v(x,t) + \beta t$. As we shall see later, applying the relationship $(\rho, u)(x,t) = (\rho, v + \beta t)(x,t)$, the Riemann solutions of the original system (1.1)–(1.2) with initial data (1.3) can be constructed from these to the modified homogeneous system (2.1) with same initial data.

Substituting the transformation $u(x,t) = v(x,t) + \beta t$ into (1.1)–(1.2) and rearranging, we get the homogeneous system as follows:

$$\begin{cases} \rho_t + (\rho(v + \beta t))_x = 0, \\ \left(\rho\left(v - \dfrac{A^2}{A+\rho}\right)\right)_t + \left(\rho(v+\beta t)\left(v - \dfrac{A^2}{A+\rho}\right)\right)_x = 0. \end{cases} \tag{2.1}$$

Meanwhile, the initial value (1.3) becomes

$$(\rho, v)(x,0) = (\rho_\pm, u_\pm) \quad (\pm x > 0). \tag{2.2}$$

System (2.1) can be written in the quasi-linear form as

$$\begin{pmatrix} 1 & 0 \\ \dfrac{A^2\rho}{(A+\rho)^2} & \rho \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v + \beta t & \rho \\ \dfrac{A^2\rho}{(A+\rho)^2}(v+\beta t) & \rho(v+\beta t) \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

After calculation, it is found that the two characteristic roots are

$$\lambda_1(\rho, v) = v + \beta t - \frac{A^2\rho}{(A+\rho)^2}, \quad \lambda_2(\rho, v) = v + \beta t.$$

As a consequence, the system (2.1) is strictly hyperbolic. The associative right characteristic vectors are given by

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$$\vec{r}_1 = \left(1, -\frac{A^2}{(A+\rho)^2}\right)^T, \quad \vec{r}_2 = (1,0)^T$$

satisfying

$$\nabla\lambda_1 \cdot \vec{r}_1 = -\frac{2A^3}{(A+\rho)^3}, \quad \nabla\lambda_2 \cdot \vec{r}_2 = 0.$$

Therefore, the waves associated to λ_1 correspond to rarefaction waves or shock waves, whereas the waves associated to λ_2 correspond to contact discontinuities.

The Riemann invariants are given as

$$w = v - \frac{A^2}{A+\rho}, \quad z = v$$

along these characteristic fields. Now, we compute the rarefaction wave by solving the integral curve of the first characteristic field. As a reminder, the Riemann invariant is conserved in the rarefaction wave. Fixing a left state (ρ_l, v_l) and considering the condition $\lambda_1(\rho_l, v_l) < \lambda_1(\rho, v)$, the rarefaction wave curve $R(\rho_l, v_l)$, which is the set of states connecting to the state (ρ_l, v_l) in the phase plane, satisfies

$$R(\rho_l, v_l): \begin{cases} \dfrac{dx}{dt} = \lambda_1(\rho, v) = v + \beta t - \dfrac{A^2\rho}{(A+\rho)^2}, \\ v - \dfrac{A^2}{A+\rho} = v_l - \dfrac{A^2}{A+\rho_l} = w_l, \quad \rho < \rho_l. \end{cases} \tag{2.3}$$

Let $\sigma(t) = x'(t)$ be the speed of a bounded discontinuity $x = x(t)$. Then, the Rankine–Hugoniot [14, 27] relation reads as

$$\begin{cases} -\sigma(t)[\rho] + [\rho(v+\beta t)] = 0, \\ -\sigma(t)\left[\rho\left(v - \dfrac{A^2}{A+\rho}\right)\right] + \left[\rho(v+\beta t)\left(v - \dfrac{A^2}{A+\rho}\right)\right] = 0, \end{cases} \tag{2.4}$$

where $[\rho] = \rho_l - \rho_r$ with $\rho_l = \rho(x(t)-0, t)$, $\rho_r = \rho(x(t)+0, t)$ and so forth. Clearly, the propagation speed of discontinuity is different from the classical systems of conservation laws because it is related to the time t.

Eliminating $\sigma(t)$ from (2.4), we get $\rho_l\rho_r(v_r - v_l)(v_r - A^2/(A+\rho_r) - v_l + A^2/(A+\rho_l)) = 0$ giving $v_r = v_l$ or $v_r - A^2/(A+\rho_r) = v_l - A^2/(A+\rho_l)$. Therefore, taking into account the lax entropy condition, the admissible shock wave S starting from (ρ_l, v_l) in the phase plane is

$$S(\rho_l, v_l): \begin{cases} \sigma_1(t) = v_l + \beta t - \dfrac{A^2\rho}{(A+\rho_l)(A+\rho)}, \\ v - \dfrac{A^2}{A+\rho} = v_l - \dfrac{A^2}{A+\rho_l}, \quad \rho > \rho_l, \end{cases} \tag{2.5}$$

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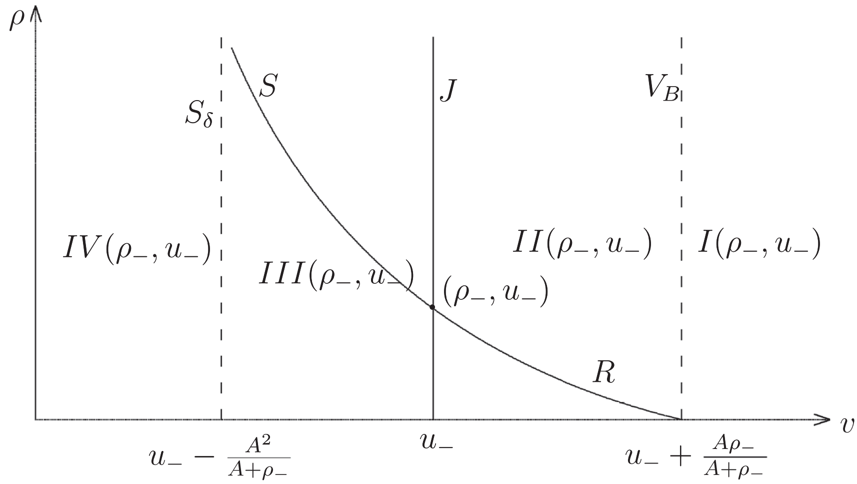

FIGURE 1. The upper right quadrant of the (ρ, v) phase plane for the modified homogeneous system (2.1).

and the contact discontinuity J starting from (ρ_l, v_l) is

$$J: \ \sigma_2(t) = v + \beta t = v_l + \beta t.$$

Successive use of (2.3) and (2.5) yields $dv/d\rho = -A^2/(A+\rho)^2 < 0$ and $d^2v/d\rho^2 = 2A^2/(A+\rho)^3 > 0$. As a result, the rarefaction wave and shock wave curves, which are expressed by the second equations of (2.3) and (2.5) in the (ρ, v) plane, are monotone decreasing and convex. Fixing a left state (ρ_-, u_-) in the upper right quadrant of the (ρ, v) phase plane, the sets of states connected on the right are composed of the contact discontinuity curve $J(\rho_-, u_-)$, rarefaction wave curve $R(\rho_-, u_-)$ and shock wave curve $S(\rho_-, u_-)$. It is clear that the contact discontinuity line $v = u_-$ is parallel to the ρ-axis. The $R(\rho_-, u_-)$ intersects the v-axis at the point $(u_- + A\rho_-/(A+\rho_-), 0)$ because of

$$\lim_{\rho\to 0} v = u_- - \frac{A^2}{A+\rho_-} + \lim_{\rho\to 0}\frac{A^2}{A+\rho} = u_- + \frac{A\rho_-}{A+\rho_-}.$$

By virtue of

$$\lim_{\rho\to +\infty} v = u_- - \frac{A^2}{A+\rho_-} + \lim_{\rho\to +\infty}\frac{A^2}{A+\rho} = u_- - \frac{A^2}{A+\rho_-},$$

the shock wave curve $S(\rho_-, u_-)$ has the asymptotic line $v = u_- - A^2/(A+\rho_-)$. (In this paper, we suppose that $u_- - A^2/(A+\rho_-) > 0$, because what we are interested in is the delta shock). We draw these wave curves which divide the upper right quadrant of the (ρ, v)-plane into four domains $I, II, III, IV(\rho_-, u_-)$ (see Figure 1).

Now, let us construct the Riemann solutions by the method of phase plane analysis.

When $(\rho_+, u_+) \in I(\rho_-, u_-)$, the Riemann solution, which is composed of a rarefaction wave R and a contact discontinuity J as well as the vacuum state, can be expressed as

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$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < \left(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\right)t + \dfrac{1}{2}\beta t^2, \\ (\rho, v), & \left(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\right)t + \dfrac{1}{2}\beta t^2 \le x \le \left(u_- + \dfrac{A\rho_-}{A+\rho_-}\right)t + \dfrac{1}{2}\beta t^2, \\ \text{Vac}, & \left(u_- + \dfrac{A\rho_-}{A+\rho_-}\right)t + \dfrac{1}{2}\beta t^2 < x < u_+ t + \dfrac{1}{2}\beta t^2, \\ (\rho_+, u_+), & x > u_+ t + \dfrac{1}{2}\beta t^2, \end{cases}$$

where

$$(\rho, v) = \left(\sqrt{\frac{A^3}{x/t - w_- - \beta t/2}} - A, \quad w_- + \sqrt{A}\sqrt{\frac{x}{t} - w_- - \frac{1}{2}\beta t}\right). \tag{2.6}$$

When $(\rho_+, u_+) \in II(\rho_-, u_-)$, the Riemann solution, which is formed from a rarefaction wave R and a contact discontinuity J with the nonvacuum intermediate state (ρ_*, v_*) between them, can be shown as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < \left(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\right)t + \dfrac{1}{2}\beta t^2, \\ (\rho, v), & \left(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\right)t + \dfrac{1}{2}\beta t^2 \le x \le \left(v_* - \dfrac{A^2\rho_*}{(A+\rho_*)^2}\right)t + \dfrac{1}{2}\beta t^2, \\ (\rho_*, v_*), & \left(v_* - \dfrac{A^2\rho_*}{(A+\rho_*)^2}\right)t + \dfrac{1}{2}\beta t^2 < x < u_+ t + \dfrac{1}{2}\beta t^2, \\ (\rho_+, u_+) & x > u_+ t + \dfrac{1}{2}\beta t^2, \end{cases}$$

where (ρ, v) is shown in (2.6),

$$(\rho_*, v_*) = \left(\frac{A^2}{u_+ - u_- + A^2/(A+\rho_-)} - A,\ u_+\right). \tag{2.7}$$

When $(\rho_+, u_+) \in III(\rho_-, u_-)$, the Riemann solution is made up of a shock wave S and a contact discontinuity J with the nonvacuum intermediate state (ρ_*, v_*) between them, which can be represented as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < \left(u_- - \dfrac{A^2\rho_*}{(A+\rho_-)(A+\rho_*)}\right)t + \dfrac{1}{2}\beta t^2, \\ (\rho_*, v_*), & \left(u_- - \dfrac{A^2\rho_*}{(A+\rho_-)(A+\rho_*)}\right)t + \dfrac{1}{2}\beta t^2 < x < u_+ t + \dfrac{1}{2}\beta t^2, \\ (\rho_+, u_+), & x > u_+ t + \dfrac{1}{2}\beta t^2, \end{cases}$$

in which (ρ_*, v_*) is given by (2.7).

However, when $(\rho_+, u_+) \in IV(\rho_-, u_-)$, that is, $0 \le u_+ \le u_- - A^2/(A+\rho_-)$, the solution cannot be constructed by classical waves. Motivated by [18, 37, 40], we seek a solution containing a weighted δ-measure supported on a curve so that the Riemann problem (2.1)–(2.2) is solved in the framework of nonclassical solutions.

DEFINITION 2.1. A two-dimensional weighted delta function $\omega(s)\delta_S$ supported on a smooth curve $S = \{(x(s), t(s)) : a \leq s \leq b\}$ is defined by

$$< \omega(s)\delta_S, \phi(x,t) >= \int_a^b \omega(s)\phi(x(s), t(s))\, ds$$

for every $\phi \in C_0^\infty(R \times R^+)$.

DEFINITION 2.2. A pair (ρ, v) is called a delta shock type solution to (2.1) in the sense of distributions if there exist a smooth curve S and a weight $\omega \in C^1(S)$ such that ρ and v are represented in the following form:

$$\rho(x,t) = \rho_0(x,t) + \omega(t)\delta_S, \quad v(x,t) = v_0(x,t), \quad v(x,t)|_S = v_\delta(t),$$

where $\rho_0(x,t) = \rho_l(x,t) - [\rho]H(x - x(t))$, $v_0(x,t) = v_l(x,t) - [v]H(x - x(t))$, in which $(\rho_l, v_l)(x,t)$ and $(\rho_r, v_r)(x,t)$ are piecewise smooth solutions to (2.1), $H(x)$ is the Heaviside function whose value is 0 for $x < 0$, 1 for $x > 0$, $-A^2/(A+\rho)$ is equal to $-A^2/(A+\rho_l)$ for $x < x(t)$, 0 for $x = x(t)$, $-A^2/(A+\rho_r)$ for $x > x(t)$, and it satisfies

$$\begin{cases} \langle \rho, \phi_t \rangle + \langle \rho(v + \beta t), \phi_x \rangle = 0, \\ \Big\langle \rho\Big(v - \dfrac{A^2}{A+\rho}\Big), \phi_t \Big\rangle + \Big\langle \rho(v+\beta t)\Big(v - \dfrac{A^2}{A+\rho}\Big), \phi_x \Big\rangle = 0 \end{cases}$$

for all test functions $\phi \in C_0^\infty(R \times R^+)$, where

$$\Big\langle \rho\Big(v - \frac{A^2}{A+\rho}\Big), \phi \Big\rangle = \int_{R^+}\int_R \rho_0\Big(v_0 - \frac{A^2}{A+\rho_0}\Big)\phi\, dx\, dt + \langle \omega(t) v_\delta(t)\delta_S, \phi \rangle.$$

By the two definitions, we can also rewrite the delta-shock solution of the system (2.1) in the form

$$(\rho, v)(x,t) = \begin{cases} (\rho_l, v_l)(x,t), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_r, v_r)(x,t), & x > x(t). \end{cases} \tag{2.8}$$

If a pair (ρ, v) of the form (2.8) is a solution to the system (2.1) in the sense of distributions, then the relation

$$\begin{cases} \dfrac{dx(t)}{dt} = \sigma(t) = v_\delta(t) + \beta t, \\ \dfrac{d\omega(t)}{dt} = -[\rho]\sigma(t) + [\rho(v + \beta t)], \\ \dfrac{d(\omega(t)v_\delta(t))}{dt} = -\Big[\rho\Big(v - \dfrac{A^2}{A+\rho}\Big)\Big]\sigma(t) + \Big[\rho(v+\beta t)\Big(v - \dfrac{A^2}{A+\rho}\Big)\Big] \end{cases} \tag{2.9}$$

holds. In a similar way as that in [21], the proof can be given.

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Relation (2.9), reflecting the exact relationship among the limit states on two sides of the discontinuity, the weight, propagation speed and the location of the discontinuity, is referred to as the generalized Rankine–Hugoniot relation. To ensure uniqueness, the following over-compressive entropy condition

$$\lambda_1(\rho_r, v_r) < \lambda_2(\rho_r, v_r) \le \frac{dx(t)}{dt} \le \lambda_1(\rho_l, v_l) < \lambda_2(\rho_l, v_l) \tag{2.10}$$

should be obeyed. The over-compressive entropy condition (2.10) means that all the characteristics on both sides of the discontinuity curve are out-coming.

Next, we are going to solve the Riemann problem (2.1)–(2.2) for the case $(\rho_+, u_+) \in IV(\rho_-, u_-)$ by using the generalized Rankine–Hugoniot [14, 27] relation (2.9). In this case, the solution is a delta-shock wave in the form

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_+, u_+), & x > x(t). \end{cases}$$

Taking into account the entropy condition (2.10), namely, $u_+ \le v_\delta(t) \le u_- - A^2\rho_-/(A+\rho_-)^2$, we solve the generalized Rankine–Hugoniot relation (2.9) with initial condition $x(0) = \omega(0) = 0$ to determine the $x(t)$, $\omega(t)$ and $v_\delta(t)$. In view of the knowledge concerning delta shocks in [37, 40], it can be found that $v_\delta(t)$ is a constant. From (2.9), we get

$$\begin{cases} \dfrac{d\omega(t)}{dt} = -v_\delta(\rho_- - \rho_+) + (\rho_- u_- - \rho_+ u_+), \\ \dfrac{d\omega(t)}{dt} v_\delta = -v_\delta\Big(\rho_-\Big(u_- - \dfrac{A^2}{A+\rho_-}\Big) - \rho_+\Big(u_+ - \dfrac{A^2}{A+\rho_+}\Big)\Big) \\ \qquad\qquad + \Big(\rho_- u_-\Big(u_- - \dfrac{A^2}{A+\rho_-}\Big) - \rho_+ u_+\Big(u_+ - \dfrac{A^2}{A+\rho_+}\Big)\Big), \end{cases}$$

which, on simplification, yields

$$(\rho_- - \rho_+)v_\delta^2 - Ev_\delta + F = 0, \tag{2.11}$$

where

$$\begin{cases} E = 2\rho_- u_- - 2\rho_+ u_+ - \dfrac{A^2\rho_-}{A+\rho_-} + \dfrac{A^2\rho_+}{A+\rho_+}, \\ F = \rho_- u_-\Big(u_- - \dfrac{A^2}{A+\rho_-}\Big) - \rho_+ u_+\Big(u_+ - \dfrac{A^2}{A+\rho_+}\Big). \end{cases}$$

After a direct calculation, from (2.11) and (2.9), we have

$$\begin{aligned} v_\delta &= \frac{u_- + u_+ - A^2/(A+\rho_-)}{2}, \\ x(t) &= \frac{u_- + u_+ - A^2/(A+\rho_-)}{2} t + \frac{1}{2}\beta t^2, \\ \omega(t) &= \rho_-(u_- - u_+)t \end{aligned} \tag{2.12}$$

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for $\rho_- = \rho_+$, and

$$\begin{cases} v_\delta = \dfrac{E - \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)}, \\ \omega(t) = \dfrac{\{A^2\rho_-/(A+\rho_-) - A^2\rho_+/(A+\rho_+)\} + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2} t, \\ x(t) = \dfrac{E - \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)} t + \dfrac{1}{2}\beta t^2 \end{cases} \tag{2.13}$$

and

$$\begin{cases} v_\delta = \dfrac{E + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)}, \\ \omega(t) = \dfrac{\{A^2\rho_-/(A+\rho_-) - A^2\rho_+/(A+\rho_+)\} - \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2} t, \\ x(t) = \dfrac{E + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)} t + \dfrac{1}{2}\beta t^2 \end{cases} \tag{2.14}$$

for $\rho_- \neq \rho_+$, in which

$$\begin{aligned} &E^2 - 4(\rho_- - \rho_+)F \\ &\quad = 4\rho_-\rho_+(u_- - u_+)\Big(u_- - \frac{A^2}{A+\rho_-} - u_+ + \frac{A^2}{A+\rho_+}\Big) + \Big(\frac{A^2\rho_+}{A+\rho_+} - \frac{A^2\rho_-}{A+\rho_-}\Big)^2 > 0. \end{aligned}$$

For the solution (2.12), one can calculate

$$\begin{cases} v_\delta - u_+ = \dfrac{u_- - u_+ - A^2/(A+\rho_-)}{2} \geq 0, \\ v_\delta - \Big(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\Big) = \dfrac{u_+ - u_- + A^2\rho_-/(A+\rho_-)^2 - A^3/(A+\rho_-)^2}{2} \leq 0, \end{cases}$$

which shows that the solution (2.12) satisfies the entropy condition when $\rho_- = \rho_+$.

By virtue of $(\rho_+, u_+) \in IV(\rho_-, u_-)$, that is, $0 \leq u_+ \leq u_- - A^2/(A+\rho_-)$, we have

$$(1)\ (\rho_- - \rho_+)u_+^2 - Eu_+ + F = \rho_-(u_- - u_+)\Big(u_- - \frac{A^2}{A+\rho_-} - u_+\Big) \geq 0;$$

$$\begin{aligned} (2)\ & -2(\rho_- - \rho_+)\Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big) + E + \sqrt{E^2 - 4(\rho_- - \rho_+)F} \\ &= 2\rho_+\Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2} - u_+\Big) + \Big(\frac{A^2\rho_+}{A+\rho_+} - \frac{A^2\rho_-}{A+\rho_-}\Big) + \frac{2A^2\rho_-^2}{(A+\rho_-)^2} + \sqrt{E^2 - 4(\rho_- - \rho_+)F} \\ &\geq 2\rho_+\Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2} - u_+\Big) + \frac{2A^2\rho_-^2}{(A+\rho_-)^2} > 0; \end{aligned}$$

$$(3)\ -2(\rho_- - \rho_+)u_+ + E = 2\rho_-\Big(u_- - u_+ - \frac{A^2}{A+\rho_-}\Big) + \Big(\frac{A^2\rho_-}{A+\rho_-} + \frac{A^2\rho_+}{A+\rho_+}\Big)$$

$$\geq \Big(\frac{A^2\rho_-}{A+\rho_-} + \frac{A^2\rho_+}{A+\rho_+}\Big) > 0;$$

$$(4)\ (\rho_- - \rho_+)\Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big)^2 - E\Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big) + F$$

$$= -\rho_+\Big(u_- - u_+ - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big)\Big(u_- - u_+ - \frac{A^2\rho_-}{(A+\rho_-)^2} + \frac{A^2}{A+\rho_+}\Big) - \frac{A^5\rho_-^2}{(A+\rho_-)^4}$$

$$\leq -\frac{A^5\rho_-^2}{(A+\rho_-)^4} < 0. \tag{2.15}$$

With the use of (2.15), we calculate

$$v_\delta - u_+ = \frac{E - \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)} - u_+$$

$$= \frac{2((\rho_- - \rho_+)u_+^2 - Eu_+ + F)}{(-2(\rho_- - \rho_+)u_+ + E) + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}$$

$$\geq 0$$

and

$$v_\delta - \Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big) = \frac{E - \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)} - \Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big)$$

$$= \frac{2[(\rho_- - \rho_+)\{u_- - A^2\rho_-/(A+\rho_-)^2\}^2 - E\{u_- - A^2\rho_-/(A+\rho_-)^2\} + F]}{[-2(\rho_- - \rho_+)\{u_- - A^2\rho_-/(A+\rho_-)^2\} + E] + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}$$

$$\leq 0,$$

which show that the solution (2.13) satisfies the entropy condition. Next, we show that for the solution (2.14), the entropy condition is not met. Indeed, we calculate

$$v_\delta - u_+ = \frac{E + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)} - u_+$$

$$= \frac{-2(\rho_- - \rho_+)u_+ + E + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)}$$

$$< 0$$

when $\rho_- < \rho_+$, and

$$\begin{aligned}v_\delta - \Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big) &= \frac{E + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)} - \Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big)\\ &= \frac{-2(\rho_- - \rho_+)\{u_- - A^2\rho_-/(A+\rho_-)^2\} + E + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2(\rho_- - \rho_+)}\\ &> 0\end{aligned}$$

when $\rho_- > \rho_+$. As a consequence, the following conclusions are obtained.

THEOREM 2.3. *When $0 \le u_+ \le u_- - A^2/(A+\rho_-)$, the Riemann problem (2.1)–(2.2) has a unique entropy solution in the sense of distributions of the form*

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < v_\delta t + \frac{1}{2}\beta t^2,\\ (\omega(t)\delta(x - (v_\delta t + \frac{1}{2}\beta t^2)), v_\delta), & x = v_\delta t + \frac{1}{2}\beta t^2,\\ (\rho_+, u_+), & x > v_\delta t + \frac{1}{2}\beta t^2,\end{cases}$$

where v_δ and $\omega(t)$ are shown in (2.12) for $\rho_- = \rho_+$ and (2.13) for $\rho_- \neq \rho_+$.

THEOREM 2.4. *The Riemann problem (2.1)–(2.2) admits a unique entropy solution, which is composed of a rarefaction wave and a contact discontinuity as well as a vacuum state when $u_- + A\rho_-/(A+\rho_-) \le u_+$, a rarefaction wave and a contact discontinuity when $u_- < u_+ < u_- + A\rho_-/(A+\rho_-)$, a shock wave and a contact discontinuity when $u_- - A^2/(A+\rho_-) < u_+ < u_-$, and a delta shock when $0 \le u_+ \le u_- - A^2/(A+\rho_-)$.*

3. Riemann solutions to the original system (1.1) with equation of state (1.2)

In this section, we will construct Riemann solutions for the model (1.1)–(1.2) with initial data (1.3) by means of $(\rho, u)(x,t) = (\rho, v + \beta t)(x,t)$.

When $(\rho_+, u_+) \in I(\rho_-, u_-)$, the Riemann solution of (1.1)–(1.2) and (1.3) is expressed as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < \Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big)t + \frac{1}{2}\beta t^2,\\ (\rho, v + \beta t), & \Big(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\Big)t + \frac{1}{2}\beta t^2 \le x \le \Big(u_- + \frac{A\rho_-}{A+\rho_-}\Big)t + \frac{1}{2}\beta t^2,\\ \text{Vac}, & \Big(u_- + \frac{A\rho_-}{A+\rho_-}\Big)t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2,\\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2.\end{cases} \tag{3.1}$$

This situation is illustrated in Figure 2.

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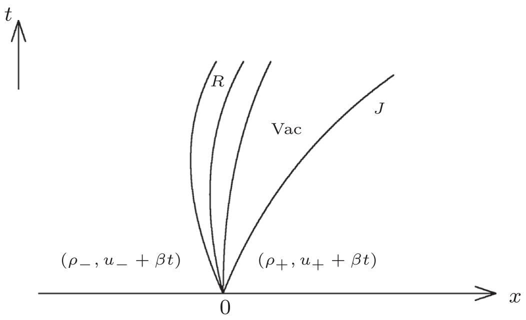

FIGURE 2. The R + Vac + J solution of (1.1)–(1.3) when $u_- - A^2\rho_-/(A+\rho_-)^2 < 0 < u_+$ and $\beta > 0$.

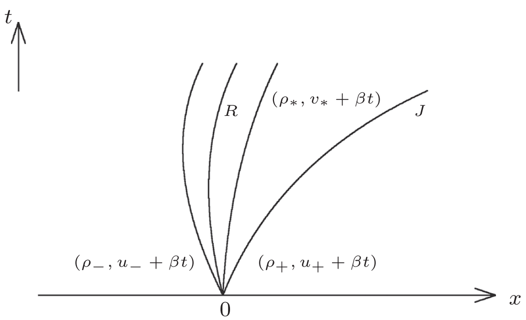

FIGURE 3. The $R + J$ solution of (1.1)–(1.3) when $u_- - A^2\rho_-/(A+\rho_-)^2 < 0 < u_+$ and $\beta > 0$.

When $(\rho_+, u_+) \in II(\rho_-, u_-)$, the Riemann solution contains a rarefaction wave R and a contact discontinuity J, which can be shown as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t) & x < \Big(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\Big)t + \dfrac{1}{2}\beta t^2, \\ (\rho, v + \beta t), & \Big(u_- - \dfrac{A^2\rho_-}{(A+\rho_-)^2}\Big)t + \dfrac{1}{2}\beta t^2 \le x \le \Big(v_* - \dfrac{A^2\rho_*}{(A+\rho_*)^2}\Big)t + \dfrac{1}{2}\beta t^2, \\ (\rho_*, v_* + \beta t), & \Big(v_* - \dfrac{A^2\rho_*}{(A+\rho_*)^2}\Big)t + \dfrac{1}{2}\beta t^2 < x < u_+ t + \dfrac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \dfrac{1}{2}\beta t^2, \end{cases} \tag{3.2}$$

where (ρ, v) is shown in (2.6) and (ρ_*, v_*) is given by (2.7). This case is illustrated in Figure 3.

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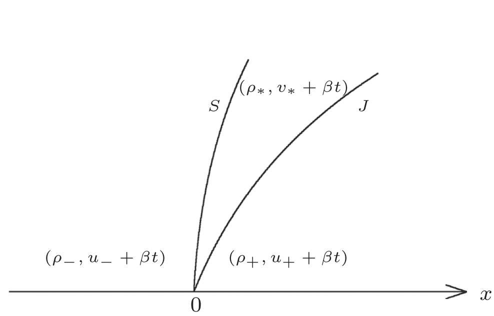

FIGURE 4. The $S + J$ solution of (1.1)–(1.3) when $0 < u_- - A^2\rho_*/((A + \rho_-)(A + \rho_*)) < u_+$ and $\beta > 0$.

When $(\rho_+, u_+) \in III(\rho_-, u_-)$, the Riemann solution includes a shock wave S and a contact discontinuity J, which can be represented as

$$
\begin{aligned}
&(\rho, u)(x, t)\\
&= \begin{cases} (\rho_-, u_- + \beta t), x < \Big(u_- - \dfrac{A^2\rho_*}{(A + \rho_-)(A + \rho_*)}\Big)t + \dfrac{1}{2}\beta t^2, \\ (\rho_*, v_* + \beta t),\ \Big(u_- - \dfrac{A^2\rho_*}{(A + \rho_-)(A + \rho_*)}\Big)t + \dfrac{1}{2}\beta t^2 < x < u_+ t + \dfrac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), x > u_+ t + \dfrac{1}{2}\beta t^2, \end{cases}
\end{aligned}
\tag{3.3}
$$

in which (ρ_*, v_*) is shown in (2.7). This situation is illustrated in Figure 4.

If $(\rho_+, u_+) \in IV(\rho_-, u_-)$, we attempt, just as before, to define the weak solution of the Riemann problem for the system (1.1)–(1.2) with initial data (1.3) in the distributional sense.

DEFINITION 3.1. A pair (ρ, u) is referred to as a delta shock type solution to (1.1)–(1.2) in the sense of distributions if there exist a smooth curve S and a weight $\omega \in C^1(S)$ such that ρ and u are represented in the following form:

$$
\rho(x, t) = \rho_0(x, t) + \omega(t)\delta_S, \quad u(x, t) = u_0(x, t), \quad u(x, t)|_S = u_\delta(t),
$$

where $\rho_0(x, t) = \rho_l(x, t) - [\rho]H(x - x(t))$, $u_0(x, t) = u_l(x, t) - [u]H(x - x(t))$, in which $(\rho_l, u_l)(x, t)$, $(\rho_r, u_r)(x, t)$ are piecewise smooth solutions to the (1.1)–(1.2), and it satisfies

$$
\begin{cases} \langle \rho, \phi_t \rangle + \langle \rho u, \phi_x \rangle = 0, \\ \Big\langle \rho\Big(u - \dfrac{A^2}{A + \rho}\Big), \phi_t \Big\rangle + \Big\langle \rho u\Big(u - \dfrac{A^2}{A + \rho}\Big), \phi_x \Big\rangle = -\langle \beta\rho, \phi \rangle \end{cases}
\tag{3.4}
$$

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for all $\phi \in C_0^\infty(R \times R^+)$, where

$$\left\langle \rho\Big(u - \frac{A^2}{A+\rho}\Big), \phi \right\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho_0\Big(u_0 - \frac{A^2}{A+\rho_0}\Big)\phi\, dx\, dt + \langle \omega(t)u_\delta(t)\delta_S, \phi\rangle.$$

When $u_+ \leq u_- - A^2/(A+\rho_-)$, the following delta shock wave solution of (1.1)–(1.3)

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x(t), \\ (\omega(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+ + \beta t), & x > x(t) \end{cases} \tag{3.5}$$

should be sought, where $u_\delta(t) - \beta t$ is a constant based on the analysis in Section 2. Moreover, the delta shock wave solution of the form (3.5) satisfies the generalized Rankine–Hugoniot relation

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\ \dfrac{d\omega(t)}{dt} = -[\rho]u_\delta(t) + [\rho u], \\ \dfrac{d(\omega(t)u_\delta(t))}{dt} = -\Big[\rho\Big(u - \dfrac{A^2}{A+\rho}\Big)\Big]u_\delta(t) + \Big[\rho u\Big(u - \dfrac{A^2}{A+\rho}\Big)\Big] + \beta\omega(t), \end{cases} \tag{3.6}$$

where the jump across the discontinuity is

$$\Big[\rho\Big(u - \frac{A^2}{A+\rho}\Big)\Big] = \rho_-\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big) - \rho_+\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big).$$

In addition to this, the over-compressive entropy condition for the delta-shock wave

$$u_+ + \beta t - \frac{A^2\rho_+}{(A+\rho_+)^2} < u_+ + \beta t \leq u_\delta(t) \leq u_- - \frac{A^2\rho_-}{(A+\rho_-)^2} + \beta t < u_- + \beta t \tag{3.7}$$

should be satisfied to ensure the uniqueness.

Under the entropy condition (3.7), $x(t)$, $u_\delta(t)$ and $\omega(t)$ can be derived uniquely by solving (3.6) with the initial value condition $x(0) = \omega(0) = 0$. To sum up, when $u_+ \leq u_- - A^2/(A+\rho_-)$, we depict the delta-shock solution of (1.1)–(1.3) by the following theorem (see Figure 5).

THEOREM 3.2. *The Riemann solution of* (1.1)–(1.3) *containing a delta shock can be shown as* (3.5) *provided that the initial data satisfy* $0 \leq u_+ \leq u_- - A^2/(A+\rho_-)$, *in which*

$$u_\delta(t) = v_\delta + \beta t, \omega(t) = \frac{\{A^2\rho_-/(A+\rho_-) - A^2\rho_+/(A+\rho_+)\} + \sqrt{E^2 - 4(\rho_- - \rho_+)F}}{2}t,$$
$$x(t) = v_\delta t + \frac{1}{2}\beta t^2, \tag{3.8}$$

where v_δ *is displayed in* (2.12) *for* $\rho_- = \rho_+$, *and* (2.13) *for* $\rho_- \neq \rho_+$.

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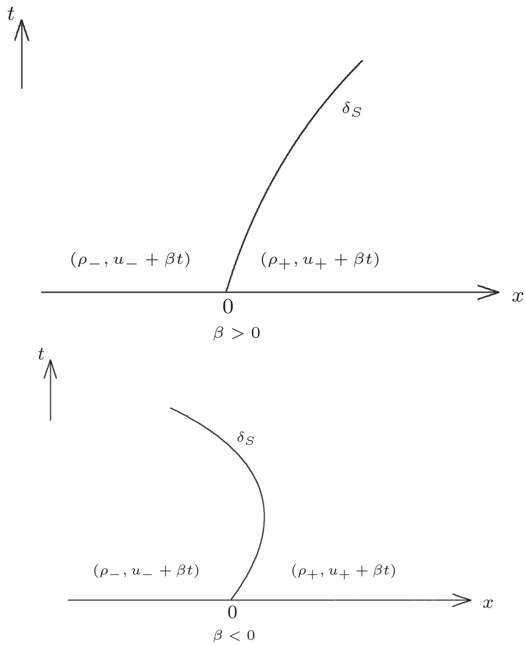

FIGURE 5. Delta-shock solution of (1.1)–(1.3).

PROOF. Bearing in mind that $u_\delta(t) - \beta t$ is a constant, from (3.6), it yields that

$$\frac{d\omega(t)}{dt} = -u_\delta(t)(\rho_- - \rho_+) + (\rho_-(u_- + \beta t) - \rho_+(u_+ + \beta t)) \tag{3.9}$$

and

$$\begin{aligned}\frac{d\omega(t)}{dt}u_\delta(t) = &-u_\delta(t)\Big(\rho_-\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big) - \rho_+\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\Big)\\ &+ \Big(\rho_-(u_- + \beta t)\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big) - \rho_+(u_+ + \beta t)\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\Big).\end{aligned} \tag{3.10}$$

Substituting (3.9) in (3.10) and rearranging terms, we get

$$\begin{aligned}(\rho_- - \rho_+)(u_\delta(t) - \beta t)^2 - \Big(2(\rho_- u_- - \rho_+ u_+) - \frac{A^2\rho_-}{A+\rho_-} + \frac{A^2\rho_+}{A+\rho_+}\Big)(u_\delta(t) - \beta t)\\ + \Big(\rho_- u_-\Big(u_- - \frac{A^2}{A+\rho_-}\Big) - \rho_+ u_+\Big(u_+ - \frac{A^2}{A+\rho_+}\Big)\Big) = 0,\end{aligned}$$

which, by virtue of the entropy condition (3.7), yields $u_\delta(t) = v_\delta + \beta t$. Again, we make use of the generalized Rankine–Hugoniot relation (3.6) to obtain (3.8). This ends the proof. □

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From (3.1)–(3.3) and (3.8), it is found that the rarefaction wave, shock wave and contact discontinuity as well as delta shock are parabolic curves. Moreover, we also can prove that the constructed delta-shock solution satisfies (1.1)–(1.2) in the sense of distributions. For the process of proof, please see Appendix A.

4. Limiting behaviour of solutions to (1.1)–(1.3) as $A \to 0$

This section first reviews the Riemann solutions for the zero-pressure Euler equations with friction. For the detailed research, please see [31]. Then, we prove that the Riemann solutions of the model (1.1)–(1.2) with initial data (1.3) converge to the corresponding Riemann solutions of nonhomogeneous zero-pressure Euler system (1.4) with (1.3) as $A \to 0$.

4.1. Riemann solutions of (1.4) with (1.3) In this subsection, let us review the Riemann solutions of (1.4) and (1.3). By the transformation $u(x,t) = v(x,t) + \beta t$, (1.4) and (1.3) are transformed into

$$\begin{cases} \rho_t + (\rho(v + \beta t))_x = 0, \\ (\rho v)_t + (\rho v(v + \beta t))_x = 0, \end{cases} \tag{4.1}$$

and (2.2). System (4.1) possesses a repeated eigenvalue $\lambda = v + \beta t$, which means that the system (4.1) is nonstrictly hyperbolic. The corresponding right eigenvector is $\overrightarrow{r} = (1,0)^T$ satisfying $\nabla\lambda \cdot \overrightarrow{r} = 0$. Therefore, λ is linearly degenerate and the elementary waves only include contact discontinuities.

A bounded discontinuity at $x = x(t)$ satisfies the Rankine–Hugoniot relation

$$\begin{cases} -\sigma(t)[\rho] + [\rho(v + \beta t)] = 0, \\ -\sigma(t)[\rho v] + [\rho v(v + \beta t)] = 0, \end{cases} \tag{4.2}$$

where $[\rho] = \rho_- - \rho$. From (4.2), it follows that the contact discontinuity

$$J : \sigma(t) = v + \beta t = u_- + \beta t.$$

Clearly, the states (ρ_-, u_-) and (ρ_+, u_+) can be connected by a contact discontinuity J, provided that $u_- = u_+$.

The Riemann solutions can be constructed in two cases. When $u_- < u_+$, the Riemann solution of (4.1) and (2.2) is made up of two contact discontinuities and a vacuum state (denoted by Vac), which can be given as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < u_- t + \frac{1}{2}\beta t^2, \\ \text{Vac}, & u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+), & x > u_+ t + \frac{1}{2}\beta t^2. \end{cases}$$

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When $u_- > u_+$, the Riemann solution contains a delta shock, which can be shown as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta), & x = x(t), \\ (\rho_+, u_+), & x > x(t), \end{cases}$$

where $x(t)$, $\omega(t)$ and v_δ satisfy the generalized Rankine–Hugoniot relation

$$\begin{cases} \dfrac{dx(t)}{dt} = \sigma(t) = v_\delta + \beta t, \\ \dfrac{d\omega(t)}{dt} = -[\rho]\sigma(t) + [\rho(v + \beta t)], \\ \dfrac{d(\omega(t)v_\delta)}{dt} = -[\rho v]\sigma(t) + [\rho v(v + \beta t)], \end{cases} \tag{4.3}$$

and the delta-entropy condition

$$u_+ < v_\delta < u_-$$

with $[\rho v] = \rho_- u_- - \rho_+ u_+$. Solving (4.3) with $x(0) = \omega(0) = 0$ gives

$$\begin{cases} v_\delta = \dfrac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ x(t) = \dfrac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_-} + \sqrt{\rho_+}} t + \dfrac{1}{2}\beta t^2, \\ \omega(t) = \sqrt{\rho_- \rho_+}(u_- - u_+)t. \end{cases} \tag{4.4}$$

By means of $u(x,t) = v(x,t) + \beta t$, two kinds of nonself-similar Riemann solutions for the nonhomogeneous zero-pressure Euler equations (1.4) with initial data (1.3) can be constructed.

Case 1. When $u_- < u_+$, the vacuum solution of (1.4) with (1.3) can be represented as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ \text{Vac}, & u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2. \end{cases}$$

Case 2. When $u_- > u_+$, the delta-shock solution of (1.4) with (1.3) can be given as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < v_\delta t + \frac{1}{2}\beta t^2, \\ (\omega(t)\delta(x - (v_\delta t + \frac{1}{2}\beta t^2)), v_\delta + \beta t), & x = v_\delta t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > v_\delta t + \frac{1}{2}\beta t^2, \end{cases} \tag{4.5}$$

where v_δ and $\omega(t)$ are shown in (4.4).

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4.2. Vanishing pressure limits of solutions of (1.1)–(1.2) with initial data (1.3)

This subsection analyses the vanishing pressure limits of Riemann solutions to (1.1)–(1.2) with initial data (1.3). It is shown that the Riemann solutions of (1.1)–(1.2) with initial data (1.3) converge to these of the pressureless Euler model with friction (1.4) with initial data (1.3) as $A \to 0$.

Case 4.1. $u_- < u_+$.

LEMMA 4.1. *If $u_- < u_+ < u_- + \rho_-$, then there exists a positive number A_1 such that $(\rho_+, u_+) \in II(\rho_-, u_-)$ when $A > A_1$, and $(\rho_+, u_+) \in I(\rho_-, u_-)$ when $0 < A < A_1$. If $u_+ \geq u_- + \rho_-$, then one has $(\rho_+, u_+) \in I(\rho_-, u_-)$ for any $A > 0$.*

PROOF. If $u_- < u_+ < u_- + A\rho_-/(A + \rho_-)$, we have $(\rho_+, u_+) \in II(\rho_-, u_-)$. If $u_+ > u_- + A\rho_-/(A + \rho_-)$, we have $(\rho_+, u_+) \in I(\rho_-, u_-)$. When $u_- < u_+ < u_- + \rho_-$,

$$A_1 = \frac{\rho_-(u_- - u_+)}{u_+ - u_- - \rho_-} > 0$$

can be chosen. If $u_+ \geq u_- + \rho_-$, then we have $(\rho_+, u_+) \in I(\rho_-, u_-)$ by virtue of

$$u_+ \geq u_- + \rho_- > u_- + A\rho_-/(A + \rho_-).$$

The proof is completed. □

To better understand the process of the formation of the vacuum, we only discuss $u_- < u_+ < u_- + \rho_-$. The case $u_+ \geq u_- + \rho_-$ can be treated in a similar manner. For any $A > A_1$, the solution consists of a rarefaction wave R and a contact discontinuity J connecting two constant states $(\rho_\pm, u_\pm)$, which can be given by (3.2), where the nonvacuum intermediate state (ρ_*, v_*) is shown in (2.7). Using (2.7), we calculate

$$\lim_{A\to A_1} \rho_* = \lim_{A\to A_1} \left(\frac{A^2}{u_+ - u_- + A^2/(A + \rho_-)} - A \right) = 0, \quad \lim_{A\to A_1} v_* = u_+,$$

which shows that the vacuum appears when A falls to A_1. Moreover, a direct calculation gives

$$\lim_{A\to A_1} \left(u_- - \frac{A^2\rho_-}{(A + \rho_-)^2} \right) = u_- - \frac{(u_- - u_+)^2}{\rho_-}.$$

Hence, we arrive at

$$\lim_{A\to A_1} (\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), x < \left(u_- - \dfrac{(u_- - u_+)^2}{\rho_-} \right) t + \dfrac{1}{2}\beta t^2, \\ (\rho, v + \beta t), \quad \left(u_- - \dfrac{(u_- - u_+)^2}{\rho_-} \right) t + \dfrac{1}{2}\beta t^2 \leq x < u_+ t + \dfrac{1}{2}\beta t^2, \\ (0, u_+ + \beta t), \quad x = u_+ t + \dfrac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), \ x > u_+ t + \dfrac{1}{2}\beta t^2, \end{cases} \tag{4.6}$$

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where

$$(\rho, v) = \left(\sqrt{\frac{A_1^3}{x/t - w_-(A_1) - \beta t/2}} - A_1,\ w_-(A_1) + \sqrt{A_1}\sqrt{\frac{x}{t} - w_-(A_1) - \frac{1}{2}\beta t}\right).$$

System (4.6) shows that when $(\rho_+, u_+) \in II(\rho_-, u_-)$, the $R + J$ solution of (1.1)–(1.2) and (1.3) converges to the vacuum solution of (1.1)–(1.2) and (1.3) for $(\rho_+, u_+) \in V_B$ as $A \to A_1$, where the curve V_B is exactly the boundary between the region $I(\rho_-, u_-)$ and the region $II(\rho_-, u_-)$.

When $0 < A < A_1$, the solution is $R + \text{Vac} + J$, which can be shown as (3.1). By means of (3.1), we conclude

$$\lim_{A\to 0}\left(u_- - \frac{A^2\rho_-}{(A+\rho_-)^2}\right) = u_-, \quad \lim_{A\to 0}\left(u_- - \frac{A\rho_-}{A+\rho_-}\right) = u_-,$$

which shows that the rarefaction wave R eventually evolves into contact discontinuity as A goes to zero. As a consequence, we have

$$\lim_{A\to 0}(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ \left(0, \frac{x}{t} + \frac{1}{2}\beta t\right), & u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2, \end{cases}$$

which corresponds to the vacuum Riemann solution of (1.4) with (1.3).

Case 4.2. $u_- > u_+$.

LEMMA 4.2. *Let $u_- > u_+$. Then, there exists a positive number A_2 such that $(\rho_+, u_+) \in IV(\rho_-, u_-)$ when $0 < A < A_2$, and $(\rho_+, u_+) \in III(\rho_-, u_-)$ when $A > A_2$.*

PROOF. Suppose that $u_- > u_+$. When $(\rho_+, u_+) \in IV(\rho_-, u_-)$, we have

$$0 \le u_+ \le u_- - \frac{A^2}{A + \rho_-}.$$

When $u_- - A^2/(A+\rho_-) < u_+ < u_-$, we have $(\rho_+, u_+) \in III(\rho_-, u_-)$. Therefore,

$$A_2 = \frac{(u_- - u_+) + \sqrt{(u_- - u_+)^2 + 4\rho_-(u_- - u_+)}}{2}$$

can be taken. The proof is completed. □

When $A > A_2$, the Riemann solution consists of a shock wave S and a contact discontinuity J with the nonvacuum intermediate state (ρ_*, v_*) in addition to the constant states $(\rho_\pm, u_\pm)$, which is given by (3.3). Using (2.7), it can be concluded that

$$\lim_{A\to A_2} \rho_* = \lim_{A\to A_2}\left(\frac{A^2}{u_+ - u_- + A^2/(A+\rho_-)} - A\right) = +\infty, \quad \lim_{A\to A_2} v_* = u_+. \tag{4.7}$$

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LEMMA 4.3. *Set*

$$x_1(t,A) = \left(u_- - \frac{A^2\rho_*}{(A+\rho_-)(A+\rho_*)}\right)t + \frac{1}{2}\beta t^2, \quad x_2(t,A) = u_+ t + \frac{1}{2}\beta t^2. \tag{4.8}$$

Let $dx_1(t)/dt = \sigma_1(t)$, $dx_2(t)/dt = \sigma_2(t)$, *then, we have*

$$\begin{aligned} &\lim_{A\to A_2} \sigma_1(t) = \lim_{A\to A_2} \sigma_2(t) = u_+ + \beta t, \\ &\lim_{A\to A_2} \int_{x_1(t,A)}^{x_2(t,A)} \rho_*\,dx = \rho_-(u_- - u_+)t, \\ &\lim_{A\to A_2} \int_{x_1(t,A)}^{x_2(t,A)} \rho_*(v_* + \beta t)\,dx = \rho_-(u_- - u_+)(u_+ + \beta t)t. \end{aligned} \tag{4.9}$$

PROOF. By virtue of (4.7) and Lemma 4.2, we have

$$\begin{cases} \lim\limits_{A\to A_2} \sigma_1(t) = \lim\limits_{A\to A_2} \left(u_- - \dfrac{A^2\rho_*}{(A+\rho_-)(A+\rho_*)}\right) + \beta t = \left(u_- - \dfrac{(A_2)^2}{A_2+\rho_-}\right) + \beta t \\ \qquad\qquad = u_+ + \beta t, \\ \lim\limits_{A\to A_2} \sigma_2(t) = u_+ + \beta t, \end{cases}$$

which indicates that when $A \to A_2$, the shock wave S and the contact discontinuity J will coincide.

Using (4.8), we calculate

$$\begin{aligned} \lim_{A\to A_2} \int_{x_1(t,A)}^{x_2(t,A)} \rho_*\,dx &= \lim_{A\to A_2} \rho_*(x_2(t,A) - x_1(t,A)) \\ &= \lim_{A\to A_2} \left[\left(\frac{A^2}{u_+ - u_- + A^2/(A+\rho_-)} - A\right)\right. \\ &\quad \left.\times \left(u_+ - u_- + \frac{A^2 - A(u_+ - u_- + A^2/(A+\rho_-))}{A+\rho_-}\right)t\right] \\ &= \rho_-(u_- - u_+)t. \end{aligned}$$

Then, by means of the above formula, we get

$$\lim_{A\to A_2} \int_{x_1(t,A)}^{x_2(t,A)} \rho_*(v_* + \beta t)\,dx = (u_+ + \beta t) \lim_{A\to A_2} \int_{x_1(t,A)}^{x_2(t,A)} \rho_*\,dx = \rho_-(u_- - u_+)(u_+ + \beta t)t,$$

and the proof is complete. □

Lemma 4.3 indicates that the shock wave S and the contact discontinuity J coincide as A drops to A_2. Formula (4.7) and (4.9) show that ρ_* possesses the singularity, which

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is a weighed Dirac delta function with speed $u_+ + \beta t$. Thereby, a delta shock appears when A arrives at A_2. Substituting

$$A = A_2 = \frac{1}{2}\Big\{(u_- - u_+) + \sqrt{(u_- - u_+)^2 + 4\rho_-(u_- - u_+)}\Big\}$$

into (3.8) gives

$$u_\delta(t, A_2) = v_\delta(A_2) + \beta t = u_+ + \beta t, \quad \omega(t, A_2) = \rho_-(u_- - u_+)t.$$

As a result, as A falls to A_2, the limit of the Riemann solution of (1.1)–(1.2) and (1.3) is exactly the delta-shock solution of (1.1)–(1.2) and (1.3) in the boundary case $(\rho_+, u_+) \in S_\delta$.

Next, the case $0 < A < A_2$ is discussed. Under such circumstances, the Riemann solution of (1.1)–(1.3) contains a delta shock, which can be expressed by (3.5), in which v_δ is given by (2.12) when $\rho_- = \rho_+$, or (2.13) when $\rho_- \neq \rho_+$. When $\rho_- \neq \rho_+$, by (2.13), we take $A \to 0$ in (3.8) to get

$$\begin{cases} \lim\limits_{A\to 0} u_\delta(t, A) = \dfrac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_-} + \sqrt{\rho_+}} + \beta t, \\ \lim\limits_{A\to 0} \omega(t, A) = \sqrt{\rho_-\rho_+}(u_- - u_+)t, \\ \lim\limits_{A\to 0} x(t, A) = \dfrac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_-} + \sqrt{\rho_+}}t + \dfrac{1}{2}\beta t^2. \end{cases}$$

Analogously, when $\rho_- = \rho_+$, sending $A \to 0$ in (3.8) yields

$$\begin{cases} \lim\limits_{A\to 0} u_\delta(t, A) = \dfrac{u_- + u_+}{2} + \beta t, \\ \lim\limits_{A\to 0} \omega(t, A) = \rho_-(u_- - u_+)t, \\ \lim\limits_{A\to 0} x(t, A) = \dfrac{u_- + u_+}{2}t + \dfrac{1}{2}\beta t^2. \end{cases} \tag{4.10}$$

Comparing with (4.5), it can be asserted that the delta-shock solution of (1.1)–(1.3) converges to the delta-shock solution of (1.4) with (1.3) as $A \to 0$.

In conclusion, when $A > A_1$, the $R + J$ solution of (1.1)–(1.3) converges to the vacuum solution of (1.1)–(1.3) at $A = A_1$. When A decreases gradually and goes to zero, the vacuum solution of (1.1)–(1.3) converges to the vacuum solution of (1.4) with (1.3). When $0 < A < A_1$, the $R + \text{Vac} + J$ solution of (1.1)–(1.3) converges to the vacuum solution of (1.4) with (1.3) as $A \to 0$. When $A > A_2$, the $S + J$ solution of (1.1)–(1.3) converges to a delta-shock solution of (1.1)–(1.3) at $A = A_2$. As A continues to decrease and goes to zero, the delta-shock solution of (1.1)–(1.3) converges to the delta-shock solution of (1.4) with (1.3). When $0 < A < A_2$, the delta-shock solution of (1.1)–(1.3) converges to the delta-shock solution of (1.4) with (1.3) as $A \to 0$.

5. Conclusion

In this work, a Born–Infeld type fluid with equation of state (1.2) is introduced into the Aw–Rascle model of traffic flow with friction. Four kinds of nonself-similar Riemann solutions are derived. To be exact, the solutions contain a rarefaction wave and a contact discontinuity with vacuum intermediate state when $u_+ \geq u_- + A\rho_-/(A+\rho_-)$, a rarefaction wave and a contact discontinuity with a nonvacuum intermediate state when $u_- < u_+ < u_- + A\rho_-/(A+\rho_-)$, a shock wave and a contact discontinuity with a nonvacuum intermediate state when $u_- - A^2/(A+\rho_-) < u_+ < u_-$, and a single delta shock when $0 \leq u_+ \leq u_- - A^2/(A+\rho_-)$. The traffic-free zone can be illustrated by the vacuum. The congestion traffic situation and the traffic jam and even the traffic accident can be illustrated by the state with high traffic density and even the delta shock. More importantly, it is proved that as the traffic pressure goes to zero, the solution containing a shock wave and a contact discontinuity and the solution including a single delta shock of the nonhomogeneous Aw–Rascle model tend to be the delta-shock solution of the zero-pressure Euler system with friction; the solutions containing a rarefaction wave and a contact discontinuity of the nonhomogeneous Aw–Rascle model tend to be the solution including vacuum for the zero-pressure Euler system with friction. The vanishing traffic pressure may signify that the driver loses the acceleration and deceleration behaviour in the traffic flow, which usually leads to either a serious traffic accident such as the rear-end collision accident or creates a car-free zone. Consequently, it is of great significance to investigate the vanishing pressure limits of Riemann solutions for the nonhomogeneous Aw–Rascle traffic flow model with Born–Infeld equation of state, which are manifestations of gradually decreasing driving pressure and entering indulgence driving.

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Appendix A. Proof that the delta-shock solution in Section 3 satisfies (1.1)–(1.2)

We will prove the constructed delta-shock solution satisfying (1.1)–(1.2) in the sense of distributions. We only check the second equation in (3.4), the first equation in (3.4) could be verified in a similar way. Due to $x(t) = v_\delta t + \beta t^2/2$, we conclude $x'(t) = v_\delta + \beta t$. Clearly, when $\beta > 0$, there exists an inverse function of $x(t)$ for all time t, which may be written as

$$t(x) = \sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}.$$

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However, for $\beta < 0$, the sign of $x'(t)$ changes when the critical point $(-v_\delta^2/(2\beta), -v_\delta/\beta)$ on the delta shock wave curve is crossed. So, the inverse function of $x(t)$ exists for $t \le -v_\delta/\beta$ and $t > -v_\delta/\beta$. After a calculation, it can be found that

$$t(x) = \begin{cases} -\dfrac{v_\delta}{\beta} - \sqrt{\dfrac{v_\delta^2}{\beta^2} + \dfrac{2x}{\beta}}, & t \le -\dfrac{v_\delta}{\beta}, \\ -\dfrac{v_\delta}{\beta} + \sqrt{\dfrac{v_\delta^2}{\beta^2} + \dfrac{2x}{\beta}}, & t > -\dfrac{v_\delta}{\beta}. \end{cases}$$

For simplicity, we assume that $\beta > 0$. The case $\beta < 0$ can be treated similarly. Under this supposition, by exchanging the ordering of integral and integration by parts, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(\rho\Big(u - \frac{A^2}{A+\rho}\Big)\phi_t + \rho u\Big(u - \frac{A^2}{A+\rho}\Big)\phi_x\Big)\,dx\,dt \\
&= \int_0^{+\infty}\int_{-\infty}^{x(t)}\Big(\rho_-\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big)\phi_t + \rho_-(u_- + \beta t)\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big)\phi_x\Big)\,dx\,dt \\
&\quad + \int_0^{+\infty}\int_{x(t)}^{+\infty}\Big(\rho_+\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\phi_t + \rho_+(u_+ + \beta t)\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\phi_x\Big)\,dx\,dt \\
&\quad + \int_0^{+\infty}\omega(t)u_\delta(t)(\phi_t + u_\delta(t)\phi_x)\,dt \\
&= \int_0^{+\infty}\int_{t(x)}^{+\infty}\Big(\rho_-\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big)\phi_t + \rho_-(u_- + \beta t)\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big)\phi_x\Big)\,dt\,dx \\
&\quad + \int_0^{+\infty}\int_0^{t(x)}\Big(\rho_+\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\phi_t + \rho_+(u_+ + \beta t)\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\phi_x\Big)\,dt\,dx \\
&\quad + \int_0^{+\infty}\omega(t)u_\delta(t)(\phi_t + u_\delta(t)\phi_x)\,dt \\
&= \int_0^{+\infty}\Big(\rho_+\Big(u_+ + \beta t(x) - \frac{A^2}{A+\rho_+}\Big) - \rho_-\Big(u_- + \beta t(x) - \frac{A^2}{A+\rho_-}\Big)\Big)\phi(x, t(x))\,dx \\
&\quad + \int_0^{+\infty}\Big(\rho_-(u_- + \beta t)\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big) \\
&\quad - \rho_+(u_+ + \beta t)\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\Big)\phi(x(t), t)\,dt \\
&\quad - \int_0^{+\infty}\int_{t(x)}^{+\infty}(\beta\rho_-\phi)\,dt\,dx - \int_0^{+\infty}\int_0^{t(x)}\beta\rho_+\phi\,dt\,dx + \int_0^{+\infty}\omega(t)u_\delta(t)\,d\phi \\
&= \int_0^{+\infty}N(t)\phi(x(t), t)\,dt - \int_0^{+\infty}\int_{-\infty}^{x(t)}\beta\rho_-\phi\,dx\,dt - \int_0^{+\infty}\int_{x(t)}^{+\infty}\beta\rho_+\phi\,dx\,dt,
\end{aligned}$$

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in which

$$\begin{aligned} N(t) &= -u_\delta(t)\Big(\rho_-\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big) - \rho_+\Big(u_+ + \beta t - \tfrac{A^2}{A+\rho_+}\Big)\Big) - \frac{d(\omega(t)u_\delta(t))}{dt} \\ &\quad + \Big(\rho_-(u_- + \beta t)\Big(u_- + \beta t - \frac{A^2}{A+\rho_-}\Big) - \rho_+(u_+ + \beta t)\Big(u_+ + \beta t - \frac{A^2}{A+\rho_+}\Big)\Big) \\ &= -\beta\omega(t). \end{aligned}$$

This shows that the second equation of (3.4) holds in the sense of distributions.

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