CORRIGENDUM

Energy dissipation rate limits for flow through rough channels and tidal flow across topography – CORRIGENDUM

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1. The problem

The upper bound derived in Kerswell (2016) is incorrect. This is because the I_4 and I_5 integrals centred at the top boundary are not exactly analogous to their counterparts at the lower boundary, counter to what is written just under equation (2.23) (in Kerswell 2016). Instead, in both cases, due to the roughness, the full volume integrals must be included which scale differently with ℓ. Specifically, the estimates for the I_4 and I_5 integrals centred at the top boundary are

$$
\begin{aligned}
I_5^{top} &:= \left\langle \int_{1-\ell}^{1} \iiint_{\mathcal{V}(\Lambda)} |\boldsymbol{u} \boldsymbol{\cdot} \nabla^2 \boldsymbol{a}| \, \mathrm{d}V \, \mathrm{d}\Lambda \right\rangle \\
&\leqslant \ell \left\langle \iiint_{\mathcal{V}(1)} |\boldsymbol{u} \boldsymbol{\cdot} \nabla^2 \boldsymbol{a}| \, \mathrm{d}V \right\rangle - \left\langle \int_{1-\ell}^{1} \iiint_{\overline{\mathcal{V}}(\Lambda)} |\boldsymbol{u} \boldsymbol{\cdot} \nabla^2 \boldsymbol{a}| \, \mathrm{d}\Lambda \right\rangle \\
&\leqslant O(\ell\sqrt{\epsilon}) + O(\ell^{5/2}\sqrt{\epsilon}),
\end{aligned}
$$

where $\overline{\mathcal{V}}(\Lambda) := \mathcal{V}(1) - \mathcal{V}(\Lambda)$ so the last term can be estimated as in expression (2.23), and

$$
\begin{aligned}
I_4^{top} &:= \left\langle \int_{1-\ell}^{1} \iiint_{\mathcal{V}(\Lambda)} |\boldsymbol{u} \boldsymbol{\cdot} (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{a}| \, \mathrm{d}V \, \mathrm{d}\Lambda \right\rangle \\
&\leqslant \ell \left\langle \iiint_{\mathcal{V}(1)} |\boldsymbol{u} \boldsymbol{\cdot} (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{a}| \, \mathrm{d}V \right\rangle - \left\langle \int_{1-\ell}^{1} \iiint_{\overline{\mathcal{V}}(\Lambda)} |\boldsymbol{u} \boldsymbol{\cdot} (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{a}| \, \mathrm{d}\Lambda \right\rangle \\
&\leqslant O(\ell\epsilon) + O(\ell^3\epsilon),
\end{aligned}
$$

where again the last term can be estimated as in expression (2.22). The first term of $O(\ell\epsilon)$ on the right-hand side in this modified estimate of I_4^{top} is the key new addition which breaks the bound. To see this, it is best to start with equation (2.16) rearranged slightly as follows

$$(1-\ell)Gr + \frac{1}{A}\left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a} + \boldsymbol{u}\cdot\nabla^2\boldsymbol{a}\,dV\right\rangle$$

$$= \frac{1}{A}\left\langle \frac{1}{\ell}\int_{1-\ell}^{1}\int_{\mathcal{S}(\Lambda)} (\boldsymbol{a}\cdot\boldsymbol{u})\boldsymbol{u}\cdot\hat{\boldsymbol{n}} + \boldsymbol{u}\cdot(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{a} - \boldsymbol{a}\cdot(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{u}\,dS\,d\Lambda\right.$$

$$-\frac{1}{\ell}\int_{0}^{\ell}\int_{\mathcal{S}(\Lambda)} (\boldsymbol{a}\cdot\boldsymbol{u})\boldsymbol{u}\cdot\hat{\boldsymbol{n}} + \boldsymbol{u}\cdot(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{a} - \boldsymbol{a}\cdot(\hat{\boldsymbol{n}}\cdot\nabla)\boldsymbol{u}\,dS\,d\Lambda$$

$$+\frac{1}{\ell}\int_{1-\ell}^{1}\iiint_{\overline{\mathcal{V}}(\Lambda)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a} + \boldsymbol{u}\cdot\nabla^2\boldsymbol{a}\,dV\,d\Lambda$$

$$\left.+\frac{1}{\ell}\int_{0}^{\ell}\iiint_{\mathcal{V}(\Lambda)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a} + \boldsymbol{u}\cdot\nabla^2\boldsymbol{a}\,dV\,d\Lambda\right\rangle \tag{1.1}$$

to highlight the full volume integrals present (now on the left). Then the arguments presented in the paper are correct to reach (2.24) which now reads

$$(1-\ell)Gr + \frac{1}{A}\left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a} + \boldsymbol{u}\cdot\nabla^2\boldsymbol{a}\,dV\right\rangle$$

$$\leqslant \frac{1}{\ell}\left\{(B_1\ell^2 + B_4\ell^3)\varepsilon + (B_2\ell + B_3 + B_5\ell^2)\sqrt{\ell\varepsilon}\right\}. \tag{1.2}$$

For $\ell \to 0$, this is (using the fact that $Gr = \varepsilon/U$)

$$\left\{\frac{1}{U} + \frac{\left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a}\,dV\right\rangle}{\left\langle \iiint_{\mathcal{V}(1)} |\nabla\boldsymbol{u}|^2\,dV\right\rangle}\right\}\varepsilon \leqslant B_1\ell\varepsilon + B_3\sqrt{\varepsilon/\ell} + \text{h.o.t.} \tag{1.3}$$

since

$$\left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u}\cdot\nabla^2\boldsymbol{a}\,dV\right\rangle \leqslant O(\sqrt{A\varepsilon}) \ll B_3\sqrt{\varepsilon/\ell}. \tag{1.4}$$

The right-hand side is minimised as before by $\ell = \varepsilon^{-1/3}$ so that

$$\left\{\frac{1}{U} + \frac{\left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a}\,dV\right\rangle}{\left\langle \iiint_{\mathcal{V}(1)} |\nabla\boldsymbol{u}|^2\,dV\right\rangle}\right\}\varepsilon \leqslant C\varepsilon^{2/3}, \tag{1.5}$$

where C is an $O(1)$ constant or rewriting

$$\varepsilon \leqslant C^3 U^3 \left/ \left\{1 + \frac{U\left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{a}\,dV\right\rangle}{\left\langle \iiint_{\mathcal{V}(1)} |\nabla\boldsymbol{u}|^2\,dV\right\rangle}\right\}^3\right. . \tag{1.6}$$

Since no lower bound is available on the denominator, this does not provide a bound on ε. In fact, the better way to view (1.5) is that it presents an upper bound on the denominator

$$\left\{ 1 + \frac{U \left\langle \iiint_{\mathcal{V}(1)} \boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{a} \, \mathrm{d}V \right\rangle}{\left\langle \iiint_{\mathcal{V}(1)} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}V \right\rangle} \right\} \leqslant O(U\varepsilon^{-1/3}) \tag{1.7}$$

rather than a bound on ε.

2. Why there is no quick fix

It became apparent that there must be a problem with the bound in Kerswell (2016) when a connection was very recently made (Chernyshenko 2017) between the 'boundary layer' method of Otto & Seis (Seis 2015) and the background method (Doering & Constantin 1994). It is worthwhile illustrating this connection in the simpler context of the smooth-walled channel flow problem before giving the background velocity field corresponding to the Otto–Seis 'boundary layer' bounding analysis presented in Kerswell (2016). This background field has shears throughout the interior and so cannot ever satisfy the spectral constraint necessary to get a bound in the background approach. This, unfortunately, makes it clear that there is no simple fix of the flawed bound in Kerswell (2016).

We adopt Seis's (2015) notation (see his § 4) so that if

$$(NS) := \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \nabla^2 \boldsymbol{u} - Gr\hat{\boldsymbol{x}} \tag{2.1}$$

then $(NS) = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{u} = 0$ with $\boldsymbol{u}(x, y, 0) = \boldsymbol{u}(x, y, 1) = \boldsymbol{0}$ define the channel flow problem.

2.1. *The background method*

The background method is to construct the functional

$$\mathcal{L}[\boldsymbol{u}, \boldsymbol{v}] := \left\langle \int_0^1 \overline{|\nabla \boldsymbol{u}|^2} \, \mathrm{d}z \right\rangle - \alpha \left\langle \int_0^1 \overline{\boldsymbol{v} \cdot (NS)} \, \mathrm{d}z \right\rangle, \tag{2.2}$$

where α is a balance (scalar) parameter (usually 'a' in past work), $\boldsymbol{v}(\boldsymbol{x}, t)$ is a Lagrange multiplier field and

$$\overline{(\cdot)} := \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} (\cdot) \, \mathrm{d}y \, \mathrm{d}x \tag{2.3}$$

is an average over the rectangle $(x, y) \in [0, L_x] \times [0, L_y]$. The key step is to restrict the difference between $\boldsymbol{u}$ and $\boldsymbol{v}$ by defining a background field $\phi(z)$ such that

$$\boldsymbol{u}(\boldsymbol{x}, t) = \phi(z)\hat{\boldsymbol{x}} + \boldsymbol{v}(\boldsymbol{x}, t), \tag{2.4}$$

where ϕ carries the mass flux of the flow but vanishes at the boundaries. In particular, if the energy dissipation rate is sought in terms of the mean flow U rather than the imposed pressure gradient (Gr) then

$$U := \int_0^1 \overline{\boldsymbol{u} \cdot \hat{\boldsymbol{x}}} \, \mathrm{d}z = \int_0^1 \phi \, \mathrm{d}z \tag{2.5}$$

(see (2.29), Kerswell 2016). Rewriting $\mathcal{L}$ in terms of $\boldsymbol{v}$ and ϕ, and then integrating by parts, the boundedness of the kinetic energy and the fact that both ϕ and $\boldsymbol{v}$ vanish on $z = 0$ and 1 leads to the simplified expression

$$\mathcal{L}[\boldsymbol{v}, \phi] = \int_0^1 \phi'^2 \, \mathrm{d}z - \langle \mathcal{G}(\boldsymbol{v}; \phi, \alpha) \rangle, \tag{2.6}$$

where

$$\mathcal{G} := \int_0^1 (\alpha - 1)\overline{|\boldsymbol{\nabla} \boldsymbol{v}|^2} + \overline{\alpha v_1 v_3 \phi'} - (\alpha - 2)\overline{v_1 \phi''} \, \mathrm{d}z. \tag{2.7}$$

Then the background method (Doering & Constantin 1994) is the observation that

$$\mathcal{L} \leqslant \int_0^1 \phi'^2 \, \mathrm{d}z - \min_{\boldsymbol{v}} \mathcal{G}(\boldsymbol{v}; \phi, \alpha), \tag{2.8}$$

where only steady fields now need to be considered. The important point is $\min \mathcal{G}$ only exists for $\alpha > 1$ and ϕ which satisfy the spectral constraint (Doering & Constantin 1994). The best bound is found by then minimising the whole right-hand side over the (convex) set of such ϕ and α.

2.2. *The Otto–Seis 'boundary layer' method*

The starting point for the Otto–Seis 'boundary layer' approach is again (2.2) and the same decomposition $\boldsymbol{u} = \phi(z)\hat{\boldsymbol{x}} + \boldsymbol{v}$ is used. The key difference now is that $\mathcal{L}$ is re-expressed in terms of $\boldsymbol{u}$ and ϕ rather than $\boldsymbol{v}$ and ϕ. So

$$\begin{aligned}
\mathcal{L}[\boldsymbol{u}, \phi] &= \left\langle \int_0^1 \overline{|\boldsymbol{\nabla} \boldsymbol{u}|^2} \, \mathrm{d}z \right\rangle - \alpha \left\langle \int_0^1 \overline{(\boldsymbol{u} - \phi(z)\hat{\boldsymbol{x}}) \cdot (\boldsymbol{NS})} \, \mathrm{d}z \right\rangle \\
&= \left\langle \int_0^1 \overline{|\boldsymbol{\nabla} \boldsymbol{u}|^2} \, \mathrm{d}z \right\rangle - \alpha \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \overline{\frac{1}{2} \boldsymbol{u}^2} \, \mathrm{d}z + \int_0^1 \overline{|\boldsymbol{\nabla} \boldsymbol{u}|^2} \, \mathrm{d}z - GrU \right\rangle \\
&\quad - \alpha \left\langle \int_0^1 \phi'(\overline{uw} - \overline{u}_z) \, \mathrm{d}z \right\rangle - \alpha GrU \tag{2.9} \\
&= (1 - \alpha) \left\langle \int_0^1 \overline{|\boldsymbol{\nabla} \boldsymbol{u}|^2} \, \mathrm{d}z \right\rangle - \alpha \left\langle \int_0^1 \phi'(\overline{uw} - \overline{u}_z) \, \mathrm{d}z \right\rangle. \tag{2.10}
\end{aligned}$$

At this point, the Euler–Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}} = \boldsymbol{0} \quad \text{and} \quad \frac{\delta \mathcal{L}}{\delta \phi} = \frac{\delta \mathcal{L}}{\delta \alpha} = 0 \tag{2.11}$$

contain the background method bound as a solution but there is no means to identify it as such (i.e. appreciate that the associated value of $\mathcal{L}$ is a bound on the dissipation rate). Instead, the Otto–Seis approach appears to be to select a simplifying value of $\alpha = 1$ so that the right-hand side reduces to

$$\varepsilon = - \left\langle \int_0^1 \phi'(\overline{uw} - \overline{u}_z) \, \mathrm{d}z \right\rangle \tag{2.12}$$

and then to choose a simple trial function

$$\phi(z) := \frac{U}{1-\ell} \times \begin{cases} (1-z)/\ell & 1-\ell \leqslant z \leqslant 1 \\ 1 & \ell \leqslant z \leqslant 1-\ell \\ z/\ell & 0 \leqslant z \leqslant \ell \end{cases} \tag{2.13}$$

(designed so that $\int_0^1 \phi \, dz = U$ with boundary layers of size ℓ). This converts (2.12) into

$$(1-\ell)Gr = \frac{1}{\ell} \int_{1-\ell}^1 (\overline{uw} - \overline{u}_z) \, dz - \frac{1}{\ell} \int_0^\ell (\overline{uw} - \overline{u}_z) \, dz \tag{2.14}$$

(after using $\varepsilon = UGr$) which is (4.10) in Seis (2015) and then the strategy is to bound the terms on the right-hand side using powers of ε. The fundamental observation is that the derivative of the background field is what appears in the Otto–Seis 'boundary layer' method (Chernyshenko 2017).

2.3. *Background field for the rough problem*

In the rough channel flow problem, the background (vector) field corresponding to the 'boundary layer' method as applied in Kerswell (2016) is $\phi(\lambda)\boldsymbol{a}$ (generalised from $\phi(z)\hat{\boldsymbol{x}}$ in the smooth-walled problem) where

$$\lambda := \frac{z - f(x, y)}{g(x, y) - f(x, y)}, \quad \boldsymbol{a} := \frac{\hat{\boldsymbol{x}} + F_x(x, y, \lambda)\hat{\boldsymbol{z}}}{g(x, y) - f(x, y)}$$
$$\text{and} \quad F(x, y, \lambda) := (1-\lambda)f(x, y) + \lambda g(x, y).$$

For example

$$\iiint_{\mathcal{V}(\Lambda)} \phi(\lambda)\boldsymbol{a} \cdot (\boldsymbol{NS}) \, dV = \int_0^1 \phi(\lambda) \int_0^{L_y} \int_{\mathcal{C}(\lambda, y)} \hat{\boldsymbol{s}} \cdot (\boldsymbol{NS}) \, ds \, dy \, d\lambda, \tag{2.15}$$

where the right-hand side is (2.6) of Kerswell (2016) before integrating with a general weight $\phi(\lambda)$ over $\lambda \in [0, 1]$. Taking $\phi(\lambda)$ again as the piecewise-linear trial function defined in (2.13) allows the shears associated with it to be controllable. However, $\boldsymbol{a}$ varies spatially throughout the domain and so the shears associated with the combination do not vanish at some controlled distance from the boundary. This is what prevents the background method working and also has to break the boundary layer method. Unfortunately, this is a known limitation of the background method with no work-around currently on the horizon.

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