BOOK REVIEWS

BONSALL, F. F. AND DUNCAN, J., *Numerical Ranges II* (Cambridge University Press, 1973), vii + 179 pp., £2·40.

The notion of numerical range for operators on a Banach space or elements of a Banach algebra has been intensely studied during the last decade and much additional work was stimulated by the authors' first book on the subject. In the present volume the authors report on some of this work and begin with the improvement of the Bishop-Phelps theorem on support functionals due to Bollobás and on its application to the spatial numerical range. A result of Zenger that the spatial numerical range contains the convex hull of the point spectrum is also proved along with a related result of Crabb and Sinclair.

Algebra numerical ranges are considered in the next chapter with mapping theorems, inequalities, and extremal algebras for these inequalities being touched upon first. Results on hermitian elements of Banach algebras and spectral operators are also obtained along with a presentation of Harris' elementary proof that the closed convex hull of the unitary elements in a unital B^*-algebra is the set of contractive elements.

Lastly, the authors describe some work on essential numerical ranges, joint numerical ranges, and matrix-valued numerical ranges. The main thrust of most of this work concerns operators on Hilbert space. The authors conclude with some proposed axioms for the notion of numerical range for operators. The book is quite readable, despite the technical nature of some of the material, covers many interesting topics including many omitted in the above description, and is highly recommended to all mathematicians interested in operator theory or the theory of Banach algebras.

R. G. DOUGLAS

GREENSPAN, D., *Discrete Models* (Addison-Wesley, 1973), 181 pp., £8·80.

The subject matter of this book is perched, somewhat precariously perhaps, between the mathematical analysis demanded of the physical applied mathematician and the finite difference methods of the numerical analyst. It is unquestionably of interest to workers in both fields. As indicated by its title it is concerned with discrete mathematics. However, in spite of the physical fields from which it draws its examples, it most emphatically does not accept the conventional viewpoint that discretisation is basically an approximation, more or less accurate according to the care taken and efforts made, to some " exact " expression of physical laws in continuum language. By contrast, it argues right from the outset that, although the usual approach is to postulate a continuous model from discrete data and then (frequently) to approximate to the differential equations inherent in this model by first differences, the intermediate continuous model can in many cases be replaced by a discrete model and that this intermediate intrusion of continuum language is logically inconsistent. Accordingly, laws governing problems in particle, continuum and even relativistic mechanics are formulated in terms of their applicability at a finite number of space and time points.

Such an approach has both advantages and disadvantages. The advantages include the shrugging off of the concept of approximations and errors—what is laid down are discrete *laws* and within the axioms the mathematical development is self-contained and exact. Even the concept of stability is rigorously defined in Definition 1·7—rigorously within the axioms, that is—but the purist may wince at a formal definition of stability in which it is demanded that the dependent variable is nowhere " in absolute

https://doi.org/10.1017/S0013091500015613 Published online by Cambridge University Press

value larger than the largest number in one's computer ". Discrete analogues are provided in the early chapters for energy and momentum conservations for particle mechanics and in later chapters the same basic principles are used in order to model shock waves, heat diffusion, elastic phenomena and even turbulence in the mixing of fluids. At the very end of the book the author lists 23 further research problems in which he invites the reader to adapt and develop the methods of the text.

There are also disadvantages, some of them serious, in this formal reliance on the discrete approach. For one thing, the precise form of discretisation has to be carefully chosen, both for stability reasons and for comparison with equivalent " continuous " properties. As an example, the differential equation representing the simple harmonic oscillator is replaced on p. 29 by the relation $F_k = -\frac{1}{2}\omega^2 (X_{k+1}+X_k)$ where X_k is the position at time t_k and F_k is the force acting on the particle at that time. In this form a potential energy V_k can be defined such that K_k+V_k is the same at all time steps t_k, K_k being the kinetic energy of the particle at time t_k. But if the more natural choice of $F_k = \omega^2 X_k$ were made, no such equivalent of this energy conservation law exists. The author is clearly well aware of this problem and discusses the various alternative forms of discretisation in Chapter IX. It would seem, however, that such examples serve to emphasise the reliance indirectly placed on the continuous model whose formulation involves differential and not algebraic equations and thereby cast some doubt on the scientific merits of the assertion that discrete measurement of physical phenomena should logically lead directly to a discrete model formulation. Again, the attempt to model a gas as in Chapter VII by a finite number of particles subject to some mutually repulsive law and to calculate each particle's motion by the laws of Newtonian mechanics is undoubtedly interesting, but one is bound to note the limitations, even with the most powerful computer, and to wonder whether it contributes anything significant to one's understanding of the mathematical principles of gas dynamics.

The book is produced by a photo-offset process from an original typed manuscript. It has been carefully and accurately done (one or two minor errors were noted but not many) but with the range of symbols needed it is difficult to avoid the conclusion that in a book such as this there is no real substitute for the professional printer's armoury of type and general expertise. A small but irritating feature is the shape of the comma, which, when it occurs after a symbol with a subscript as it frequently does, makes it appear that the subscript has a prime on it. Only from p. 96 on, where genuinely primed subscripts appear for the first time, can one see the difference.

Despite these shortcomings, there is much thought-provoking material in this book which ought to tempt workers in various fields of mathematics, especially applied mathematics, to add it to their libraries. Not all will agree with the author but most will find his presentation interesting and refreshingly different from more conventional approaches.

A. G. MACKIE

LANG, SERGE, *Elliptic Functions*, (Addison-Wesley/W. A. Benjamin, Inc., 1974), xii+326 pp., $17.50.

Most recent books on elliptic functions (Tricomi, Neville, etc.) treat the subject from the point of view of complex analysis. Not so the book under review which explores ramifications in arithmetic, algebra, and algebraic geometry. Elliptic integrals and the problem of inversion do not fall within the scope of the book which concentrates on elliptic curves, their parametrisation, homomorphisms, isomorphisms, and other algebraic properties. Characteristically enough, there is no item in common in the list of books and monographs given in the present work, and the list of references in the National Bureau of Standards' " Handbook of mathematical functions ".

https://doi.org/10.1017/S0013091500015613 Published online by Cambridge University Press