

A RELATIONSHIP BETWEEN LEFT EXACT AND REPRESENTABLE FUNCTORS

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1. Introduction. Our aim in this paper is to demonstrate a relationship between left exact and representable functors. More precisely, in the functor category $\mathfrak{Ab}^{\mathfrak{A}^{\text{op}}}$ whose objects are the additive functors from the dual of an abelian category \mathfrak{A} to the category of abelian groups \mathfrak{Ab} and whose morphisms are the natural transformations between them, the left exact functors can be characterized as those equivalent to a direct limit of representable functors taken over a directed class. The proof will proceed in the following manner. Lambek [3] and Ulmer [7] have shown that any functor T in $\mathfrak{Ab}^{\mathfrak{A}^{\text{op}}}$ can be expressed as a direct limit of representable functors taken over a comma category. When T is left exact, it is easily observed that this comma category is a filtered category. We shall show that for any filtered category \mathfrak{D}_f there exists a directed class I and a cofinal functor $F: I \rightarrow \mathfrak{D}_f$. Our result then follows.

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2. Filtered categories and directed classes. Let us fix some terminology and prove a proposition which will be useful later on. Our aim will be to show that, given any filtered category (small filtered category) \mathfrak{D}_f , one can construct a directed class (directed set) I and a cofinal functor $F: I \rightarrow \mathfrak{D}_f$. But first, let us note the precise definitions involved.

A *filtered category* is a category \mathfrak{D}_f satisfying the following two axioms:

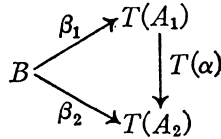
(i) given $D_1, D_2 \in |\mathfrak{D}_f|$, there exists $D_3 \in |\mathfrak{D}_f|$ and maps $\delta_1: D_1 \rightarrow D_3$ and $\delta_2: D_2 \rightarrow D_3$ in \mathfrak{D}_f , and,

(ii) given two maps $\delta_1, \delta_2: D_1 \rightarrow D_2$ in \mathfrak{D}_f , there exists a third map $\delta_3: D_2 \rightarrow D_3$ in \mathfrak{D}_f such that $\delta_3 \circ \delta_1 = \delta_3 \circ \delta_2$.

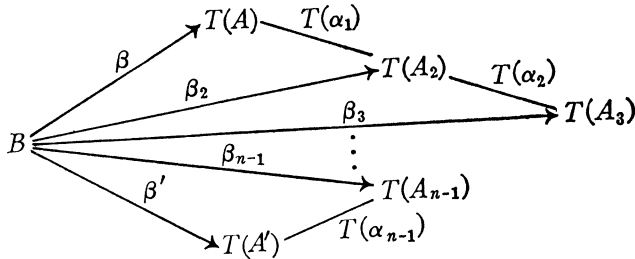
Given a directed set I , a subset $J \subseteq I$ is called *cofinal* if and only if for each $i \in I$ there is an element $j \in J$ such that $i \leq j$ (cf. [5, pp. 47-48]). We wish to generalize this notion to include filtered categories. But first, we need to define the *comma category* (B, T) , where $T: \mathfrak{A} \rightarrow \mathfrak{B}$ is a functor and $B \in |\mathfrak{B}|$ (cf. [4, pp. 13-14]). The objects of (B, T) are maps $\beta: B \rightarrow T(A)$ in \mathfrak{B} , $A \in |\mathfrak{A}|$. The morphisms of (B, T) from $\beta_1: B \rightarrow T(A_1)$ to $\beta_2: B \rightarrow T(A_2)$ are maps

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$\alpha: A_1 \rightarrow A_2$ in \mathfrak{A} which yield the commutative diagram:



Given any category \mathfrak{C} , we say that two objects C and C' are *connected* if and only if there exists a finite sequence of objects $C_1 = C, C_2, C_3, \dots, C_n = C'$ and maps $\gamma_i: C_i \rightarrow C_{i+1}$ or $C_{i+1} \rightarrow C_i, 1 \leq i \leq n - 1$, in \mathfrak{C} . A category \mathfrak{C} is a *connected category* if and only if any two objects in \mathfrak{C} are connected. Then objects $\beta: B \rightarrow T(A)$ and $\beta': B \rightarrow T(A')$ in the comma category (B, T) are connected if and only if there exist objects $A_1 = A, A_2, A_3, \dots, A_n = A'$ and maps $\alpha_i: A_i \rightarrow A_{i+1}$ or $A_{i+1} \rightarrow A_i, 1 \leq i \leq n - 1$, in \mathfrak{A} and maps $\beta_i: B \rightarrow T(A_i), 1 \leq i \leq n$, in \mathfrak{B} with $\beta_1 = \beta$ and $\beta_n = \beta'$ yielding the following commutative diagram:



Now, a functor $F: \mathfrak{X} \rightarrow \mathfrak{D}_f$, where \mathfrak{X} is any category and \mathfrak{D}_f is a filtered category, is called *cofinal* if and only if for each $D \in |\mathfrak{D}_f|$ the comma category (D, F) is non-empty and connected.

One final definition. Let us say that a directed class I is *pointwise finitely preceded* (pfp) if and only if, for each $i_0 \in I$, the subclass $\{i \mid i \leq i_0 \text{ in } I\}$ is finite.

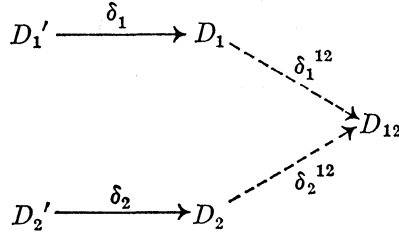
Now we are ready for the proposition.

PROPOSITION 2.1. *Let \mathfrak{D}_f be a filtered category. Then there exist a directed class I and a functor $F: I \rightarrow \mathfrak{D}_f$ which is cofinal. Furthermore, I is pfp. As a function, $F: I \rightarrow |\mathfrak{D}_f|$ is onto. If \mathfrak{D}_f is small, then I is a set.*

Proof. The objects of the category I are all finite non-empty sets of maps from \mathfrak{D}_f . The maps in I are the set inclusions. It is immediately clear that I is a pfp directed class. If \mathfrak{D}_f is small, then I is a directed set. It remains to construct the cofinal functor $F: I \rightarrow \mathfrak{D}_f$.

For $n > 0$, let \mathfrak{X}_n be the category whose objects are subsets $s \subseteq \{1, 2, \dots, n\}, s \neq \emptyset$, and whose morphisms are the set inclusions. We proceed to define F inductively. For each object $\{\delta\}$ in I consisting of one map $\delta: D' \rightarrow D$ in \mathfrak{D}_f

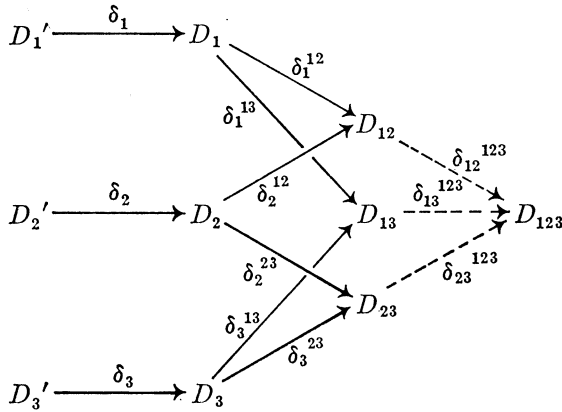
we define $F(\{\delta\}) = D$. For each object $\{\delta_1, \delta_2\}$ in I consisting of two maps, we use axiom (i) for a filtered category to fill in the diagram:



This yields a functor $D: \mathfrak{Q}_2 \rightarrow \mathfrak{D}_f$ given by

$$s \not\subseteq s' \quad D \rightsquigarrow_s \delta_s^{s'} \longrightarrow D_{s'}.$$

Define $F(\{\delta_1, \delta_2\}) = D_{12}$ and $F(\{\delta_i\} \subset \{\delta_1, \delta_2\}) = \delta_i^{12}$, $i = 1, 2$. Next, for each $\{\delta_1, \delta_2, \delta_3\}$ in I , take the previously designated objects and maps and use the two axioms for a filtered category to fill in the following commutative diagram:



This yields a functor $D: \mathfrak{Q}_3 \rightarrow \mathfrak{D}_f$. Define $F(\{\delta_1, \delta_2, \delta_3\}) = D_{123}$ and $F(\{\delta_i, \delta_j\} \subset \{\delta_1, \delta_2, \delta_3\}) = \delta_{ij}^{123}$, $1 \leq i < j \leq 3$. The other inclusions into $\{\delta_1, \delta_2, \delta_3\}$ can be handled by composing the obvious δ maps, a process which is well-defined since D is a functor.

Proceed inductively, defining F on each finite set of maps $\{\delta_1, \delta_2, \dots, \delta_i\}$ and its subset inclusions, $i = 1, 2, \dots, n - 1$. Now consider the object $\{\delta_1, \delta_2, \dots, \delta_n\}$ in I . Let $t = \{1, 2, \dots, n\}$ and $t_i = \{1, 2, \dots, \hat{i}, \dots, n\}$ in \mathfrak{Q}_n . Using the two axioms for a filtered category, construct a diagram consisting of the previously designated objects

$$D_i', \quad 1 \leq i \leq n, \quad D_s, \quad s \not\subseteq t,$$

and maps

$$\delta_i: D_i' \rightarrow D_i, \quad 1 \leq i \leq n, \quad \delta_s^{s'}, \quad s \not\subseteq s' \not\subseteq t,$$

plus object D_t and maps $\delta_{t_i}{}^t: D_{t_i} \rightarrow D_t, 1 \leq i \leq n$, such that:

- (i) the diagram is commutative (i.e. $D: \mathfrak{L}_n \rightarrow \mathfrak{D}_f$ defines a functor), and
- (ii) given $\delta_i: D_i' \rightarrow D_i$ and $\delta_j: D_j' \rightarrow D_j$ with $D_i' = D_j', 1 \leq i, j \leq n$, $\delta_s^t \circ \delta_i = \delta_{s'}^t \circ \delta_j$ whenever the compositions are defined (we want this condition to hold for $n \geq 4$).

Define $F(\{\delta_1, \delta_2, \dots, \delta_n\}) = D_t$ and

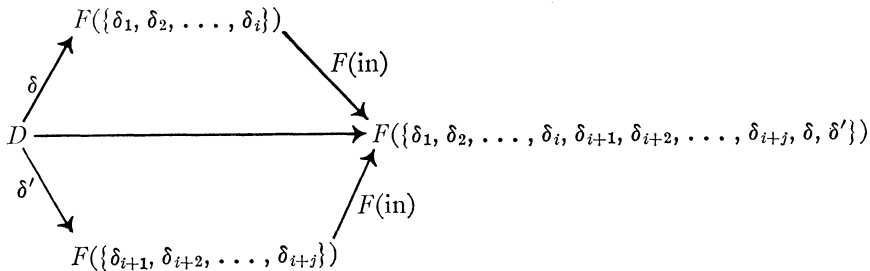
$$F(\{\delta_1, \delta_2, \dots, \hat{\delta}_i, \dots, \delta_n\} \subset \{\delta_1, \delta_2, \dots, \delta_n\}) = \delta_{t_i}{}^t.$$

Define F on the other inclusions into $\{\delta_1, \delta_2, \dots, \delta_n\}$ by taking compositions, a well-defined process since (i) holds. (ii) will ensure the cofinality of F .

It is clear that F is a functor. It remains to show that $F: I \rightarrow \mathfrak{D}_f$ is cofinal. For each $D \in |\mathfrak{D}_f|$, the comma category (D, F) is certainly non-empty, since $F: I \rightarrow |\mathfrak{D}_f|$ considered as a function is onto; i.e. $F(\{id_D\}) = D$ for every $D \in |\mathfrak{D}_f|$, where id_D is the identity map. We still need to show that (D, F) is connected for each $D \in |\mathfrak{D}_f|$. That is, we need to show that any two objects

$$\delta: D \rightarrow F(\{\delta_1, \delta_2, \dots, \delta_i\}) \quad \text{and} \quad \delta': D \rightarrow F(\{\delta_{i+1}, \delta_{i+2}, \dots, \delta_{i+j}\})$$

in (D, F) are connected. But F was constructed to satisfy the following commutative diagram:



where “in” are the inclusions. Hence, (D, F) is connected for each $D \in |\mathfrak{D}_f|$, and we have completed the proof of the proposition.

3. Left exact and representable functors. Let \mathfrak{A} be an abelian category. It is “well known” that every functor in \mathfrak{A}^{op} is a direct limit of representable functors. Let us sketch this result, referring the reader to [3; 7, pp. 79–82] for the details.

Let $J: \mathfrak{A} \rightarrow \mathfrak{A}'$ be a functor between abelian categories, and, for each $A' \in |\mathfrak{A}'|$, let (J, A') be the comma category. The objects of (J, A') are pairs (A, α') where $A \in |\mathfrak{A}|$ and $\alpha': J(A) \rightarrow A'$ is a map in \mathfrak{A}' . A map in (J, A') from (A_1, α'_1) to (A_2, α'_2) is a map $\alpha: A_1 \rightarrow A_2$ in \mathfrak{A} such that $\alpha'_2 \circ J(\alpha) = \alpha'_1$. There exists a forgetful functor $F_J(A'): (J, A') \rightarrow \mathfrak{A}$ defined by $(A, \alpha') \rightsquigarrow A$.

The functor $J: \mathfrak{A} \rightarrow \mathfrak{A}'$ is called *dense* if and only if, for each $A' \in |\mathfrak{A}'|$, the natural transformation $\phi_J(A'): J \circ F_J(A') \rightarrow \text{const}_{A'}$, the constant functor, defined by

$$\phi_J(A')[(A, \alpha')] = \alpha',$$

is universal. Then

$$\text{inj lim } J \circ F_J(A') = A'.$$

Let $Y_{\mathfrak{A}'}: \mathfrak{A}' \hookrightarrow \mathfrak{A}b^{(\mathfrak{A}')\text{op}}$ denote the Yoneda embedding defined by

$$A' \rightsquigarrow \text{Hom}_{\mathfrak{A}'}(_, A').$$

Then we have the following result.

LEMMA 3.1. *Let $J: \mathfrak{A} \rightarrow \mathfrak{A}'$ be a functor between abelian categories. J is dense if and only if the composite functor*

$$\mathfrak{A}b^J \circ Y_{\mathfrak{A}'}: \mathfrak{A}' \hookrightarrow \mathfrak{A}b^{(\mathfrak{A}')\text{op}} \rightarrow \mathfrak{A}b^{\mathfrak{A}\text{op}}$$

defined by

$$A' \rightarrow \text{Hom}_{\mathfrak{A}'}(J(_), A')$$

is full and faithful.

Using this we can conclude the following result.

PROPOSITION 3.2. *The Yoneda embedding $Y_{\mathfrak{A}}: \mathfrak{A} \hookrightarrow \mathfrak{A}b^{\mathfrak{A}\text{op}}$ is dense; i.e. each functor from \mathfrak{A}^{op} to $\mathfrak{A}b$ is a direct limit of representable functors.*

For, the Yoneda lemma implies that the composite functor $M: \mathfrak{A}b^{\mathfrak{A}\text{op}} \rightarrow \mathfrak{A}b^{\mathfrak{A}\text{op}}$ defined by

$$T \rightsquigarrow \text{Hom}_{\mathfrak{A}b}^{\mathfrak{A}\text{op}}(Y_{\mathfrak{A}}(_), T)$$

is full and faithful.

Now let us proceed to the main result of this paper. We have noted that each functor T in $\mathfrak{A}b^{\mathfrak{A}\text{op}}$ is a direct limit of representable functors:

$$T = \text{inj lim } Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(T),$$

where $Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(T): (Y_{\mathfrak{A}}, T) \rightarrow \mathfrak{A} \hookrightarrow \mathfrak{A}b^{\mathfrak{A}\text{op}}$ is the composite functor. Let us examine the comma category $(Y_{\mathfrak{A}}, T)$. Using the Yoneda lemma again, it is clear that the objects of $(Y_{\mathfrak{A}}, T)$ are the pairs (A, a) where $A \in |\mathfrak{A}|$ and $a \in T(A)$. A map in $(Y_{\mathfrak{A}}, T)$ from (A_1, a_1) to (A_2, a_2) is a map $\alpha: A_1 \rightarrow A_2$ such that $T(\alpha)[a_2] = a_1$.

LEMMA 3.3. *Let \mathfrak{A} be abelian. Then any left exact functor L in $\mathfrak{A}b^{\mathfrak{A}\text{op}}$ is a direct limit of representable functor over a filtered category. If \mathfrak{A} is small, then so is the filtered category.*

Proof. We have shown that

$$L = \text{inj lim } Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(L),$$

where the direct limit is taken over the comma category $(Y_{\mathfrak{A}}, L)$ described

above. We now show that L left exact implies $(Y_{\mathfrak{A}}, L)$ filtered. A left exact functor preserves biproducts (cf. [1, pp. 64–65]). Thus, given

$$(A_1, a_1), (A_2, a_2) \in |(Y_{\mathfrak{A}}, L)|,$$

the object $(A_1 \oplus A_2, a_1 \oplus a_2)$ is well-defined, since

$$a_1 \oplus a_2 \in L(A_1) \oplus L(A_2) = L(A_1 \oplus A_2).$$

Furthermore, we have the two maps $\iota^1: (A_1, a_1) \rightarrow (A_1 \oplus A_2, a_1 \oplus a_2)$ and $\iota^2: (A_2, a_2) \rightarrow (A_1 \oplus A_2, a_1 \oplus a_2)$ in $(Y_{\mathfrak{A}}, L)$ induced from the biproduct injections, since $L(\iota^1)[a_1 \oplus a_2] = \pi^1[a_1 \oplus a_2] = a_1$ and $L(\iota^2)[a_1 \oplus a_2] = \pi^2[a_1 \oplus a_2] = a_2$, where π^1 and π^2 are the biproduct projections. Hence axiom (i) for a filtered category is satisfied. Given maps

$$\alpha_1, \alpha_2: (A_1, a_1) \rightarrow (A_2, a_2)$$

in $(Y_{\mathfrak{A}}, L)$, let $\alpha: A_2 \rightarrow C$ be the coequalizer of $\alpha_1, \alpha_2: A_1 \rightarrow A_2$ in \mathfrak{A} . Then by the left exactness of L , the equalizer of $L(\alpha_1), L(\alpha_2): L(A_2) \rightarrow L(A_1)$ is $L(\alpha): L(C) \rightarrow L(A_2)$. Since $L(\alpha_1)[a_2] = L(\alpha_2)[a_2] = a_1$, (C, a_2) is well-defined as an object in $(Y_{\mathfrak{A}}, L)$. Furthermore, we have the commutative diagram

$$(A_1, a_1) \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} (A_2, a_2) \xrightarrow{\alpha} (C, a_2)$$

in $(Y_{\mathfrak{A}}, L)$. Thus axiom (ii) for a filtered category is satisfied. Note that if \mathfrak{A} is small, then $(Y_{\mathfrak{A}}, L)$ is also small.

The following lemma is “well known” (cf. [6, p. 225]); in fact, it has motivated the definition of cofinal. Its proof can be left to the reader.

LEMMA 3.4. *Let $F: \mathfrak{X} \rightarrow \mathfrak{D}_f$ be a cofinal functor from any category to a filtered category. Then, for any functor $G: \mathfrak{D}_f \rightarrow \mathfrak{A}$, where \mathfrak{A} is any category, we have*

$$\text{inj lim } G = \text{inj lim } G \circ F.$$

Now we are ready for the main result.

THEOREM 3.5. *Let \mathfrak{A} be an abelian category (a small abelian category). Then a functor L in $\mathfrak{Ab}^{\text{op}}$ is left exact if and only if L is a direct limit of representable functors over a directed class (directed set) I ; i.e.*

$$L = \text{inj lim}_I \text{Hom}_{\mathfrak{A}}(\quad, A.) = \text{inj lim}_I \text{Hom}_{\mathfrak{A}}(\quad, A_i).$$

Proof. The necessity follows from Lemma 3.3, Proposition 2.1, and Lemma 3.4. For let $F: I \rightarrow (Y_{\mathfrak{A}}, L)$ be a cofinal functor with I a directed class (directed set). Then $A. = F_{Y_{\mathfrak{A}}}(L) \circ F: I \rightarrow \mathfrak{A}$. The sufficiency follows, since a representable functor is left exact and the direct limit over a directed class (directed set) is exact in \mathfrak{Ab} .

Remark. Note that the directed class (directed set) I constructed in Proposition 2.1 is also a lattice class (lattice). Hence Theorem 3.5 could be restated in terms of lattice classes (lattices).

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