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A RELATIONSHIP BETWEEN LEFT EXACT AND REPRESENTABLE FUNCTORS

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1. Introduction. Our aim in this paper is to demonstrate a relationship between left exact and representable functors. More precisely, in the functor category $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ whose objects are the additive functors from the dual of an abelian category $\mathfrak{A}$ to the category of abelian groups $\mathfrak{Ab}$ and whose morphisms are the natural transformations between them, the left exact functors can be characterized as those equivalent to a direct limit of representable functors taken over a directed class. The proof will proceed in the following manner. Lambek [**3**] and Ulmer [**7**] have shown that any functor T in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ can be expressed as a direct limit of representable functors taken over a comma category. When T is left exact, it is easily observed that this comma category is a filtered category. We shall show that for any filtered category $\mathfrak{D}_f$ there exists a directed class I and a cofinal functor $F: I \to \mathfrak{D}_f$. Our result then follows.

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2. Filtered categories and directed classes. Let us fix some terminology and prove a proposition which will be useful later on. Our aim will be to show that, given any filtered category (small filtered category) $\mathfrak{D}_f$, one can construct a directed class (directed set) I and a cofinal functor $F: I \to \mathfrak{D}_f$. But first, let us note the precise definitions involved.

A *filtered category* is a category $\mathfrak{D}_f$ satisfying the following two axioms:

(i) given $D_1, D_2 \in |\mathfrak{D}_f|$, there exists $D_3 \in |\mathfrak{D}_f|$ and maps $\delta_1: D_1 \to D_3$ and $\delta_2: D_2 \to D_3$ in $\mathfrak{D}_f$, and,

(ii) given two maps $\delta_1, \delta_2: D_1 \to D_2$ in $\mathfrak{D}_f$, there exists a third map $\delta_3: D_2 \to D_3$ in $\mathfrak{D}_f$ such that $\delta_3 \circ \delta_1 = \delta_3 \circ \delta_2$.

Given a directed set I, a subset $J \subseteq I$ is called *cofinal* if and only if for each $i \in I$ there is an element $j \in J$ such that $i \leqq j$ (cf. [**5**, pp. 47–48]). We wish to generalize this notion to include filtered categories. But first, we need to define the *comma category* (B, T), where $T: \mathfrak{A} \to \mathfrak{B}$ is a functor and $B \in |\mathfrak{B}|$ (cf. [**4**, pp. 13–14]). The objects of (B, T) are maps $\beta: B \to T(A)$ in $\mathfrak{B}$, $A \in |\mathfrak{A}|$. The morphisms of (B, T) from $\beta_1: B \to T(A_1)$ to $\beta_2: B \to T(A_2)$ are maps

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$\alpha\colon A_1 \to A_2$ in $\mathfrak{A}$ which yield the commutative diagram:

$$\begin{tikzcd} & T(A_1) \arrow[dd, "T(\alpha)"] \\ B \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & \\ & T(A_2) \end{tikzcd}$$

Given any category $\mathfrak{C}$, we say that two objects C and C' are *connected* if and only if there exists a finite sequence of objects $C_1 = C, C_2, C_3, \ldots, C_n = C'$ and maps $\gamma_i\colon C_i \to C_{i+1}$ or $C_{i+1} \to C_i$, $1 \leqq i \leqq n-1$, in $\mathfrak{C}$. A category $\mathfrak{C}$ is a *connected category* if and only if any two objects in $\mathfrak{C}$ are connected. Then objects $\beta\colon B \to T(A)$ and $\beta'\colon B \to T(A')$ in the comma category (B, T) are connected if and only if there exist objects $A_1 = A, A_2, A_3, \ldots, A_n = A'$ and maps $\alpha_i\colon A_i \to A_{i+1}$ or $A_{i+1} \to A_i$, $1 \leqq i \leqq n-1$, in $\mathfrak{A}$ and maps $\beta_i\colon B \to T(A_i)$, $1 \leqq i \leqq n$, in $\mathfrak{B}$ with $\beta_1 = \beta$ and $\beta_n = \beta'$ yielding the following commutative diagram:

$$\begin{tikzcd} & T(A) \arrow[dr, "T(\alpha_1)"] & & \\ & & T(A_2) \arrow[dr, "T(\alpha_2)"] & \\ B \arrow[uur, "\beta"] \arrow[urr, "\beta_2"] \arrow[rrr, "\beta_3"] \arrow[ddrr, "\beta_{n-1}"] \arrow[dddr, "\beta'"'] & & & T(A_3) \\ & & \vdots & \\ & & T(A_{n-1}) & \\ & T(A') \arrow[ur, dash, "T(\alpha_{n-1})"'] & & \end{tikzcd}$$

Now, a functor $F\colon \mathfrak{X} \to \mathfrak{D}_f$, where $\mathfrak{X}$ is any category and $\mathfrak{D}_f$ is a filtered category, is called *cofinal* if and only if for each $D \in |\mathfrak{D}_f|$ the comma category (D, F) is non-empty and connected.

One final definition. Let us say that a directed class I is *pointwise finitely preceded* (pfp) if and only if, for each $i_0 \in I$, the subclass $\{i \mid i \leqq i_0 \text{ in } I\}$ is finite.

Now we are ready for the proposition.

PROPOSITION 2.1. *Let $\mathfrak{D}_f$ be a filtered category. Then there exist a directed class I and a functor $F\colon I \to \mathfrak{D}_f$ which is cofinal. Furthermore, I is pfp. As a function, $F\colon I \to |\mathfrak{D}_f|$ is onto. If $\mathfrak{D}_f$ is small, then I is a set.*

Proof. The objects of the category I are all finite non-empty sets of maps from $\mathfrak{D}_f$. The maps in I are the set inclusions. It is immediately clear that I is a pfp directed class. If $\mathfrak{D}_f$ is small, then I is a directed set. It remains to construct the cofinal functor $F\colon I \to \mathfrak{D}_f$.

For $n > 0$, let $\mathfrak{L}_n$ be the category whose objects are subsets $s \subseteq \{1, 2, \ldots, n\}$, $s \neq \emptyset$, and whose morphisms are the set inclusions. We proceed to define F inductively. For each object $\{\delta\}$ in I consisting of one map $\delta\colon D' \to D$ in $\mathfrak{D}_f$

we define $F(\{\delta\}) = D$. For each object $\{\delta_1, \delta_2\}$ in I consisting of two maps, we use axiom (i) for a filtered category to fill in the diagram:

$$\begin{tikzcd}
D_1' \arrow[r, "\delta_1"] & D_1 \arrow[dr, dashed, "\delta_1^{12}"] & \\
 & & D_{12} \\
D_2' \arrow[r, "\delta_2"] & D_2 \arrow[ur, dashed, "\delta_2^{12}"'] &
\end{tikzcd}$$

This yields a functor $D\colon \mathfrak{L}_2 \to \mathfrak{D}_f$ given by

$$s \not\subseteq s' \quad D_s \xrightarrow{\delta_s^{s'}} D_{s'}.$$

Define $F(\{\delta_1, \delta_2\}) = D_{12}$ and $F(\{\delta_i\} \subset \{\delta_1, \delta_2\}) = \delta_i^{12}$, $i = 1, 2$. Next, for each $\{\delta_1, \delta_2, \delta_3\}$ in I, take the previously designated objects and maps and use the two axioms for a filtered category to fill in the following commutative diagram:

$$\begin{tikzcd}
D_1' \arrow[r, "\delta_1"] & D_1 \arrow[dr, "\delta_1^{12}"] \arrow[ddr, "\delta_1^{13}"'] & & \\
 & & D_{12} \arrow[dr, dashed, "\delta_{12}^{123}"] & \\
D_2' \arrow[r, "\delta_2"] & D_2 \arrow[ur, "\delta_2^{12}"'] \arrow[dr, "\delta_2^{23}"'] & D_{13} \arrow[r, dashed, "\delta_{13}^{123}"] & D_{123} \\
 & & D_{23} \arrow[ur, dashed, "\delta_{23}^{123}"'] & \\
D_3' \arrow[r, "\delta_3"] & D_3 \arrow[uur, "\delta_3^{13}"] \arrow[ur, "\delta_3^{23}"'] & &
\end{tikzcd}$$

This yields a functor $D\colon \mathfrak{L}_3 \to \mathfrak{D}_f$. Define $F(\{\delta_1, \delta_2, \delta_3\}) = D_{123}$ and $F(\{\delta_i, \delta_j\} \subset \{\delta_1, \delta_2, \delta_3\}) = \delta_{ij}^{123}$, $1 \leqq i < j \leqq 3$. The other inclusions into $\{\delta_1, \delta_2, \delta_3\}$ can be handled by composing the obvious δ maps, a process which is well-defined since D is a functor.

Proceed inductively, defining F on each finite set of maps $\{\delta_1, \delta_2, \ldots, \delta_i\}$ and its subset inclusions, $i = 1, 2, \ldots, n - 1$. Now consider the object $\{\delta_1, \delta_2, \ldots, \delta_n\}$ in I. Let $t = \{1, 2, \ldots, n\}$ and $t_i = \{1, 2, \ldots, \hat{\imath}, \ldots, n\}$ in $\mathfrak{L}_n$. Using the two axioms for a filtered category, construct a diagram consisting of the previously designated objects

$$D_i', \quad 1 \leqq i \leqq n, \qquad D_s, \quad s \not\subseteq t,$$

and maps

$$\delta_i: D_i' \to D_i, \qquad 1 \leqq i \leqq n, \qquad \delta_s{}^{s'}, \qquad s \nsubseteq s' \nsubseteq t,$$

plus object D_t and maps $\delta_{ti}{}^t: D_{ti} \to D_t$, $1 \leqq i \leqq n$, such that:

(i) the diagram is commutative (i.e. $D: \mathfrak{L}_n \to \mathfrak{D}_f$ defines a functor), and

(ii) given $\delta_i: D_i' \to D_i$ and $\delta_j: D_j' \to D_j$ with $D_i' = D_j'$, $1 \leqq i, j \leqq n$, $\delta_s{}^t \circ \delta_i = \delta_{s'}{}^t \circ \delta_j$ whenever the compositions are defined (we want this condition to hold for $n \geqq 4$).

Define $F(\{\delta_1, \delta_2, \ldots, \delta_n\}) = D_t$ and

$$F(\{\delta_1, \delta_2, \ldots, \hat{\delta}_i, \ldots, \delta_n\} \subset \{\delta_1, \delta_2, \ldots, \delta_n\}) = \delta_{ti}{}^t.$$

Define F on the other inclusions into $\{\delta_1, \delta_2, \ldots, \delta_n\}$ by taking compositions, a well-defined process since (i) holds. (ii) will ensure the cofinality of F.

It is clear that F is a functor. It remains to show that $F: I \to \mathfrak{D}_f$ is cofinal. For each $D \in |\mathfrak{D}_f|$, the comma category (D, F) is certainly non-empty, since $F: I \to |\mathfrak{D}_f|$ considered as a function is onto; i.e. $F(\{\mathrm{id}_D\}) = D$ for every $D \in |\mathfrak{D}_f|$, where id_D is the identity map. We still need to show that (D, F) is connected for each $D \in |\mathfrak{D}_f|$. That is, we need to show that any two objects

$$\delta: D \to F(\{\delta_1, \delta_2, \ldots, \delta_i\}) \quad \text{and} \quad \delta': D \to F(\{\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{i+j}\})$$

in (D, F) are connected. But F was constructed to satisfy the following commutative diagram:

$$\begin{tikzcd}
& F(\{\delta_1, \delta_2, \ldots, \delta_i\}) \arrow[dr, "F(\mathrm{in})"] & \\
D \arrow[ur, "\delta"] \arrow[rr] \arrow[dr, "\delta'"'] & & F(\{\delta_1, \delta_2, \ldots, \delta_i, \delta_{i+1}, \delta_{i+2}, \ldots, \delta_{i+j}, \delta, \delta'\}) \\
& F(\{\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{i+j}\}) \arrow[ur, "F(\mathrm{in})"'] &
\end{tikzcd}$$

where "in" are the inclusions. Hence, (D, F) is connected for each $D \in |\mathfrak{D}_f|$, and we have completed the proof of the proposition.

3. Left exact and representable functors. Let $\mathfrak{A}$ be an abelian category. It is "well known" that every functor in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is a direct limit of representable functors. Let us sketch this result, referring the reader to [**3**; **7**, pp. 79–82] for the details.

Let $J: \mathfrak{A} \to \mathfrak{A}'$ be a functor between abelian categories, and, for each $A' \in |\mathfrak{A}'|$, let (J, A') be the comma category. The objects of (J, A') are pairs (A, α') where $A \in |\mathfrak{A}|$ and $\alpha': J(A) \to A'$ is a map in $\mathfrak{A}'$. A map in (J, A') from (A_1, α_1') to (A_2, α_2') is a map $\alpha: A_1 \to A_2$ in $\mathfrak{A}$ such that $\alpha_2' \circ J(\alpha) = \alpha_1'$. There exists a forgetful functor $F_J(A'): (J, A') \to \mathfrak{A}$ defined by $(A, \alpha') \rightsquigarrow A$.

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The functor $J: \mathfrak{A} \to \mathfrak{A}'$ is called *dense* if and only if, for each $A' \in |\mathfrak{A}'|$, the natural transformation $\phi_J(A'): J \circ F_J(A') \to \mathrm{const}_{A'}$, the constant functor, defined by

$$\phi_J(A')[(A, \alpha')] = \alpha',$$

is universal. Then

$$\mathrm{inj\,lim}\ J \circ F_J(A') = A'.$$

Let $Y_{\mathfrak{A}'}: \mathfrak{A}' \hookrightarrow \mathfrak{Ab}^{(\mathfrak{A}')^{\mathrm{op}}}$ denote the Yoneda embedding defined by

$$A' \rightsquigarrow \mathrm{Hom}_{\mathfrak{A}'}(\quad, A').$$

Then we have the following result.

LEMMA 3.1. *Let $J: \mathfrak{A} \to \mathfrak{A}'$ be a functor between abelian categories. J is dense if and only if the composite functor*

$$\mathfrak{Ab}^J \circ Y_{\mathfrak{A}'}: \mathfrak{A}' \hookrightarrow \mathfrak{Ab}^{(\mathfrak{A}')^{\mathrm{op}}} \to \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$$

defined by

$$A' \to \mathrm{Hom}_{\mathfrak{A}'}(J(\quad), A')$$

is full and faithful.

Using this we can conclude the following result.

PROPOSITION 3.2. *The Yoneda embedding $Y_{\mathfrak{A}}: \mathfrak{A} \hookrightarrow \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is dense; i.e. each functor from $\mathfrak{A}^{\mathrm{op}}$ to $\mathfrak{Ab}$ is a direct limit of representable functors.*

For, the Yoneda lemma implies that the composite functor $M: \mathfrak{Ab}^{\mathfrak{A}_{\mathrm{do}}} \to \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ defined by

$$T \rightsquigarrow \mathrm{Hom}_{\mathfrak{Ab}}{}^{\mathfrak{A}^{\mathrm{op}}}(Y_{\mathfrak{A}}(\quad), T)$$

is full and faithful.

Now let us proceed to the main result of this paper. We have noted that each functor T in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is a direct limit of representable functors:

$$T = \mathrm{inj\,lim}\ Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(T),$$

where $Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(T): (Y_{\mathfrak{A}}, T) \to \mathfrak{A} \hookrightarrow \mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is the composite functor. Let us examine the comma category $(Y_{\mathfrak{A}}, T)$. Using the Yoneda lemma again, it is clear that the objects of $(Y_{\mathfrak{A}}, T)$ are the pairs (A, a) where $A \in |\mathfrak{A}|$ and $a \in T(A)$. A map in $(Y_{\mathfrak{A}}, T)$ from (A_1, a_1) to (A_2, a_2) is a map $\alpha: A_1 \to A_2$ such that $T(\alpha)[a_2] = a_1$.

LEMMA 3.3. *Let $\mathfrak{A}$ be abelian. Then any left exact functor L in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is a direct limit of representable functor over a filtered category. If $\mathfrak{A}$ is small, then so is the filtered category.*

Proof. We have shown that

$$L = \mathrm{inj\,lim}\ Y_{\mathfrak{A}} \circ F_{Y_{\mathfrak{A}}}(L),$$

where the direct limit is taken over the comma category $(Y_{\mathfrak{A}}, L)$ described above. We now show that L left exact implies $(Y_{\mathfrak{A}}, L)$ filtered. A left exact functor preserves biproducts (cf. [**1**, pp. 64–65]). Thus, given

$$(A_1, a_1),\ (A_2, a_2) \in |(Y_{\mathfrak{A}}, L)|,$$

the object $(A_1 \oplus A_2, a_1 \oplus a_2)$ is well-defined, since

$$a_1 \oplus a_2 \in L(A_1) \oplus L(A_2) = L(A_1 \oplus A_2).$$

Furthermore, we have the two maps ι^1: $(A_1, a_1) \to (A_1 \oplus A_2, a_1 \oplus a_2)$ and ι^2: $(A_2, a_2) \to (A_1 \oplus A_2, a_1 \oplus a_2)$ in $(Y_{\mathfrak{A}}, L)$ induced from the biproduct injections, since $L(\iota^1)[a_1 \oplus a_2] = \pi^1[a_1 \oplus a_2] = a_1$ and $L(\iota^2)[a_1 \oplus a_2] = \pi^2[a_1 \oplus a_2] = a_2$, where π^1 and π^2 are the biproduct projections. Hence axiom (i) for a filtered category is satisfied. Given maps

$$\alpha_1, \alpha_2\colon\ (A_1, a_1) \to (A_2, a_2)$$

in $(Y_{\mathfrak{A}}, L)$, let α: $A_2 \to C$ be the coequalizer of α_1, α_2: $A_1 \to A_2$ in $\mathfrak{A}$. Then by the left exactness of L, the equalizer of $L(\alpha_1), L(\alpha_2)$: $L(A_2) \to L(A_1)$ is $L(\alpha)$: $L(C) \to L(A_2)$. Since $L(\alpha_1)[a_2] = L(\alpha_2)[a_2] = a_1$, (C, a_2) is well-defined as an object in $(Y_{\mathfrak{A}}, L)$. Furthermore, we have the commutative diagram

$$(A_1, a_1) \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} (A_2, a_2) \xrightarrow{\alpha} (C, a_2)$$

in $(Y_{\mathfrak{A}}, L)$. Thus axiom (ii) for a filtered category is satisfied. Note that if $\mathfrak{A}$ is small, then $(Y_{\mathfrak{A}}, L)$ is also small.

The following lemma is "well known" (cf. [**6**, p. 225]); in fact, it has motivated the definition of cofinal. Its proof can be left to the reader.

LEMMA 3.4. *Let F: $\mathfrak{X} \to \mathfrak{D}_f$ be a cofinal functor from any category to a filtered category. Then, for any functor G: $\mathfrak{D}_f \to \mathfrak{A}$, where $\mathfrak{A}$ is any category, we have*

$$\operatorname{inj\,lim} G = \operatorname{inj\,lim} G \circ F.$$

Now we are ready for the main result.

THEOREM 3.5. *Let $\mathfrak{A}$ be an abelian category (a small abelian category). Then a functor L in $\mathfrak{Ab}^{\mathfrak{A}^{\mathrm{op}}}$ is left exact if and only if L is a direct limit of representable functors over a directed class (directed set) I; i.e.*

$$L = \operatorname*{inj\,lim}_I \operatorname{Hom}_{\mathfrak{A}}(\ \ , A.) = \operatorname*{inj\,lim}_I \operatorname{Hom}_{\mathfrak{A}}(\ \ , A_i).$$

Proof. The necessity follows from Lemma 3.3, Proposition 2.1, and Lemma 3.4. For let F: $I \to (Y_{\mathfrak{A}}, L)$ be a cofinal functor with I a directed class (directed set). Then $A. = F_{Y_{\mathfrak{A}}}(L) \circ F$: $I \to \mathfrak{A}$. The sufficiency follows, since a representable functor is left exact and the direct limit over a directed class (directed set) is exact in $\mathfrak{Ab}$.

Remark. Note that the directed class (directed set) I constructed in Proposition 2.1 is also a lattice class (lattice). Hence Theorem 3.5 could be restated in terms of lattice classes (lattices).

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