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GENERALIZED SOLUTIONS OF AUTONOMOUS ALGEBRAIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In an earlier paper, the author introduced the notions of N-solutions and IN-solutions of algebraic differential equations (ADE's). Here it is shown, in contradistinction to the situation for C^n solutions, that every N-solution of an ADE is an N-solution of an autonomous ADE. The corresponding result also holds for IN-solutions.

By an ADE, we mean an equation of the form

$$P(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0, \tag{1}$$

where P is a polynomial, with complex coefficients, in its $n + 2$ variables. We often write

$$P = P(y_{-1}, y_0, y_1, \ldots, y_n), \tag{2}$$

and think of y_{-1} as the independent variable x, and $y_{i+1} = (d/dx)y_i$ for $i = 0, 1, \ldots, n$. In this way, P becomes a differential polynomial.

It is a well-known fact that if $u(x)$ is an analytic function that satisfies an ADE, then it satisfies some *autonomous* ADE, i.e. an equation (1) where P is independent of $y_{-1}(= x)$.

In [2], it was shown that the ADE

$$y'^2 = x^2 4y(1 - y) \tag{3}$$

has a C^1 solution $y = u(x)$ that does *not* satisfy any autonomous ADE. In the present paper, we show that this bad situation is rectified if one considers *generalized* solutions in the sense of [3]. We recall the definitions. First, u is called an N-solution of (1) if there is a set Ω of differential polynomials such that u is an actual solution of the system $\Omega = 0$ of ADE's and if $P \in \rho(\Omega)$, where $\rho(\Omega)$ denotes the radial differential ideal (see [1]) generated by Ω. Since we only briefly mention IN-solutions u (which could be *distributions*), we refer the reader to [3] for details about them. The idea is that some iterated integral of u should be an N-solution of a modification of (1). (Here "N" stands

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for "Nullstellensatz" and "IN" for "Integrated Nullstellensatz".) With minor changes, our results carry over to algebraic *partial* differential equations. For simplicity, we take here the domain of the independent variable to be a compact interval I in $\mathbb{R}$.

THEOREM. *If u is an N-solution of an algebraic differential equation, then it is an N-solution of an* autonomous *algebraic differential equation. The corresponding result holds also for IN-solutions.*

PROOF. *Let u* be an N-solution of (1), where we choose P to be of the smallest possible order, and of the lowest degree within that order. Let $\mathrm{d}P/\mathrm{d}x$ denote the derivative of P as a differential polynomial, i.e.

$$\frac{\mathrm{d}P}{\mathrm{d}x}(y_{-1}, y_0, \ldots, y_n, y_{n+1}) = P_{-1} + P_0 \cdot y_1 + P_1 \cdot y_2 + \cdots + P_n \cdot y_{n+1}, \tag{4}$$

where $P_j = \partial P/\partial y_j$ for $j = -1, 0, 1, \ldots, n$. In this way, for any $v(x) \in C^{\infty}(I)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} P(x, v(x), v'(x), \ldots, v^{(n)}(x)) = \frac{\mathrm{d}P}{\mathrm{d}x}(x, v(x), v'(x), \ldots, v^{(n+1)}(x)). \tag{5}$$

Now let $R = R(y_0, y_1, \ldots, y_{n+1})$ be the resultant on x of P and $\mathrm{d}P/\mathrm{d}x$

$$R = \mathrm{Res}_x\left[P, \frac{\mathrm{d}P}{\mathrm{d}x}\right]. \tag{6}$$

It is easy to show that u is an N-solution of the autonomous ADE, $R = 0$, but we shall first show that R is not the zero polynomial. (See the first few pages of [4: Part II] for the facts we shall use about resultants.) Here, we are thinking of our differential polynomials as lying in $\mathbb{C}(y_0, y_1, \ldots, y_n)[x]$, i.e. as polynomials in x whose coefficients are rational functions of $y_0, y_1, \ldots, y_n$. In this context, a "constant" is such a polynomial that is independent of x.

We claim that R is not the zero polynomial. If it were, then P and $\mathrm{d}P/\mathrm{d}x$ would have to have a common nonconstant factor, say Q. Extracting as many factors of Q as possible, we would have

$$P = Q^s S, \ \frac{\mathrm{d}P}{\mathrm{d}x} = Q^t T; \qquad s, t \geq 1, \tag{7}$$

where $(Q, S) = (Q, T) = 1$. Now if s were to equal 1, we would have

$$\frac{\mathrm{d}P}{\mathrm{d}x} = S\frac{\mathrm{d}Q}{\mathrm{d}x} + Q\frac{\mathrm{d}S}{\mathrm{d}x} = Q^t T, \tag{8}$$

from which it would follow that $\mathrm{d}Q/\mathrm{d}x$ would be divisible by Q. This is impossible (since Q is not a constant) as the following argument shows.

For then, by (5), for every *polynomial* $v(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} Q(x, v(x), v'(x), \ldots, v^{(n+1)}(x))$$

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would be divisible by

$$Q(x, v(x), v'(x), \ldots, v^{(n+1)}(x)).$$

But this last term is a polynomial in one complex variable, and hence must be a constant. But by results of elementary interpolation theory, there is, for every $x_0 \in \mathbb{C}(x_0 \neq 0)$ and every $(n + 1)$-tuple $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n+2}$, a polynomial $v(x)$ (of degree $\leq 2n + 2$) such that

$$v^{(j)}(0) = 0 \text{ and } v^{(j)}(x_0) = z_j \qquad \text{for } j = 0, 1, \ldots, n + 1. \tag{9}$$

We then would have

$$Q(x_0, z_0, z_1, \ldots, z_{n+1}) = Q(0, 0, \ldots, 0), \tag{10}$$

so that Q would actually be a constant in $\mathbb{C}$, contrary to fact.

Thus, we would have to have $s > 1$. Now, letting Ω be the set of differential polynomials that make u be an N-solution of (1), we would have $P = Q^s S \in \rho(\Omega)$, and thus $QS \in \rho(\Omega)$, since $(QS)^s = S^{s-1}(Q^s S) \in \rho(\Omega)$ since $\rho(\Omega)$ is a *radical* differential ideal. But then u would be an N-solution of $QS = 0$, which is impossible since QS is of the same order as P, but of lower degree than P.

Thus, we have established that R is not the trivial zero polynomial. By the nature of the resultant, R is free of x—i.e. $R = 0$ is an autonomous ADE. Now R belongs to the algebraic ideal generated by P and $\mathrm{d}P/\mathrm{d}x$. A fortiori, R belongs to $\rho(P)$. Hence $R \in \rho(\Omega)$, and thus u is an N-solution of the autonomous ADE, $R = 0$. The corresponding result for IN-solutions follows from the same proof, with a little extra attention to the orders of the derivatives involved. We omit the details.

References

1. Irving Kaplansky, *An Introduction to Differential Algebra*, Paris 1957 (2nd ed. 1976).
2. Lee A. Rubel, *Solutions of algebraic differential equations*, J. Diff. Eq. **49** (1983), pp. 441–452.
3. ———, *Generalized solutions of algebraic differential equations*, J. Diff. Eq., to appear.
4. B. L. van der Waerden, *Moderne Algebra*, Berlin 1931.

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