ON MAXIMAL RINGS OF RIGHT QUOTIENTS

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Utumi has shown [3, Claim 5.1] that for a certain class of rings the associated maximal rings of right quotients are isomorphic to the endomorphism rings of modules over division rings. We shall prove a generalization of this theorem and then show how it is obtained as a corollary. The following proofs do not depend on Utumi's paper; instead, they make extensive use of results proved in [1]. The terminology and notations employed here are the same as in [1].

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LEMMA: If J is a left ideal with zero left annihilator in a ring R then a maximal ring of right quotients of R is also a maximal ring of right quotients of J.

Proof. Let Q be a maximal ring of right quotients of R. We must show that $J \leq Q(Q_J)$ and Q is maximal such. Following the remarks in 1.2 of [2], take $0 \neq q$, $q' \in Q$; then since $R \leq Q(Q_R)$ there exists an $r \in R$ and an integer h with $qr + hq \neq 0$ and $q'r + hq' \in R$. We may assume that $qr + hq \in R$. Since J is a left ideal with zero left annihilator in R, there exists a $j \in J$ with $(qr + hq)j = q(rj + hj) \neq 0$ and $(q'r + hq')j = q'(rj + hj) \in J$. Now $rj + hj \in J$, hence we have proved $J \leq Q(Q_J)$. Furthermore, if Q' is a maximal ring of right quotients of J, we may assume $Q \subseteq Q'$; since $J \subseteq R \subseteq Q'$ and $J \leq Q'(Q'_J)$ we have $R \leq Q'(Q'_J)$ and it follows trivially that $R \leq Q'(Q'_R)$. Thus $Q = Q'$ and Q is a maximal ring of right quotients of J.

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PROPOSITION: If J is an idempotent left ideal with zero left annihilator in a ring R and Q a maximal ring of right quotients of R then $Q \cong \mathrm{Hom}_J(QJ, QJ)$, where QJ, the ideal generated by J in Q, is taken as a right J-module.

Proof. We first prove

$$Q \cong \mathrm{Fr}_J(J, J) \cong \mathrm{Fr}_J(J, Q) \cong \mathrm{Fr}_J(QJ, Q) = \mathrm{Hom}_J(QJ, Q) .$$

The first isomorphism follows from [1, Prop. 6.1] and the second from [1, Prop. 3.4], both in view of the Lemma.

Now, $J \subseteq QJ \subseteq Q$ by $J = J^2$, so $J \leq Q(Q_J)$ implies $J \leq QJ(Q_J)$ which leads to the third isomorphism by [1, Prop. 3.3]. The rational completeness of Q as a right J-module gives the last step. It remains to prove that $\mathrm{Hom}_J(QJ, Q) = \mathrm{Hom}_J(QJ, QJ)$. Take $\varphi \in \mathrm{Hom}_J(QJ, Q)$ and $x \in QJ$; then $x = q_1 j_1 + \ldots + q_n j_n$ with $q_i \in Q$, $j_i \in J$ for $i = 1, \ldots, n$. Since $J^2 = J$, we can write each j_i as a sum $\Sigma l_{ik} l'_{ik}$ with $l_{ik}, l'_{ik} \in J$. Then $\varphi(x) = \varphi(q_1 j_1) + \ldots + \varphi(q_n j_n)$ where for each i, $\varphi(q_i j_i) = \varphi(q_i \Sigma l_{ik} l'_{ik}) = \Sigma \varphi(q_i l_{ik}) l'_{ik} \in QJ$. Hence $\varphi \in \mathrm{Hom}_J(QJ, QJ)$ and we have equality since clearly $\mathrm{Hom}_J(QJ, QJ) \subseteq \mathrm{Hom}_J(QJ, Q)$. Thus $Q \cong \mathrm{Hom}_J(QJ, QJ)$.

COROLLARY: If R is a prime ring with a minimal left or right ideal, then the maximal ring of right quotients of R is isomorphic to the endomorphism ring of a right module over a division ring.

Proof. Let J_o be a minimal left (right) ideal, then $J_o = Re$ (resp. $= eR$) for some $e = e^2 \in R$. In either case, let $J = Re$. Then $(Re)^2 = Re$ and Re has zero left annihilator in R since R is a prime ring. Hence, if Q is a maximal ring of right quotients of R, then by the Proposition, $Q \cong \mathrm{Hom}_{Re}(Qe, Qe)$ since $QRe = Qe$. We claim

$\mathrm{Hom}_{Re}(Qe, Qe) = \mathrm{Hom}_{eRe}(Qe, Qe)$. Clearly $\mathrm{Hom}_{Re}(Qe, Qe) \subseteq \mathrm{Hom}_{eRe}(Qe, Qe)$ and for any $\varphi \in \mathrm{Hom}_{eRe}(Qe, Qe)$ and any $q = qe \in Qe$, $\varphi(qre) = \varphi(qere) = \varphi(q)ere = \varphi(q)re$, showing that $\varphi \in \mathrm{Hom}_{Re}(Qe, Qe)$. Since either Re is a minimal left ideal or eR is a minimal right ideal, eRe is a division ring, and our proof is complete.

Remark: We can show $Re = Qe$ as a consequence of $Re \leq Qe(Q_{Re})$ and the fact that eRe is a division ring. Therefore also $Q \cong \mathrm{Hom}_{eRe}(Re, Re)$.

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