Canad. J. Math. Vol. **55** (6), 2003 pp. 1231–1263

Admissible Majorants for Model Subspaces of H^2, Part I: Slow Winding of the Generating Inner Function

Victor Havin and Javad Mashreghi

Abstract. A model subspace K_Θ of the Hardy space $H^2 = H^2(\mathbb{C}_+)$ for the upper half plane $\mathbb{C}_+$ is $H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$ where Θ is an inner function in $\mathbb{C}_+$. A function $\omega\colon \mathbb{R} \mapsto [0, \infty)$ is called *an admissible majorant* for K_Θ if there exists an $f \in K_\Theta$, $f \not\equiv 0$, $|f(x)| \le \omega(x)$ almost everywhere on $\mathbb{R}$. For some (mainly meromorphic) Θ's some parts of $\operatorname{Adm}\Theta$ (the set of all admissible majorants for K_Θ) are explicitly described. These descriptions depend on the rate of growth of $\arg\Theta$ along $\mathbb{R}$. This paper is about slowly growing arguments (slower than x). Our results exhibit the dependence of $\operatorname{Adm} B$ on the geometry of the zeros of the Blaschke product B. A complete description of $\operatorname{Adm} B$ is obtained for B's with purely imaginary ("vertical") zeros. We show that in this case a unique minimal admissible majorant exists.

1 Introduction

1.1 Historical Background

Let Θ be an inner function in the upper half plane $\mathbb{C}_+$. The *model subspace* K_Θ of the Hardy space $H^2(\mathbb{C}_+)$ generated by Θ is, by definition, the orthogonal complement of $\Theta H^2(\mathbb{C}_+)$:

$$K_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+).$$

By Beurling's famous theorem the spaces $\Theta H^2(\mathbb{C}_+)$ are the only shift invariant closed subspaces of $H^2(\mathbb{C}_+)$, *i.e.*, invariant with respect to multiplication by any exponential $e^{i\sigma z}$, $\sigma > 0$ [4]. This is why K_Θ is often called a shift *coinvariant subspace* of $H^2(\mathbb{C}_+)$. We prefer the shorter term model subspace which appeared due to connections of K_Θ's with the Nagy-Foiaş model of contractions in a Hilbert space [28], [29].

The model subspaces are an important theme of complex and harmonic analysis. Their properties and numerous connections with various topics in analysis can be found, *e.g.*, in the work of Douglas, Shapiro, Shields [11], Cohn [9], Dyakonov [13], Volberg [32], Treil [33], Nikolski [29], Ahern, Clark [1], Alexandrov [2], and in the monograph of Cima, Ross [8]. The spaces K_Θ generated by meromorphic Θ are

Received by the editors May 14, 2002; revised November 1, 2002.
This work was supported by NSERC (Canada) and FCAR (Québec).
AMS subject classification: Primary: 30D55; secondary: 47A15.
Keywords: Hardy space, inner function, shift operator, model subspace, Hilbert transform, admissible majorant.
©Canadian Mathematical Society 2003.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

closely related to the de Branges Hilbert spaces of entire functions [6]. A very particular but extremely important case is $K_{e^{i\sigma z}}$, since $e^{-i\sigma z/2}K_{e^{i\sigma z}}$ is the Paley-Wiener space of entire functions of type at most $\sigma/2$ and square summable along the real line $\mathbb{R}$.

We call a measurable non-negative function $\omega\colon \mathbb{R} \mapsto [0,\infty)$ an *admissible majorant* for K_Θ, and we write $\omega \in \operatorname{Adm}\Theta$, if there exists a non-zero function $f \in K_\Theta$ satisfying

$$|f(x)| \leq \omega(x) \tag{1.1}$$

almost everywhere on $\mathbb{R}$. Here $f(x)$ denotes $\lim_{\varepsilon\to 0^+} f(x+i\varepsilon)$ wherever the limit exists. It is well known that $f(x)$ is defined almost everywhere on $\mathbb{R}$ and $f \in L^2(\mathbb{R})$ [22, page 114]. The subspace of $L^2(\mathbb{R})$ formed by all boundary traces of elements of $H^2(\mathbb{C}_+)$ is isometric to $H^2(\mathbb{C}_+)$ and is denoted by $H^2(\mathbb{R})$ [12, pages 190–191].

Our aim is to describe some classes of admissible majorants for some classes of model subspaces. A necessary condition for an ω to be in $\operatorname{Adm}\Theta$ is the convergence of its logarithmic integral

$$\mathcal{L}(\omega) = \int_{-\infty}^{\infty} \frac{\Omega^+(x)}{1+x^2}\,dx, \tag{1.2}$$

where

$$\Omega(x) = -\log\omega(x). \tag{1.3}$$

If $\mathcal{L}(\omega) = \infty$, then the only $f \in H^2(\mathbb{R})$ satisfying (1.1) is zero [12, pages 189–190]. The convergence of $\mathcal{L}(\omega)$ is also sufficient for the existence of a non-zero $f \in H^2(\mathbb{R})$ satisfying (1.1), or even $|f(x)| = \omega(x)$ a.e., provided $\omega \in L^2(\mathbb{R})$ [22, page 120]. But functions in a K_Θ are *much more analytic* than an average element of $H^2(\mathbb{R})$. Namely, the elements of K_Θ admit pseudo-analytic (or rather pseudo-meromorphic) continuations to the lower half plane $\mathbb{C}_-$: for any $f \in K_\Theta$ there exists a function g, meromorphic and of the Nevanlinna class in $\mathbb{C}_-$, such that

$$\lim_{\varepsilon\to 0^+} g(x-i\varepsilon) = f(x)$$

almost everywhere on $\mathbb{R}$. If Θ is analytic on an interval $I \subset \mathbb{R}$, then g is the classical analytic continuation of f across I. So it is natural to expect that the mere convergence of $\mathcal{L}(\omega)$ is too weak to ensure the inclusion $\omega \in \operatorname{Adm}\Theta$. It may happen that for a *nice* ω (say, decreasing on $[0,\infty)$, even and smooth) the integral $\mathcal{L}(\omega)$ is finite, but the decrease of ω is still too fast to let ω be in $\operatorname{Adm}\Theta$. We provide two examples. The first one is quite simple.

Example 1.1 Θ is a finite Blaschke product with zeros $z_1,\dots,z_n \in \mathbb{C}_+$. Then K_Θ is the set of all rational functions P/Q where $Q(z) = (z-z_1)\cdots(z-z_n)$ and P is a polynomial of degree at most $n-1$. In this case

$$\frac{1}{(1+|x|)^{n+1}} \notin \operatorname{Adm}\Theta$$

although $\mathcal{L}(\omega) < \infty$.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Note that $(1+|x|)^{-n} \in \operatorname{Adm}\Theta$, and this majorant is *sharp* in the sense that if $\omega(x) = o(|x|^{-n})$ as $|x| \to \infty$, then $\omega \notin \operatorname{Adm}\Theta$. The following example is much more interesting and delicate.

Example 1.2 $\Theta(z) = e^{i\sigma z}$, $\sigma > 0$. The admissibility of ω means that (1.1) holds for a non-zero Paley-Wiener function f (entire, of type at most $\sigma/2$ and square summable along $\mathbb{R}$). In this case any nice ω with $\mathcal{L}(\omega) < \infty$ is in $\operatorname{Adm}\Theta$, but the regularity assumptions cannot be dropped. *(A concrete form of regularity of ω entailing its admissibility is stated at the end of this subsection.)*

The characterization of $\operatorname{Adm} e^{i\sigma z}$ is a difficult problem related to the uncertainty principle in harmonic analysis (see Chapter 3 of Part II of [17]). A complete and palpable description of $\operatorname{Adm} e^{i\sigma z}$ is hardly possible, but Beurling and Malliavin found sufficient conditions for an ω to be in that class (actually in $\cap_{\sigma>0} \operatorname{Adm} e^{i\sigma z}$). The theorem of Beurling and Malliavin (the so called *multiplier theorem*) is one of the deepest results of harmonic analysis of the twentieth century [5]. Several proofs are known now. For the present state of this topic see books [23], [24] and [25].

1.2 Our Approach

In the present paper and in [18], we discuss $\operatorname{Adm}\Theta$ for certain inner Θ's. We concentrate mainly on the special case of *meromorphic* Θ's, that is we assume Θ coincides in $\mathbb{C}_+$ with a meromorphic function whose poles are in $\mathbb{C}_-$. In other words

$$\Theta(z) = e^{i\sigma z} B(z) \tag{1.4}$$

where $\sigma \geq 0$ and B is a meromorphic Blaschke product for $\mathbb{C}_+$ (either B is finite or its zeros tend to infinity). The case $B(z) \equiv 1$, *i.e.*, the Blaschke product with the empty set of zeros, is exactly the Beurling-Malliavin case. Our results *here* are devoted mainly to the case $\sigma = 0$. (Note that $K_{\Theta_1\Theta_2} = K_{\Theta_1} \oplus \Theta_1 K_{\Theta_2}$ [1], whence $\operatorname{Adm}\Theta_1\Theta_2 \supset \operatorname{Adm}\Theta_1 + \operatorname{Adm}\Theta_2$). The Beurling-Malliavin case ($\sigma > 0$ and $B \equiv 1$) and some other similar cases will be considered in [18].

We turn to the case $\sigma = 0$, *i.e.*, $\Theta = B$ in (1.4). The set $\operatorname{Adm} B$ depends on $B^{-1}(0)$, or to be more precise, on the divisor of B, *i.e.*, $B^{-1}(0)$ and the multiplicities of zeros. We obtain a quite satisfactory description of $\operatorname{Adm} B$ for purely imaginary (*vertical*) zeros. The *horizontal* case (say, zeros on a line $\Im z = c$, $c > 0$) is much more difficult and for certain sets $B^{-1}(0)$ is similar to the Beurling-Malliavin case. In [18] we obtain some partial results in this direction.

Any meromorphic inner function Θ can be written as $\Theta(x) = e^{i\varphi(x)}$ on $\mathbb{R}$, where φ is real and continuous (in fact, real analytic). We call φ a continuous argument of Θ and denote it by $\arg\Theta$. Thus $\arg\Theta$ is defined up to a constant. This function is increasing. In this paper we consider situations gravitating to our Example 1.1: $\Theta = B$, and $\arg B$ grows slowly (so that the unit vector $B(x)$ is winding slowly as x grows from $-\infty$ to ∞; note that in Example 1.1, an extreme case, $\arg B$ is just bounded). In this paper, as a rule, $(\arg B)'(x) = o(1)$ as $|x| \to \infty$. On the other hand, in the Beurling-Malliavin case (Example 1.2) $\arg\Theta(x) = \sigma x$ is linear. Some

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

inner functions Θ with $\arg \Theta(x)$ growing almost linearly (and even faster) will be analyzed in [18]. The technique used there is different from that of the present paper.

The statements of our main results involve comparison of functions on $\mathbb{R}$. Let ω_1 and ω_2 be such functions. We write

$$\omega_1 \prec \omega_2,$$

if $\omega_1(x) \leq C\omega_2(x)$ for all $x \in \mathbb{R}$ and a positive number C. We say that ω_1 and ω_2 are *comparable*, and write

$$\omega_1 \asymp \omega_2,$$

if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$.

An element ω of $\operatorname{Adm} \Theta$ is called a *minimal majorant* for K_Θ if any $\omega_1 \in \operatorname{Adm} \Theta$ satisfying $\omega_1 \prec \omega$ is comparable with ω. We will be also interested in the uniqueness of a minimal majorant. We say that the minimal majorant $\omega \in \operatorname{Adm} \Theta$ is unique if it is strictly positive, continuous, and any minimal, strictly positive and continuous majorant for K_Θ is comparable with ω.

In this paper we prove the *existence of unique minimal majorants* for some spaces K_B, give their explicit expressions and prove their uniqueness. (Note that if $\arg \Theta$ grows fast, then, as a rule, the minimal majorant for K_Θ does not exist, see [18]).

1.3 Our Main Themes

The main results of this paper are as follows. First, we completely characterize the unique minimal admissible majorant for model subspaces generated by a meromorphic Blaschke product with zeros on the imaginary axis.

Theorem 1.3 *Let $\{b_k\}_{k\geq 1}$ be an increasing sequence of positive numbers, and $\sum_{k=1}^{\infty} 1/b_k < \infty$. Let B be the Blaschke product with zeros $\{ib_k\}_{k\geq 1}$. Put*

$$E(z) = \prod_{k=1}^{\infty} \Big(1 + \frac{z}{ib_k} \Big).$$

Then $1/|E(x)|$ is in $\operatorname{Adm} B$ and it is the unique minimal majorant for K_B. Moreover,

$$\log |E(x)| \asymp \int_0^x \frac{n(t)}{t}\, dt + x^2 \int_x^\infty \frac{n(t)}{t^3}\, dt,$$

where $n(t)$ is the counting function of the sequence $\{b_k\}_{k\geq 1}$.

The convergence of $\sum_{k=1}^{\infty} 1/b_k$ coincides with the Blaschke condition and cannot be weakened. But to obtain a similar result for more general sets of zeros in $\mathbb{C}_+$ (not necessarily vertical) we need somewhat stronger conditions.

Theorem 1.4 *Let $\{z_k\}_{k\geq 1}$ be a sequence in the upper half plane $\mathbb{C}_+$ such that $\lim_{k\to\infty} |z_k| = \infty$ and*

$$\sum_{k=1}^{\infty} \frac{\log |z_k|}{|z_k|} < \infty.$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Let B be the Blaschke product with zeros $\{z_k\}_{k\geq 1}$. Put

$$E(z) = \prod_{k=1}^{\infty}\Big(1 - \frac{z}{\bar{z}_k}\Big),$$

so that $B(z) = E^*(z)/E(z)$ where $E^*(z) = \overline{E(\bar{z})}$. If

$$\frac{1}{|E(x)|} \in L^2(\mathbb{R}),$$

then $1/|E(x)| \in \operatorname{Adm} B$. Moreover, the majorant $1/|E(x)|$ is minimal and unique.

We provide some examples to show that the condition $1/|E(x)| \in L^2(\mathbb{R})$ is not a consequence of the assumption $\sum_{k=1}^{\infty} \log|z_k|/|z_k| < \infty$ and hence we have to insert it in the theorem (criteria for $1/E(x) \in L^2(\mathbb{R})$ can be deduced from [7] and [34]). The next result is our only theorem dealing with a general (not necessarily meromorphic) inner function Θ. It generalizes an essential part of Theorem 1.4.

Theorem 1.5 *Suppose there exists an outer function $O \in H^1(\mathbb{C}_+)$ such that*

$$O(x) = |O(x)|\Theta(x)$$

almost everywhere on $\mathbb{R}$. Then $\sqrt{|O(x)|}$ is a minimal majorant for K_Θ.

As a matter of fact a stronger assertion is proved in Theorem 5.2. Theorem 1.5 is an easy corollary of the complete description of moduli of elements of K_Θ obtained by Dyakonov in [13] (in [13] the L^p-analogs of K_Θ are also considered; we only need a particular case of Dyakonov's result). From Dyakonov's criterion we deduce a complete description of $\operatorname{Adm}\Theta$ (Theorem 4.4). This result yields a parameterization of $\operatorname{Adm}\Theta$: putting $\Omega = \log 1/\omega$ we obtain a representation of $e^{2i\tilde{\Omega}}$ in terms of *free parameters* m and I where m is an arbitrary element of $L^\infty(dt)$ such that $m\omega \in L^2(dt)$, $\log m \in L^1\big(dt/(1+t^2)\big)$, and I is an arbitrary inner function. This parametrization is used in the proof of Theorem 5.2; it is an important element of [18].

2 Representations of K_Θ

In this section we discuss several aspects of model subspaces generated by a meromorphic inner function.

2.1 Reminder on Blaschke Products

Let $\{z_k\}_{k\geq 1}$ be a sequence of complex numbers in the upper half plane $\mathbb{C}_+$. (Sometimes we allow the index k to range through $\mathbb{Z}$.) Let

$$b_k(z) = e^{i\alpha_k}\cdot\frac{z - z_k}{z - \bar{z}_k},$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

where a real α_k is so chosen that

$$e^{i\alpha_k} \cdot \frac{i - z_k}{i - \bar{z}_k} \geq 0.$$

The rational function $B_K = \prod_{k=1}^{K} b_k$ is called a *finite Blaschke product* for the upper half plane; B_K is analytic at each point of the real line and $|B_K(x)| = 1$ for $x \in \mathbb{R}$. The relation

$$\sum_{k=1}^{\infty} \frac{\Im z_k}{|z_k + i|^2} < \infty$$

is a necessary and sufficient condition for the uniform convergence of B_K on compact sets, disjoint from the closure of $\{\bar{z}_k \,;\, k \geq 1\}$, to a non-zero analytic function

$$B(z) = \prod_{k=1}^{\infty} \Big(e^{i\alpha_k} \cdot \frac{z - z_k}{z - \bar{z}_k} \Big) = \lim_{K \to \infty} B_K(z),$$

and we call B an *infinite Blaschke product* for the upper half plane [22, page 120]. We have $|B(z)| < 1$ for $z \in \mathbb{C}_+$. Therefore, by Fatou's theorem [22, page 57], for almost all $x \in \mathbb{R}$, $\lim_{z \not\to x} B(z)$ exists. Denoting that limit by $B(x)$ (wherever it exists), one has $|B(x)| = 1$ almost everywhere [22, page 66].

2.2 Meromorphic Blaschke products

A Blaschke sequence in the upper half plane, $\{z_k\}_{k \geq 1}$, has no accumulation point on the real line if and only if

$$\lim_{k \to \infty} |z_k| = \infty.$$

Here, since the z_k stay away from zero, a necessary and sufficient condition for the uniform convergence of B_K to B on compact sets disjoint from $\{\bar{z}_k \,;\, k \geq 1\}$ is

$$\sum_{k=1}^{\infty} \frac{\Im z_k}{|z_k|^2} < \infty. \tag{2.1}$$

In this case, B is a meromorphic function with poles at the $\bar{z}_k$. For this reason, it is called a *meromorphic Blaschke product*. The function B is analytic at each point of $\mathbb{R}$, and

$$|B(x)| = 1 \quad \text{for } x \in \mathbb{R}.$$

Let us multiply B by a constant of modulus one to get $B(0) = 1$. Then for each z different from all the $\bar{z}_k$,

$$B(z) = \prod_{k=1}^{\infty} \frac{1 - z/z_k}{1 - z/\bar{z}_k}.$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

2.3 Representation of a Meromorphic Blaschke Product as $E^*(z)/E(z)$

The following result is a direct corollary of a theorem of M. G. Krein on entire functions of the Hermite-Biehler class [26, pages 317–318]. We give a direct proof.

Lemma 2.1 *Every meromorphic Blaschke product can be represented as*

$$B(z) = \frac{\overline{E(\bar{z})}}{E(z)} \quad \text{for } z \in \mathbb{C},$$

where E is an entire function with zeros at the $\bar{z}_k$. The order of $\bar{z}_k$ as a zero of E is the same as its order as a pole of B.

Proof Put

$$E_k(z) = \Big(1 - \frac{z}{\bar{z}_k}\Big) \exp\Big\{ \Re\Big(\frac{1}{\bar{z}_k}\Big) z + \cdots + \frac{1}{k}\Re\Big(\frac{1}{\bar{z}_k^k}\Big) z^k \Big\}. \tag{2.2}$$

Suppose $|z| \le R$. Then, for $|z_k| \ge 2R$,

$$\begin{aligned}
\log E_k(z) &= -\frac{z}{\bar{z}_k} - \frac{1}{2}\Big(\frac{z}{\bar{z}_k}\Big)^2 - \cdots - \frac{1}{k}\Big(\frac{z}{\bar{z}_k}\Big)^k - \cdots \\
&\qquad + \Re\Big(\frac{1}{\bar{z}_k}\Big) z + \frac{1}{2}\Re\Big(\frac{1}{\bar{z}_k^2}\Big) z^2 + \cdots + \frac{1}{k}\Re\Big(\frac{1}{\bar{z}_k^k}\Big) z^k \\
&= -i\Im\Big(\frac{1}{\bar{z}_k}\Big) z - \frac{i}{2}\Im\Big(\frac{1}{\bar{z}_k^2}\Big) z^2 - \cdots - \frac{i}{k}\Im\Big(\frac{1}{\bar{z}_k^k}\Big) z^k \\
&\qquad - \frac{1}{k+1}\cdot\Big(\frac{z}{\bar{z}_k}\Big)^{k+1} - \frac{1}{k+2}\cdot\Big(\frac{z}{\bar{z}_k}\Big)^{k+2} - \cdots .
\end{aligned}$$

Here, we are using the branch of the logarithm which is zero at 1. Since $|\Im(w^n)| \le n|w|^{n-1}|\Im w|$ for every $w \in \mathbb{C}$, we have

$$\begin{aligned}
|\log E_k(z)| &\le \Big|\Im\Big(\frac{1}{\bar{z}_k}\Big)\Big| R + \cdots + \frac{1}{k}\Big|\Im\Big(\frac{1}{\bar{z}_k^k}\Big)\Big| R^k \\
&\qquad + \frac{1}{k+1}\cdot\frac{1}{2^{k+1}} + \frac{1}{k+2}\cdot\frac{1}{2^{k+2}} + \cdots \\
&\le \Big|\Im\Big(\frac{1}{\bar{z}_k}\Big)\Big| R + \cdots + \Big|\Im\Big(\frac{1}{\bar{z}_k}\Big)\Big| \frac{1}{|\bar{z}_k|^{k-1}} R^k + \frac{1}{2^k} \\
&\le \frac{\Im z_k}{|z_k|^2}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\Big) R + \frac{1}{2^k} \\
&\le 2R\cdot\frac{\Im z_k}{|z_k|^2} + \frac{1}{2^k}.
\end{aligned}$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

By this inequality and by (2.1), $\sum_{|z_k|\ge 2R}\log E_k(z)$ converges absolutely and uniformly for $|z|\le R$. Thus $\prod_{|z_k|\ge 2R} E_k(z)$ converges uniformly for such values of z. Therefore,

$$E(z)=\prod_{k=1}^{\infty}\Big(1-\frac{z}{\bar z_k}\Big)\,\exp\Big\{\Re\Big(\frac{1}{\bar z_k}\Big)z+\cdots+\frac{1}{k}\Re\Big(\frac{1}{\bar z_k^k}\Big)z^k\Big\}$$

is an entire function and

$$\overline{E(\bar z)}=\prod_{k=1}^{\infty}\Big(1-\frac{z}{z_k}\Big)\,\exp\Big\{\Re\Big(\frac{1}{\bar z_k}\Big)z+\cdots+\frac{1}{k}\Re\Big(\frac{1}{\bar z_k^k}\Big)z^k\Big\}.$$

The relation $\overline{E(\bar z)}/E(z)=B(z)$ is now clear by inspection. ■

In the general case, E is not necessarily of exponential type. But if more is known about the growth of the z_k as $k\to\infty$, the degrees of the polynomials figuring in the exponential factors in (2.2) can be diminished. If, for instance,

$$\sum_{k=1}^{\infty}\frac{1}{|z_k|}<\infty,$$

all of those polynomials can be taken equal to zero (and the exponential factors dropped altogether). See also [6, page 14].

2.4 Representation of Meromorphic Inner Functions

Meromorphic inner functions are generalizations of meromorphic Blaschke products. We call an inner function $\Theta\in H^\infty(\mathbb{C}_+)$ meromorphic if it is continuous (or equivalently, analytic) up to $\mathbb{R}$. It is easy to see that the set of meromorphic inner functions coincides with the set of all products $B(z)e^{i\sigma z}$ where B is a meromorphic Blaschke product and σ is a non-negative real number.

The entire function E constructed in Lemma 2.1 enjoys the following property:

$$|E^*(z)|<|E(z)| \tag{2.3}$$

for all $z\in\mathbb{C}_+$ and all its zeros are in the lower half plane. On the other hand, any such E generates a meromorphic inner function Θ, namely

$$\Theta(z)=E^*(z)/E(z) \tag{2.4}$$

for all $z\in\mathbb{C}_+$. Indeed Lemma 2.1 shows that any meromorphic inner function Θ can be represented by (2.4) with a suitable entire function satisfying (2.3) and having zeros only in the lower half plane. For if $\Theta(z)=B(z)e^{i\sigma z}$, then by Lemma 2.1,

$$\Theta(z)=\frac{E^*(z)}{E(z)}e^{i\sigma z}=\frac{E^*(z)e^{i\sigma z/2}}{E(z)e^{-i\sigma z/2}}=\frac{E_1^*(z)}{E_1(z)},$$

and $E_1(z)=E(z)e^{-i\sigma z/2}$ satisfies (2.3) and all its zeros are in the lower half plane.

Subsections 2.5 and 2.6 contain some well known facts on the spaces K_Θ. We state them (with short proofs) for reader's convenience.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

2.5 Various Descriptions of K_Θ

Let Θ be an arbitrary inner function for the upper half plane. Then $\Theta H^2(\mathbb{R})$ is a closed subspace of the Hilbert space $H^2(\mathbb{R})$. According to the notation introduced in Section 1, the orthogonal complement of $\Theta H^2(\mathbb{R})$ in $H^2(\mathbb{R})$ is denoted by K_Θ. Thus

$$H^2(\mathbb{R}) = \Theta H^2(\mathbb{R}) \oplus K_\Theta.$$

The following lemma gives another characterization of K_Θ which can be used as the definition of it in all Hardy spaces $H^p(\mathbb{R})$, $0 < p \leq \infty$.

Lemma 2.2 *For each inner function Θ*

$$K_\Theta = H^2(\mathbb{R}) \cap \Theta\, \overline{H^2(\mathbb{R})}.$$

Proof We use the properties $\Theta \in H^\infty$ and $\Theta\bar{\Theta} = 1$. By definition, $f \in K_\Theta$ if and only if $f \in H^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} f(x)\overline{\Theta(x)g(x)}\, dx = 0$$

for each $g \in H^2(\mathbb{R})$. Thus, $f \in K_\Theta$ if and only if $f \in H^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} \frac{f(x)}{\Theta(x)}\overline{g(x)}\, dx = 0$$

for each $g \in H^2(\mathbb{R})$. This condition is equivalent to $f/\Theta \in \overline{H^2(\mathbb{R})}$. Therefore $f \in K_\Theta$ if and only if $f \in H^2(\mathbb{R})$ and also $f \in \Theta\overline{H^2(\mathbb{R})}$. ■

An inner function Θ is already defined in the upper half plane and it is analytic there. Its nontangential limits at the points of the real line define a measurable unimodular function there. It can be extended to the lower half plane by putting

$$\Theta(z) = \frac{1}{\overline{\Theta(\bar{z})}} \quad \text{for } z \in \mathbb{C}_-.$$

Let $h \in L^2(\mathbb{R})$. Then the Poisson integral formula

$$P_h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2} h(t)\, dt, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

gives an extension of h to the upper and to the lower half planes. It can be shown that $h \in H^2(\mathbb{R})$ if and only if P_h, as a function defined in the upper half plane, is in $H^2(\mathbb{C}_+)$. Similarly, $h \in \overline{H^2(\mathbb{R})}$ if and only if P_h, as a function defined in the lower half plane, is in $H^2(\mathbb{C}_-)$. An $f \in K_\Theta$ belongs in particular to $H^2(\mathbb{R})$. Therefore it has an extension $f(z)$ to the upper half plane, belonging to $H^2(\mathbb{C}_+)$ and given there by the formula

$$f(z) = P_f(z) \quad \text{for } z \in \mathbb{C}_+.$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

The extension of an $f \in K_\Theta$ to the lower half plane is indirect (depending on Θ). For such an f we have $\bar{\Theta} f \in \overline{H^2(\mathbb{R})}$ by Lemma 2.2, so, by the preceding observation, $\bar{\Theta} f$ has an analytic extension to the lower half plane, equal there to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\overline{\Theta(t)} f(t)\, dt = P_{\bar{\Theta} f}(z), \quad z \in \mathbb{C}_-.$$

We then *define* the extension of $f \in K_\Theta$ to $\mathbb{C}_-$ by putting

$$f(z) = \Theta(z) P_{\bar{\Theta} f}(z) \quad \text{for } z \in \mathbb{C}_-,$$

with $\Theta(z)$ defined as above in $\mathbb{C}_-$ [11]. This extension is at least meromorphic in the lower half plane. We have $\lim_{z \not\to x} \Theta(z) = \Theta(x)$ and $\lim_{z \not\to x} = f(x)$ for almost all $x \in \mathbb{R}$. In these limits, z is allowed to tend to x non-tangentially from *either* half plane. With these definitions, Lemma 2.2 yields the following characterization of K_Θ.

Theorem 2.3 *The space K_Θ consists precisely of the functions $f \in L^2(\mathbb{R})$ having extension to the whole complex plane $\mathbb{C}$, as defined above, so that $f \in H^2(\mathbb{C}_+)$ and $f/\Theta \in H^2(\mathbb{C}_-)$.*

A function $f \in K_\Theta$ can be continued analytically across intervals of $\mathbb{R}$ on which Θ is analytic. This result has important consequences in characterizing elements of K_B when B is a meromorphic Blaschke product.

Theorem 2.4 *If Θ is analytic in a neighborhood of the interval $(a, b) \subset \mathbb{R}$ then any $f \in K_\Theta$ is also analytic there.*

2.6 Paley-Wiener Spaces

Let $\sigma > 0$. Then $\Theta(x) = \exp(i\sigma x)$ is an *entire* inner function. In this case, the functions $f(x) \in K_\Theta$ differ by the factor $\exp(i\sigma x/2)$ from those in a Paley-Wiener space.

Theorem 2.5 *Let $\sigma > 0$. Then $f \in K_{e^{i\sigma x}}$ if and only if f is an entire function of exponential type, square integrable on the real line, with $-\sigma \le h_+ \le 0$ and $0 \le h_- \le \sigma$, where*

$$h_+ = \limsup_{y\to\infty} \frac{\log|f(iy)|}{y} \quad \text{and} \quad h_- = \limsup_{y\to\infty} \frac{\log|f(-iy)|}{y}.$$

Proof Since $\Theta(x) = \exp(i\sigma x)$ is analytic across $\mathbb{R}$, each $f \in K_{e^{i\sigma x}}$ is also analytic there. Furthermore, $f \in H^2(\mathbb{C}_+)$ and $f/\Theta \in H^2(\mathbb{C}_-)$ imply that f is analytic on $\mathbb{C}_+$ and also on $\mathbb{C}_-$, that $f \in L^2(\mathbb{R})$, and besides that the support of the Fourier-Plancherel transform of f is a subset of $[0, \sigma]$. Thus $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and for each $z = x \in \mathbb{R}$, $f(z) = \int_0^\sigma \hat{f}(t) e^{izt}\, dt$. By the uniqueness theorem for analytic functions, equality holds everywhere. Therefore f is an entire function of exponential type with the indicated growth conditions on the imaginary axis.

The if part is an easy consequence of the celebrated Paley-Wiener theorem. ∎

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Corollary 2.6 *Each $f \in K_{e^{i\sigma x}}$ has the representation*

$$f(z) = \int_0^\sigma \hat{f}(t) e^{izt}\, dt, \tag{2.5}$$

where $\hat{f} \in L^2(0, \sigma)$.

2.7 The Model Subspace K_Θ With a Meromorphic Θ

Let $\{z_k\}_{k=1}^\infty$ be a sequence in the upper half plane with $z_k \to \infty$ as $k \to \infty$. Let $\{m_k\}_{k=1}^\infty$ be a sequence of positive integers. Suppose that

$$\sum_{k=1}^\infty \frac{m_k \Im z_k}{|z_k|^2} < \infty.$$

Then

$$B(z) = \prod_{k=1}^\infty \Big(\frac{\bar{z}_k}{z_k} \cdot \frac{z - z_k}{z - \bar{z}_k} \Big)^{m_k}$$

is a meromorphic Blaschke product. Put $\Theta(z) = B(z) e^{i\sigma z}$, $\sigma \ge 0$.

Theorem 2.7 *Let $\Theta(z) = B(z) e^{i\sigma z}$, $\sigma \ge 0$ and B be a meromorphic Blaschke product. Then the space K_Θ consists precisely of the meromorphic functions f with poles of order at most m_k at the $\bar{z}_k$, such that $f \in H^2(\mathbb{C}_+)$ and also $f/\Theta \in H^2(\mathbb{C}_-)$.*

Proof Let $f \in K_\Theta$. Then by Theorem 2.3, f and f/Θ are respectively analytic in the upper and lower half planes. Hence $f = \Theta \cdot f/\Theta$ is a meromorphic function in the lower half plane, with poles of order at most m_k at the $\bar{z}_k$. Finally, by Theorem 2.4, f is analytic at each point of the real line.

If, on the other hand, $f \in H^2(\mathbb{C}_+)$ and $f/\Theta \in H^2(\mathbb{C}_-)$, then at least $f \in L^2(\mathbb{R})$. Thus $f \in K_\Theta$ by Theorem 2.3. ∎

Corollary 2.8 *Let $\Theta(z) = B(z) e^{i\sigma z}$, $\sigma \ge 0$ and B be a meromorphic Blaschke product with zeros of order m_k at z_k, $k \ge 1$. Then, for each j, $1 \le j \le m_k$, we have $(z - \bar{z}_k)^{-j} \in K_\Theta$.*

Corollary 2.9 *Let B be the finite Blaschke product*

$$B(z) = \prod_{k=1}^K \Big(\frac{z - z_k}{z - \bar{z}_k} \Big)^{m_k}.$$

Then K_B consists precisely of the linear combinations of the fractions $(z - \bar{z}_k)^{-j_k}$ where $1 \le k \le K$ and $1 \le j_k \le m_k$. Thus $f \in K_B$ if and only if

$$f(z) = \frac{P(z)}{\prod_{k=1}^K (z - \bar{z}_k)^{m_k}},$$

where P is a polynomial of degree at most $m_1 + \cdots + m_K - 1$.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Suppose now that Θ is any meromorphic Blaschke product. Then, according to Lemma 2.1 and the paragraph after it, $\Theta = E^*/E$, where E is an entire function satisfying (2.3). This observation enables us to give another characterization of K_Θ.

Theorem 2.10 *Let $\Theta = E^*/E$, where E is an entire function satisfying (2.3). Then the space K_Θ consists precisely of functions of the form f/E where f is an entire function with both $f/E \in H^2(\mathbb{C}_+)$ and $f/E^* \in H^2(\mathbb{C}_-)$.*

Proof Let $g \in K_\Theta$. Then by Theorem 2.7, g is a meromorphic function with poles of order at most m_k at the $\bar{z}_k$. Hence gE is an entire function, where E is the entire function furnished by Lemma 2.1. Put $f = gE$. Then $f/E = g \in H^2(\mathbb{C}_+)$, and $f/E^* = g/\Theta \in H^2(\mathbb{C}_-)$. On the other hand, if f satisfies these conditions, then $f/E \in K_\Theta$ by Theorem 2.7. ■

2.8 Model Subspaces K_Θ and the de Branges Spaces $\mathcal{H}(E)$

Any entire function E satisfying (2.3) generates the de Branges space

$$\mathcal{H}(E) = \{f : f \text{ is entire}, f/E \text{ and } f^*/E \in H^2(\mathbb{C}_+)\}$$

with norm $\|f\|_{\mathcal{H}(E)} = \|f/E\|_{L^2(\mathbb{R})}$. Theorem 2.10 shows that $\mathcal{H}(E)$ and K_Θ are isometric as Hilbert spaces. Indeed, the operator $f \mapsto f/E$ is an isometry of $\mathcal{H}(E)$ onto K_Θ with $\Theta = E^*/E$. Theorem 2.10 also enables us to estimate the growth of a function $g \in K_B$ in the complex plane (see Theorem 3.1 below).

3 What Happens if $1/E \in K_B$?

Here we turn to the main results of this paper and explicitly describe admissible majorants for spaces K_B generated by certain meromorphic Blaschke products. The results are sharp, since our majorants turn out to be the best possible ones in a sense. We use symbols $\prec$ and $\asymp$ as defined in the Introduction.

3.1 Blaschke Products $B = E^*/E$ when E is of Zero Type

Let $\Im z > 0$ and consider the finite Blaschke product

$$B(z) = \prod_{k=1}^{K} \Big(\frac{1 - z/z_k}{1 - z/\bar{z}_k} \Big)^{m_k} = \frac{\overline{E(\bar{z})}}{E(z)}, \tag{3.1}$$

where $E(z) = \prod_{k=1}^{K} (1 - z/\bar{z}_k)^{m_k}$. Then the model space K_B precisely consists of

$$f(z) = \frac{P(z)}{\prod_{k=1}^{K} (1 - z/\bar{z}_k)^{m_k}} = \frac{P(z)}{E(z)}, \tag{3.2}$$

where P is a polynomial of degree at most $m_1 + \cdots + m_K - 1$. In particular $1/E(z) \in K_B$ and we have $1/|E(x)| \asymp (1 + |x|)^{-(m_1+\cdots+m_K)}$, which, by (3.2), is the fastest possible

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

rate of decrease (along $\mathbb{R}$) for elements of K_B. That is why $1/|E(x)|$ deserves to be called the *minimal admissible majorant* for K_B. In the following we show that this idea can be appropriately generalized for a class of infinite Blaschke products.

Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in the upper half plane with $z_k \to \infty$ as $k \to \infty$. Let $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers. Suppose that $\sum_{k=1}^{\infty} \frac{m_k \Im z_k}{|z_k|^2} < \infty$. Then

$$B(z) = \prod_{k=1}^{\infty} \Big(\frac{1 - z/z_k}{1 - z/\bar{z}_k} \Big)^{m_k}$$

is a meromorphic Blaschke product. We know that the model space K_B consists precisely of the meromorphic functions $f(z)$ with poles of order at most m_k at the $\bar{z}_k$, such that $f(z) \in H^2(\mathbb{C}_+)$ and also $f(z)/B(z) \in H^2(\mathbb{C}_-)$. Thus, according to the representation $B(z) = E^*(z)/E(z)$ where $E(z)$ is an entire function with zeros of order m_k at the $\bar{z}_k$, the space K_B consists precisely of functions of the form $f(z) = g(z)/E(z)$ where $g(z)$ is an entire function with both

$$\frac{g(z)}{E(z)} \in H^2(\mathbb{C}_+) \quad \text{and} \quad \frac{g(z)}{\overline{E(\bar{z})}} \in H^2(\mathbb{C}_-). \tag{3.3}$$

Here we provide conditions to ensure $1/E(z) \in K_B$ and besides $1/|E(x)|$ to have the fastest possible rate of decrease (along $\mathbb{R}$) for elements of K_B.

This representation (3.3) enables us to estimate the growth of $g(z)$ in terms of $E(z)$ for $z \in \mathbb{C}$.

Theorem 3.1 *Let $f \in K_B$. Then, for the entire function $g(z) = f(z)E(z)$, we have*

$$|g(x + iy)| \le C \big| E(x + i|y|) \big|$$

for $|y| \ge 1$, and

$$|g(x + iy)| \le C \max\{|E(\xi + i\eta)| : |\xi - x| \le 2, 0 \le \eta \le 2\}$$

for $|y| < 1$. Here C is a constant depending on f.

Proof Since $f(z) = g(z)/E(z) \in H^2(C_+)$, we have

$$|f(x + iy)| \le \frac{\text{Const}}{\sqrt{y}}$$

for $y > 0$ [22, page 112]. We thus have

$$|g(x + iy)| \le \frac{\text{Const}}{\sqrt{y}} |E(x + iy)| \quad \text{for } y > 0.$$

Again, $g(z)/E^*(z) \in H^2(C_-)$, so we find in like manner that

$$|g(x + iy)| \le \frac{\text{Const}}{\sqrt{|y|}} |E(x - iy)| = \frac{\text{Const}}{\sqrt{|y|}} \big| E(x + i|y|) \big|$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

for $y < 0$. These two estimates give us our first relation. For the second one we use the estimates in Cauchy's formula, applied to the entire function $g(z)$. Assuming that $|\Im z| \le 1$, we can write

$$g(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$

with Γ a square of side 4 having $x = \Re z$ as its center. Since the integral $\int_{-2}^{2} \frac{d\eta}{\sqrt{|\eta|}}$ is finite, the second relation follows immediately. ■

In the following we mainly use a simple consequence of this theorem.

Corollary 3.2 *Let $f(z) = g(z)/E(z) \in K_B$. If $E(z)$ is an entire function of exponential type zero, then so is $g(z)$.*

The following result shows that for a meromorphic Blaschke product $B(z) = E^*(z)/E(z)$, the majorant $1/|E(x)|$ has, in some sense, the best possible rate of decrease as $|x| \to \infty$. But it is more interesting when $1/E(x) \in K_B$.

Theorem 3.3 *Let $B(z) = E^*(z)/E(z)$ where $E(z)$ is an entire function of exponential type zero. Let $f \in K_B$ and suppose that*

$$|f(x)| \le \frac{1}{|E(x)|} \quad \text{for } x \in \mathbb{R}.$$

If

$$\liminf_{|x| \to \infty} |f(x)E(x)| = 0,$$

then $f \equiv 0$.

Proof Referring to Corollary 3.2 we see that the entire function $g(z) = f(z)E(z)$ is in particular entire and of zero exponential type. By the hypothesis, we also have

$$|g(x)| = |f(x)E(x)| \le 1, \quad x \in \mathbb{R}.$$

A Phragmén-Lindelöf theorem therefore implies that $g(z)$ is bounded in both the upper and lower half planes [23, page 28]. It is therefore constant, so, since $g(x_n)$ tends to zero for a sequence x_n tending to $+\infty$ or $-\infty$, it is zero. ■

3.2 Sharpness of a Minimal Majorant

Let Θ be an inner function and $\omega \in \operatorname{Adm}\Theta$. A minimal majorant ω is *sharp* in the following sense: If $f \in K_\Theta$ and $|f| \prec \omega$, then either $f \equiv 0$ or $\omega \prec |f|$ (since $|f| \in \operatorname{Adm}\Theta$ whenever $f \not\equiv 0$). In particular, for a positive minimal $\omega \in \operatorname{Adm}\Theta$, if $f \in K_\Theta$ and $|f| \prec \omega$ and $\liminf_{|x|\to\infty} |f(x)|/\omega(x) = 0$, then $f \equiv 0$.

Returning to the finite Blaschke product (3.1), we conclude that $(1+|x|)^{-(m_1+\cdots+m_K)}$ is the unique *positive* and continuous minimal majorant for K_B. The positivity assumption is essential. There exist other minimal majorants not comparable with $(1+|x|)^{-(m_1+\cdots+m_K)}$, *e.g.*, $|x|(1+|x|)^{-(1+m_1+\cdots+m_K)}$.

From now on we concentrate on a situation generalizing (3.1).

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Lemma 3.4 *Let B be a Blaschke product. Suppose that $B = E^*/E$ where E is an entire function of zero exponential type whose zeros are in the lower half plane $\mathbb{C}_-$. If $f \in K_B$ and $|f| \prec 1/|E|$, then $f = \text{Const}/E$.*

Proof According to Corollary 3.2, $f = g/E$ where g is entire and of zero exponential type. Our assumption, $|f| \prec 1/|E|$, implies that g is bounded on $\mathbb{R}$. By a Phragmen-Lindelöf theorem [23, page 28] g is bounded on $\mathbb{C}$ and thus constant. ∎

Theorem 3.5 *Let B be a Blaschke product. Suppose that $B = E^*/E$ where E is an entire function of zero exponential type whose zeros are in the lower half plane $\mathbb{C}_-$. If $1/E \in K_B$, then $1/|E|$ is the unique minimal positive and continuous majorant.*

Proof Since $1/E \in K_B$ the inclusion $1/|E| \in \operatorname{Adm} B$ is immediate. Now assume that $\omega \in \operatorname{Adm} B$ and $\omega \prec 1/|E|$. Hence there exists a non-zero $f \in K_B$ satisfying $|f(x)| \le \omega(x)$ on $\mathbb{R}$ and thus $|f| \prec 1/|E|$. By Lemma 3.4, $f = C/E$ with a nonzero constant C. Therefore, $1/|E| \prec \omega$ and $\omega \asymp 1/|E|$, so that $1/|E|$ is minimal.

To prove the uniqueness property, take a minimal positive and continuous $\omega \in \operatorname{Adm} B$. Then $\omega \ge |g|/|E|$ on $\mathbb{R}$, where $g \not\equiv 0$ is entire and of zero type. We are going to prove that g is a nonzero constant, whence $\omega \succ 1/|E|$ and, by minimality of ω, $\omega \asymp 1/|E|$. Suppose g is not a constant. Then, by the Hadamard theorem [23, page 16], g has a zero, *i.e.*, $g(a) = 0$ for an $a \in \mathbb{C}$. Then, by Theorem 2.10,

$$f_1(z) = \frac{g(z)}{(z-a)E(z)} = \frac{f(z)}{z-a} \in K_B.$$

If $a \in \mathbb{C} \setminus \mathbb{R}$, then, clearly $\omega_1(x) = \omega(x)(1+|x|)^{-1} \succ |f_1(x)|$, and thus $\omega_1 \in \operatorname{Adm} B$ which is impossible since ω is minimal. If $a \in \mathbb{R}$, then still $\omega_1 \succ |f_1|$ due to the estimate $\min\{\omega(x) : a-1 \le x \le a+1\} > 0$, (by positiveness and continuity of ω), and once again we get a contradiction with the minimality of ω. ∎

3.3 Some Cases of Non-Existence of Minimal Majorants

Let $B = E^*/E$ where $E(z) = \prod_{k=1}^{\infty}(1 - z/\bar{z}_k)$ is a canonical product for the sequence $\{z_k\}_{k=1}^{\infty}$ in the upper half plane satisfying $\lim_{k\to\infty} |z_k| = \infty$ and

$$\sum_{k=1}^{\infty} \frac{\log|z_k|}{|z_k|} < \infty. \tag{3.4}$$

Here we have the following dichotomy.

Theorem 3.6 *Let B be a meromorphic Blaschke product satisfying the conditions in the last paragraph. Then, either*

(a) *$1/E(x) \in L^2(\mathbb{R})$, and $1/|E(x)|$ is a minimal and positive majorant for K_B.*
or

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

(b) $1/E(x) \notin L^2(\mathbb{R})$, *and there is no minimal continuous and positive majorant for* K_B. *Moreover, if* ω *is a positive and continuous admissible majorant for* K_B, *then so is* $\omega(x)/(1+|x|)$.

Proof In case (b) suppose $\omega \in \operatorname{Adm} B \cap \mathcal{C}(\mathbb{R})$, $\omega(x) > 0$ for all $x \in \mathbb{R}$. Then $|f(x)| \le \omega(x)$, $x \in \mathbb{R}$, for a non-zero $f \in K_B$, $f = g/E$ where g is an entire function of type zero and not identically zero (see Theorem 2.10 and Corollary 3.2). Then g cannot be a constant function, since otherwise $1/E(x) \in L^2(\mathbb{R})$, and thus $g(a) = 0$ for a point $a \in \mathbb{C}$. Then

$$\frac{\omega(x)}{1+|x|} \succ \frac{|f(x)|}{|x-a|}$$

whereas $f(z)/(z-a) \in K_B$, as in the proof of Theorem 3.5, and thus $\omega(x)/(1+|x|) \in \operatorname{Adm} B$.

In case (a) the proof will be given at the end of Subsection 3.4 after some preparation and with essential use of (3.4) (it has not been used in case (b)). ■

3.4 A Sufficient Condition for $E \in \operatorname{Cart}$ and $1/E \in K_B$

An entire function f is said to belong to the Cartwright class if it is of finite exponential type, *i.e.*,

$$|f(z)| \le Ae^{B|z|} \tag{3.5}$$

for all $z \in \mathbb{C}$ and some $A, B > 0$, and

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2}\, dx < \infty. \tag{3.6}$$

In this case we write $f \in \operatorname{Cart}$.

Suppose $\{z_k\}_{k=1}^{\infty}$ is a sequence in $\mathbb{C}_+$, $\lim_{k\to\infty} |z_k| = \infty$ and

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|} < \infty. \tag{3.7}$$

If $n(t) = \operatorname{Card}\{k : |z_k| < t\}$, then (3.7) is equivalent to $\int_1^{\infty} n(t)/t^2\, dt < \infty$. Hence the canonical product $E(z) = \prod_{k=1}^{\infty}(1 - z/\bar{z}_k)$ converges and defines an entire function of zero exponential type, *i.e.*, the constant B in (3.5) can be taken arbitrarily small. In this section we assume the stronger condition (3.4) which is equivalent to

$$\int_1^{\infty} \frac{\log t}{t^2} n(t)\, dt < \infty, \tag{3.8}$$

and ensures that E is an outer function in the upper half plane.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Lemma 3.7 *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{C}_+$, $\lim_{k\to\infty}|z_k| = \infty$ and suppose that $\sum_{k=1}^{\infty}\log|z_k|/|z_k| < \infty$. Then the entire function $E(z) = \prod_{k=1}^{\infty}(1 - z/\bar{z}_k) \in \mathrm{Cart}$ and is outer in the upper half plane, i.e.,*

$$\log|E(z)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|E(t)|\,dt$$

for each $z \in \mathbb{C}_+$.

Proof We first show that $E(z)$ is in the Cartwright class. Since

$$|E(x)| \le \prod_{k=1}^{\infty}\left(1 + \frac{|x|}{|z_k|}\right),$$

for $x \in \mathbb{R}$, we have

$$\log^+|E(x)| \le \sum_{k=1}^{\infty}\log\left(1 + \frac{|x|}{|z_k|}\right) = \int_0^{\infty}\log\left(1 + \frac{|x|}{t}\right)dn(t).$$

Integration by parts gives

$$\log^+|E(x)| \le \int_0^{\infty}\frac{|x|n(t)}{t(t+|x|)}\,dt.$$

Hence

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\log^+|E(x)|}{1+x^2}\,dt\\
&\quad\le \int_{-\infty}^{\infty}\int_0^{\infty}\frac{|x|n(t)}{t(t+|x|)(1+x^2)}\,dt\,dx\\
&\quad= \int_0^{\infty}\left\{\int_0^{\infty}\frac{2x}{t(t+x)(1+x^2)}\,dx\right\}n(t)\,dt\\
&\quad= \int_0^{\infty}\left\{\frac{1}{1+t^2}\log\left(\frac{1+x^2}{(t+x)^2}\right) + \frac{2}{t(1+t^2)}\arctan x\right\}\Bigg|_{x=0}^{x\to\infty}n(t)\,dt\\
&\quad= \int_0^{\infty}\left\{\frac{\pi}{t(1+t^2)} + \frac{2\log t}{1+t^2}\right\}n(t)\,dt,
\end{aligned}$$

which is finite by (3.8). Therefore, $E(z)$ has the representation

$$\log|E(z)| = A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|E(t)|\,dt$$

in the upper half plane, where

$$A = \lim_{y\to\infty}\frac{\log|E(iy)|}{y}.$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

This representation is comprehensively studied in the third chapter of [23]. Thus it is enough to show that $A = 0$. Since, for each $y > 0$,

$$\sqrt{1 + \frac{y^2}{|z_k|^2}} \le \Big| 1 - \frac{iy}{\bar{z}_k} \Big| \le 1 + \frac{y}{|z_k|},$$

we thus have

$$0 \le \log |E(iy)| \le \sum_{k=1}^{\infty} \log\Big(1 + \frac{y}{|z_k|} \Big).$$

Hence

$$0 \le \log |E(iy)| \le \int_0^{\infty} \log\Big(1 + \frac{y}{t} \Big) \, dn(t).$$

Integration by parts gives

$$0 \le \frac{\log |E(iy)|}{y} \le \int_0^{\infty} \frac{n(t)}{t(t+y)} \, dt.$$

Now, by the dominated convergence theorem, we have

$$\lim_{y\to\infty} \int_0^{\infty} \frac{n(t)}{t(t+y)} \, dt = 0.$$

Thus $A = \lim_{y\to\infty} \log |E(iy)|/y = 0$. ∎

The following result will be used in our investigation of minimal majorants. It is well known that an outer function square summable along $\mathbb{R}$ is in $H^2(\mathbb{C}_+)$. Combining this fact with Lemma 3.7, we arrive at the following conclusion.

Theorem 3.8 *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{C}_+$, $\lim_{k\to\infty} |z_k| = \infty$ and suppose that $\sum_{k=1}^{\infty} \log |z_k|/|z_k| < \infty$. Put $E(z) = \prod_{k=1}^{\infty}(1 - z/\bar{z}_k)$ and $B(z) = E^*(z)/E(z)$. Then $1/E(z) \in K_B$ if and only if $1/E(x) \in L^2(\mathbb{R})$.*

Proof If $1/E(x) \in L^2(\mathbb{R})$, then, by Lemma 3.7 and a variation of Smirnov's theorem, $1/E(x) \in H^2(\mathbb{R})$. At the same time,

$$\frac{1/E(x)}{B(x)} = \frac{1/E(x)}{\overline{E(x)}/E(x)} = \frac{1}{\overline{E(x)}} \in \overline{H^2(\mathbb{R})}.$$

thus by Theorem 2.3, $1/E(z) \in K_B$. ∎

Now we are ready to complete the proof of Theorem 3.6: in case (a), $1/|E(x)|$ is a minimal positive majorant for K_B just by the combination of Theorems 3.5 and 3.8.

In Subsections 3.5 and 3.6 we give examples clarifying some points in the proofs of Theorems 3.6 and 3.8.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

3.5 A Blaschke Product $B = E^*/E$ With $E \in \text{Cart}$, but $1/E \notin K_B$

The assumption $1/E \in K_B$ in our Theorem 3.6 poses a problem. As we shall see now, this inclusion may fail even if the Blaschke sequence fulfills the much stronger condition (3.4) and the zeros z_k all lie on the ray $\{y = x\} \cap \mathbb{C}_+$. The following example shows that $1/E$ can be far away from being an element of the model space K_B. Let us consider the Blaschke sequence $z_k = \sqrt{2}2^k e^{i\pi/4}$ where z_k has the multiplicity $[a^k]$ with $a < 2$ so that $\sum_{k=1}^{\infty} [a^k] \log |z_k|/|z_k| < \infty$ is fulfilled. The choice of a will be specified later (it will be close to 2). Fix $n \geq 1$ and let $2^n \leq x < 2^{n+1}$. Since $E(z) = \prod_{k=1}^{\infty} (1 - z/\bar{z}_k)^{[a^k]}$, we have

$$\log |E(x)|^2 = \sum_{k=1}^{\infty} [a^k] \log\Big(1 + \frac{x^2}{2 \cdot 4^k} - \frac{x}{2^k}\Big).$$

The terms corresponding to $1 \leq k \leq n-1$ are positive and the rest are negative. Hence

$$\begin{aligned}
\sum_{k=1}^{n-1} [a^k] \log\Big(1 + \frac{x^2}{2 \cdot 4^k} - \frac{x}{2^k}\Big) &\leq \sum_{k=1}^{n-1} [a^k] \log\Big(4 \cdot \frac{x^2}{4^k}\Big) \\
&\leq \sum_{k=1}^{n-1} a^k \log(4^{n+2-k}) \\
&= \Big((n+2) \sum_{k=1}^{n-1} a^k - \sum_{k=1}^{n-1} k a^k\Big) \log 4 \\
&= \Big((n+2)\frac{a^n - 1}{a - 1} - \frac{na^n(a-1) - a(a^n - 1)}{(a-1)^2}\Big) \log 4 \\
&= \frac{(3a-2)a^n - (n+2)(a-1) - a}{(a-1)^2} \log 4 \\
&\leq \Big(\frac{(3a-2)}{(a-1)^2} \log 4\Big) a^n.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{k=n}^{\infty} [a^k] \Big| \log\Big(1 + \frac{x^2}{2 \cdot 4^k} - \frac{x}{2^k}\Big) \Big| &\geq \sum_{k=n}^{\infty} [a^k] \Big(\frac{x}{2^k} - \frac{x^2}{2 \cdot 4^k}\Big) \\
&\geq \sum_{k=n}^{\infty} [a^k] \Big(\frac{2^n}{2^k} - \frac{4^{n+1}}{2 \cdot 4^k}\Big) = \sum_{k=0}^{\infty} [a^{k+n}] \Big(\frac{1}{2^k} - \frac{2}{4^k}\Big) \\
&\geq \sum_{k=0}^{\infty} \frac{a^{k+n}}{2} \Big(\frac{1}{2^k} - \frac{2}{4^k}\Big) = \Big(\frac{1}{2-a} - \frac{4}{4-a}\Big) a^n.
\end{aligned}$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

We choose a such that

$$\Big(\frac{1}{2-a}-\frac{4}{4-a}\Big) > 2a + \Big(\frac{(3a-2)}{(a-1)^2}\log 4\Big).$$

Thus, for $2^n \le x < 2^{n+1}$,

$$\log|E(x)| \le -a^{n+1} \le -x^{\log a/\log 2}.$$

Therefore, for each $x \ge 2$,

$$|E(x)| \le \exp(-x^{\log a/\log 2}).$$

This example shows that $1/|E(x)|$ can be very big for large positive x, so that $1/E(x)$ is not even in $L^2(\mathbb{R})$, and thus $1/E(z)$ is not in K_B.

3.6 A Blaschke Product $B = E^*/E$ With $E \notin \mathrm{Cart}$

The condition (3.4) cannot be dropped if we want E to belong to the Cartwright class. Here we give an example of E with zeros on the ray $\{y = -x\} \cap \mathbb{C}_+$ and satisfying (3.7) but

$$\int_0^\infty \frac{\log^+|E(x)|}{1+x^2}\,dx = \infty.$$

In our example of Section 3.5 the ray was bent to the right to make the zeros closer to $(0,\infty)$ and $|E(x)|$ small on that interval. Now, our ray is bent to the left, so that the zeros are far from $(0,\infty)$ and thus $|E(x)|$ is big for large positive x's.

Let us consider the Blaschke sequence $z_k = \sqrt{2}2^k e^{i3\pi/4}$ with multiplicity $[\frac{2^k}{k\log^2 k}]$, $k \ge 2$. Fix $n \ge 2$ and let $2^n \le x < 2^{n+1}$. Then, with a very generous estimate, we have

$$\begin{aligned}\log|E(x)|^2 &= \sum_{k=2}^{\infty}\Big[\frac{2^k}{k\log^2 k}\Big]\log\Big(1+\frac{x^2}{2\cdot 4^k}+\frac{x}{2^k}\Big)\\ &\ge \sum_{k=n+2}^{\infty}\Big(\frac{1}{2}\cdot\frac{2^k}{k\log^2 k}\Big)\cdot\frac{1}{2}\Big(\frac{x^2}{2\cdot 4^k}+\frac{x}{2^k}\Big)\\ &\ge \frac{x}{4}\sum_{k=n+2}^{\infty}\frac{1}{k\log^2 k} \ge \frac{x}{8\log\log x}.\end{aligned}$$

Thus $\log^+|E(x)|$ is not summable with respect to $dx/(1+x^2)$. This example shows that the condition $\sum_k 1/|z_k| < \infty$ is not enough to ensure that $E(z)$ is in the Cartwright class.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

3.7 Blaschke Products With Zeros on the Imaginary Axis

In our examples of Sections 3.5 and 3.6 we could place our zeros on any line $y = mx$ with $m > 0$ or $m < 0$ but *not* on the imaginary axis. For purely imaginary zeros the Blaschke condition (2.1) (coinciding with (3.7)) is *sufficient* for the inclusion $1/E \in K_B$. Note that (3.7) is equivalent to the Blaschke condition (2.1) for any sequence $\{z_k\}_{k=1}^\infty$ situated in a Stoltz domain and $|z_k| \to \infty$ as $k \to \infty$, but it is only for *purely vertical* zeros that it guarantees $1/E \in K_B$.

Lemma 3.9 *Let $b_k > 0$, $k \geq 1$, and suppose that $\sum_{k=1}^\infty 1/b_k < \infty$. Then the entire function $E(z) = \prod_{k=1}^\infty (1 + z/ib_k)$ is outer in the upper half plane.*

Proof Naturally, we first show that $E(z)$ is in the Cartwright class. But in this case $|E(x)|^2 = \prod_{k=1}^\infty \left(1 + \frac{x^2}{b_k^2}\right)$, $x \in \mathbb{R}$. Thus we have

$$0 \leq \log|E(x)| = \frac{1}{2}\sum_{k=1}^\infty \log\Big(1 + \frac{x^2}{b_k^2}\Big) = \frac{1}{2}\int_0^\infty \log\Big(1 + \frac{x^2}{t^2}\Big)\, dn(t).$$

Integration by parts gives

$$\log|E(x)| = \int_0^\infty \frac{x^2 n(t)}{t(t^2 + x^2)}\, dt.$$

Hence

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\log|E(x)|}{1 + x^2}\, dt &\leq \int_{-\infty}^\infty \int_0^\infty \frac{x^2 n(t)}{t(t^2 + x^2)(1 + x^2)}\, dt\, dx \\
&= \int_0^\infty \Big\{ \int_{-\infty}^\infty \frac{x^2}{t(t^2 + x^2)(1 + x^2)}\, dx \Big\}\, n(t)\, dt \\
&= \int_0^\infty \Big\{ 2\pi i \cdot \frac{i}{2t(t^2 - 1)} + 2\pi i \cdot \frac{it}{2t(1 - t^2)} \Big\}\, n(t)\, dt \\
&= \int_0^\infty \frac{\pi}{t(t + 1)} n(t)\, dt < \infty.
\end{aligned}$$

Therefore, $E(z)$ has the representation

$$\log|E(z)| = A\Im z + \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Im z}{|z - t|^2} \log|E(t)|\, dt$$

in the upper half plane. But to show that $A = 0$, we proceed as in the proof of Lemma 3.7 and use the convergence of $\int_1^\infty \frac{n(t)}{t^2}\, dt$. ∎

In contrast to Theorem 3.8, when zeros are on the imaginary axis no extra condition is needed to ensure $1/E \in K_B$.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Theorem 3.10 *Let $b_k > 0$, $k \geq 1$, and suppose that $\sum_{k=1}^{\infty} 1/b_k < \infty$. Put $E(z) = \prod_{k=1}^{\infty}(1 + z/ib_k)$ and $B(z) = \prod_{k=1}^{\infty} \frac{1-z/ib_k}{1+z/ib_k}$. Then $1/E \in K_B$.*

Proof Since

$$|E(x)|^2 = \prod_{k=1}^{\infty}\Big(1 + \frac{x^2}{b_k^2}\Big) \geq \Big(1 + \frac{x^2}{b_1^2}\Big),$$

we have $1/E \in L^2(\mathbb{R})$. Thus, by Lemma 3.9, $1/E \in H^2(\mathbb{C}_+)$, whence $1/E \in K_B$ (see the proof of Theorem 3.8). ∎

3.8 Asymptotic Behavior of E

The asymptotic behavior of the majorant $\omega(x) = \prod_{k=1}^{\infty} 1/\sqrt{1 + x^2/b_k^2}$, studied in the Section 3.7, can be made explicit if the sequence $\{b_k\}_{k=1}^{\infty}$ is regular. If, for example, $b_k = k^p$, $k \geq 1$ and $p > 1$, then the asymptotic of $\log|E(x)|$ for $|x| \to \infty$ can be found in [26, page 64]. Indeed, we will show that there are positive constants c, C and A with

$$Ae^{c|x|^{1/p}} \leq |E(x)| \leq e^{C|x|^{1/p}}$$

for $x \in \mathbb{R}$. Here we study some estimates to illustrate the following phenomenon: for purely imaginary z_k's and for some nice ω (even and decreasing on $(0, \infty)$) the mere convergence of the logarithmic integral $\mathcal{L}(\omega) = \int_{-\infty}^{\infty} \frac{\Omega^+(x)}{1+x^2}\, dx$ does not imply the inclusion $\omega \in \operatorname{Adm} B$. This is in contrast to the situation when z_k's are on a line parallel to $\mathbb{R}$ (see [18]).

Let $n(t)$ denote the number of b_k in the interval $(0, t)$. Integration by parts gives

$$\int_0^b \frac{dn(t)}{t} = \frac{n(b)}{b} + \int_0^b \frac{n(t)}{t^2}\, dt,$$

so that the convergence of $\sum_{k=1}^{\infty} 1/b_k$ implies $\int_0^{\infty} \frac{n(t)}{t^2}\, dt < \infty$ and $n(t) = o(t)$ $(t \to \infty)$. Hence,

$$\begin{aligned}\log(|E(x)|^2) &= \sum_{k=1}^{\infty} \log\Big(1 + \frac{x^2}{b_k^2}\Big) = \int_{b_1}^{\infty} \log\Big(1 + \frac{x^2}{t^2}\Big)\, dn(t)\\ &= \Big(1 + \frac{x^2}{t^2}\Big) n(t)\Big|_{t=b_1}^{\infty} + \int_{b_1}^{\infty} \frac{2x^2 n(t)}{t(t^2 + x^2)}\, dt = 2x^2 \int_{b_1}^{\infty} \frac{n(t)}{t(t^2 + x^2)}\, dt,\end{aligned}$$

and thus

$$\log|E(x)| \asymp \int_0^x \frac{n(t)}{t}\, dt + x^2 \int_x^{\infty} \frac{n(t)}{t^3}\, dt. \tag{3.9}$$

Therefore, if

$$n(t) \asymp t^{\alpha},$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

for some α in $(0,1)$, then there are positive constants c, C and A with

$$Ae^{c|x|^\alpha} \le |E(x)| \le e^{C|x|^\alpha}$$

for all $x \in \mathbb{R}$. We conclude that $e^{-c|x|^\alpha} \in \operatorname{Adm} B$ whereas $\omega \notin \operatorname{Adm} B$ if $\omega(x) = o(e^{-C|x|^\alpha})$ $(|x| \to \infty)$, since $1/|E|$ is a minimal majorant for K_B. These statements can be made more precise depending on the concrete nature of b_k. Here we only mention that for $b_k = k^2$, by a direct computation using the Euler product for $\sin z$, we obtain

$$\Big|\prod_{k=1}^{\infty}\Big(1+\frac{x}{ik^2}\Big)\Big| \approx \frac{1}{2\pi\sqrt{|x|}}e^{\frac{\pi}{\sqrt{2}}\sqrt{|x|}}$$

as $|x| \to \infty$ $(x \in \mathbb{R})$, *i.e.*, the quotient of the left and right sides tends to one. Thus

$$\sqrt{1+|x|}\exp(-\pi\sqrt{|x|/2}) \in \operatorname{Adm} B,$$

where

$$B(z) = \prod_{k=1}^{\infty}\Big(\frac{1-z/ik^2}{1+z/ik^2}\Big),$$

but

$$(1+|x|)^\varepsilon \exp(-\pi\sqrt{|x|/2}) \notin \operatorname{Adm} B, \tag{3.10}$$

for all $\varepsilon < 1/2$. Especially,

$$\exp(-|x|^\alpha) \notin \operatorname{Adm} B, \tag{3.11}$$

for all $\alpha > 1/2$.

4 Moduli of Elements In K_Θ

This section contains an important ingredient to be used in the rest of this paper and throughout [18]. Let Θ be an inner function, and write

$$|K_\Theta| = \{|f| : f \in K_\Theta\}.$$

4.1 Hilbert transform

We conclude this paper with a generalization of Theorem 1.3 to minimal majorants for K_Θ's with an arbitrary inner Θ (Theorem 5.1 in Section 5.1). To do so we need to make a digression devoted to the Hilbert transform and present it in a form we need. Sections 4.2–4.7 are mainly devoted to admissibility criteria (to be used in the proof of Theorem 5.1 and in [18]).

Let u be a real function in $L^1(\frac{dt}{1+t^2})$. Then

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}u(t)\,dt$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

is a harmonic function in the upper half plane with

$$\lim_{z \not\to x} U(z) = u(x)$$

for almost all $x \in \mathbb{R}$. Let V be a harmonic conjugate of U. Such a function V is defined up to an additive constant. It is well known that $\lim_{z \not\to x} V(z)$ exists for almost all $x \in \mathbb{R}$ [22, page 58]. This limit is called a *Hilbert transform of* u, and is denoted by $\tilde{u}$. Since $\tilde{u}$ depends on V, it is defined up to an additive constant. Furthermore, the Hilbert transform of a constant function is another constant. Hence the Hilbert transforms of u and $u+c$ are the same up to an additive constant. Thus we assume that the correspondence $u \leftrightarrow \tilde{u}$ is between two classes of functions, each class consisting of a real function and all those obtainable by adding real constants to it. The formula

$$V(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\Re z - t}{|z-t|^2} + \frac{t}{1+t^2} \right) u(t)\, dt$$

gives a harmonic conjugate of U. Here the term $\frac{t}{1+t^2}$ is included to ensure the convergence of the integral. In this case, $\lim_{z \not\to x} V(z)$ is equal to

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) u(t)\, dt$$

for almost all $x \in \mathbb{R}$ [22, page 110]. This limit is usually written as

$$\frac{1}{\pi} \fint_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) u(t)\, dt.$$

The sign $\fint_{\mathbb{R}}$ represents a *singular integral*; it is usually not an integral in the ordinary sense. We thus have a representation of the form

$$\tilde{u}(x) = \frac{1}{\pi} \fint_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) u(t)\, dt. \tag{4.1}$$

Remark Suppose u in (4.1) vanishes in $(x_0 - \varepsilon, x_0 + \varepsilon)$ where $x_0 \in \mathbb{R}$, $\varepsilon > 0$. Then $\tilde{u}$ is analytic at x_0. Indeed, the integral in (4.1) becomes

$$\int_{\mathbb{R} \setminus (x_0 - \varepsilon, x_0 + \varepsilon)} \frac{1+xt}{x-t} \cdot \frac{u(t)}{1+t^2}\, dt$$

and thus converges uniformly with respect to complex values of x satisfying $|x - x_0| < \varepsilon/2$.

The following result is an immediate consequence of the theorems of Kolmogorov [22, page 98] and Smirnov [22, page 74].

Theorem 4.1 *If u and $\tilde{u}$ are in $L^1(\frac{dt}{1+t^2})$, then*

$$\tilde{\tilde{u}} = -u.$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Under certain conditions we can drop the term $\frac{t}{1+t^2}$ in (4.1) or replace it by something else. For example, if $\int_{-\infty}^{\infty}\frac{|u(t)|}{1+|t|}\,dt<\infty$, a harmonic conjugate can be defined by the formula

$$V(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Re z-t}{|z-t|^2}u(t)\,dt.$$

In this case, $\lim_{z \not\to x} V(z)$ is equal to

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{u(t)}{x-t}\,dt$$

for almost all $x\in\mathbb{R}$, and we can write

$$\tilde{u}(x)=\frac{1}{\pi}⨍_{\mathbb{R}}\frac{u(t)}{x-t}\,dt. \tag{4.2}$$

This formula can be used when $u\in L^p(dt)$, $1\le p<\infty$, since then by Hölder's inequality $\int_{-\infty}^{\infty}\frac{|u(t)|}{1+|t|}\,dt<\infty$. When $u\in L^\infty(dt)$, formula (4.2) does not always work, and then we have to use (4.1). On the other hand, if u is bounded on $\mathbb{R}$ and $|u(t)|\le C|t|$ in a neighborhood of the origin, then

$$\tilde{u}(x)=\frac{1}{\pi}⨍_{\mathbb{R}}\Big(\frac{1}{x-t}+\frac{1}{t}\Big)\,u(t)\,dt$$

for almost all $x\in\mathbb{R}$.

The Hilbert transform appears in the construction of outer functions: given a Lebesgue measurable $h\ge 0$ on $\mathbb{R}$ with $\log h\in L^1(\frac{dt}{1+t^2})$, put

$$O(x)=O_h(x)=\exp\big(\log h(x)+i\widetilde{\log h}(x)\big)\;=h(x)\exp\big(i\widetilde{\log h}(x)\big)$$

for almost all $x\in\mathbb{R}$ (since h and $\widetilde{\log h}$ are defined almost everywhere). Obviously $|O_h(x)|=h(x)$ for almost all $x\in\mathbb{R}$, and thus $O_h\in H^p(\mathbb{R})$, $0<p\le\infty$, if and only if $h\in L^p(dt)$; O_h, or its analytic counterpart

$$O_h(z)=\exp\Big(\frac{i}{\pi}\int_{-\infty}^{\infty}\Big(\frac{1}{z-t}+\frac{t}{1+t^2}\Big)\,\log h(t)\,dt\Big),\quad z\in\mathbb{C}_+,$$

is the outer function with modulus h [22, page 120].

4.2 A Complete Characterization of $|K_\Theta|$

The following lemma connects $|K_\Theta|$ to Θ. It is a particular case of a more general result by Dyakonov [13]. We give a direct proof for reader's convenience.

Lemma 4.2 *Let the function $h(x)\ge 0$ be defined and measurable on $\mathbb{R}$. Then $h\in|K_\Theta|$ if and only if $h^2\Theta\in H^1(\mathbb{R})$. Furthermore, if $h\in|K_\Theta|$, then*

$$h\exp(i\widetilde{\log h})$$

is an outer function in K_Θ.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Proof Suppose that $h \in |K_\Theta|$. Then there is a real function φ defined on $\mathbb{R}$ such that $h\exp(i\varphi) \in K_\Theta$. Hence by Lemma 2.2, $h\exp(i\varphi) \in H^2(\mathbb{R})$ and $h\exp(i\varphi) \in \Theta\overline{H^2(\mathbb{R})}$. Thus $h\exp(i\varphi)$ and $\Theta h\exp(-i\varphi)$ are both in $H^2(\mathbb{R})$. Therefore

$$h^2\Theta = h\exp(i\varphi)\cdot\Theta h\exp(-i\varphi) \in H^1(\mathbb{R}).$$

On the other hand, suppose that $h^2\Theta \in H^1(\mathbb{R})$. Since $h^2 = |h^2\Theta| \in |H^1(\mathbb{R})|$, $O = h\exp(i\widetilde{\log h})$ is an outer function in $H^2(\mathbb{R})$, and there is, besides, an inner function I such that $h^2\Theta = O^2 I$. Thus

$$\bar{O}\Theta = h\exp(-i\widetilde{\log h})\cdot\Theta = \frac{h^2\Theta}{h\exp(i\widetilde{\log h})} = \frac{O^2 I}{O} = OI \in H^2(\mathbb{R}).$$

Therefore $O \in K_\Theta$. ■

In the following, we consider functions $\omega \geq 0$ defined on $\mathbb{R}$. We always write $\Omega(x)$ for $-\log\omega(x)$. It will be assumed throughout the remaining discussion that

$$\int_{-\infty}^{\infty} \frac{|\Omega(x)|}{1+x^2}\,dx < \infty.$$

Note that we are not, *for now*, assuming $\omega(x)$ to be bounded above, and, therefore, $\Omega(x)$ is not assumed to be bounded below.

Lemma 4.3 *Let m be a non-negative measurable function on $\mathbb{R}$ with $m \not\equiv 0$. Then the following are equivalent.*

(a) $m\omega \in |K_\Theta|$;
(b) $m\omega \in L^2(dt)$, $\log m \in L^1(\frac{dt}{1+t^2})$, *and there is an inner function I such that*

$$\Theta\exp(2i\tilde{\Omega}) = I\exp(2i\widetilde{\log m}).$$

Furthermore, if (a) or (b) holds, then

$$m\omega\exp\big(i\widetilde{\log(m\omega)}\big)$$

is an outer function in K_Θ.

Proof Suppose that $m\omega \in |K_\Theta|$. Then by Lemma 4.2, $m^2\omega^2\Theta$ is a non-zero function in $H^1(\mathbb{R})$. Thus, by the Smirnov factorization theorem, $m^2\omega^2\Theta = OI$, where O and I are respectively the outer and inner factors of $m^2\omega^2\Theta$. Hence $(m\omega)^2 = |m^2\omega^2\Theta| \in L^1(dt)$ and $\log|m^2\omega^2\Theta| = 2\log m + 2\log\omega \in L^1(\frac{dt}{1+t^2})$. Thus $m\omega \in L^2(dt)$, and $\log m \in L^1(\frac{dt}{1+t^2})$. Furthermore,

$$\begin{aligned} m^2\omega^2\Theta &= OI = m^2\omega^2\exp\big(i\widetilde{\log(m^2\omega^2)}\big)\cdot I \\ &= m^2\omega^2\exp(2i\widetilde{\log m})\exp(2i\widetilde{\log\omega})\cdot I. \end{aligned}$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Hence $\Theta \exp(2i\tilde{\Omega}) = I \exp(2i\widetilde{\log m})$.

Now suppose that (b) holds. Since $m\omega \in L^2(dt)$, and $\log(m\omega) \in L^1(\frac{dt}{1+t^2})$,

$$O = m^2\omega^2 \exp(2i\widetilde{\log m} + 2i\widetilde{\log \omega})$$

is an outer function in $H^1(\mathbb{R})$. Therefore

$$\begin{aligned} m^2\omega^2\Theta &= m^2\omega^2 \exp(-2i\tilde{\Omega}) \cdot \exp(2i\tilde{\Omega})\Theta \\ &= m^2\omega^2 \exp(-2i\tilde{\Omega}) \cdots \exp(2i\widetilde{\log m}) \cdot I \\ &= m^2\omega^2 \exp(2i\widetilde{\log m} + 2i\widetilde{\log \omega}) \cdot I = OI \in H^1. \end{aligned}$$

Hence by Lemma 4.2, $m\omega \in |K_\Theta|$, and $m\omega \exp\big(i\widetilde{\log(m\omega)}\big)$ is an outer function in K_Θ. ■

4.3 A Criterion for Admissibility

We use Lemma 4.3 to characterize admissible majorants for K_Θ.

Theorem 4.4 *Given a measurable function $\omega(x) \geq 0$ on $\mathbb{R}$, the following are equivalent.*

(a) *There exists an $f \in K_\Theta$ with $f \not\equiv 0$ and $|f| \leq \omega$, i.e., $\omega \in \operatorname{Adm}\Theta$;*
(b) *There exists an $m \in L^\infty(dt)$ with $m \geq 0$, $m\omega \in L^2(dt)$ and $\log m \in L^1(\frac{dt}{1+t^2})$, such that, for some inner function I, we have*

$$\Theta \exp(2i\tilde{\Omega}) = I \exp(2i\widetilde{\log m}).$$

Moreover, if (a) or (b) holds,

$$m\omega \exp\big(i\widetilde{\log(m\omega)}\big)$$

is an outer function in K_Θ.

Proof There exists a non-zero function $f \in K_\Theta$ with $|f| \leq \omega$ if and only if there is an $m \in L^\infty(\mathbb{R})$, $m \geq 0$, $m \not\equiv 0$, such that $m\omega \in |K_\Theta|$. Now apply Lemma 4.3. We see in that way that $f = m\omega \exp\big(i\widetilde{\log(m\omega)}\big)$ will do the job. ■

We are going to rephrase Theorem 4.4 in terms of the argument of Θ. But first we have to define this notation.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

4.4 Circular Part and Arguments of a Complex Valued Function

Let $u\colon \mathbb{R} \mapsto \mathbb{C}$ be a Lebesgue measurable unimodular function on $\mathbb{R}$, *i.e.*, $|u(x)| = 1$ almost everywhere on $\mathbb{R}$. Denote by Arg the function defined on $\mathbb{C} \setminus \{0\}$ by the identity

$$e^{i\,\mathrm{Arg}(\zeta)} = \frac{\zeta}{|\zeta|}, \quad \mathrm{Arg}(\zeta) \in (-\pi, \pi].$$

Then $\mathrm{Arg} \circ u$ ($= \mathrm{Arg}\, u$) is Lebesgue measurable on $\mathbb{R}$ and $\exp(i \,\mathrm{Arg}\, u) = u$ almost everywhere on $\mathbb{R}$. Any real Lebesgue measurable function φ satisfying $\exp(i\varphi) = u$ almost everywhere on $\mathbb{R}$ is called *an argument* of u. Let us call $\mathrm{Arg}\, u$ the *principal argument* of u. Clearly, $\mathrm{Arg}\, u \in L^\infty(dt)$, and any argument φ of u can be written as $\mathrm{Arg}\, u + 2\pi S$ where S is a Lebesgue measurable integer valued function on $\mathbb{R}$.

If u is unimodular and *continuous* on $\mathbb{R}$, then it has a continuous argument which is unique up to an additive constant $2\pi k$, $k \in \mathbb{Z}$; this argument is in $\mathcal{C}^p(\mathbb{R})$ if u is. For example, the continuous argument of $e^{i\sigma x}$, $\sigma > 0$, is σx, whereas $\mathrm{Arg}\, e^{i\sigma x}$ is a sawtooth $2\pi/\sigma$ periodic function coinciding with σx on $(-\pi/\sigma, \pi/\sigma]$.

Let $f\colon \mathbb{R} \mapsto \mathbb{C}$ be a Lebesgue measurable function on $\mathbb{R}$. Suppose $f(x) \neq 0$ almost everywhere on $\mathbb{R}$. We call $f/|f|$ the *circular part* of f (since $f(x)/|f(x)|$ is the projection of the points $f(x)$ on the unit circle $\mathbb{T}$). By definition an argument of $f/|f|$ is an argument of f.

4.5 Continuous Arguments of a Meromorphic Blaschke Product

Since a meromorphic Blaschke product B is analytic and non-vanishing on $\mathbb{R}$, there is a real C^∞ function, say $\arg B$, such that

$$B(x) = \exp\bigl(i \arg B(x)\bigr) \quad \text{for } x \in \mathbb{R}.$$

This function is unique up to an additive constant $2\pi k$, $k \in \mathbb{Z}$, so that $\arg B(x) + 2\pi k$ is the general form of continuous arguments of $B(x)$. Thus its derivative is defined uniquely. In the simple case where

$$b_{z_k}(x) = \frac{\bar{z}_k}{z_k} \cdot \frac{z - z_k}{z - \bar{z}_k} = \exp\bigl(i \arg b_{z_k}(x)\bigr),$$

we have, by taking the logarithmic derivative,

$$\frac{d \arg b_{z_k}(x)}{dx} = \frac{b'_{z_k}(x)}{i b_{z_k}(x)} = \frac{2\Im z_k}{|x - z_k|^2}. \tag{4.3}$$

Since $b_{z_k}(0) = B(0) = 1$, we can (and do) always assume that $\arg b_{z_k}(0) = 0$, and similarly that $\arg B(0) = 0$. Then,

$$\arg b_{z_k}(x) = \int_0^x \frac{2\Im z_k}{|t - z_k|^2}\, dt = 2\arctan\Bigl(\frac{x - \Re z_k}{\Im z_k}\Bigr) + 2\arctan\Bigl(\frac{\Re z_k}{\Im z_k}\Bigr). \tag{4.4}$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Lemma 4.5 *If $B(z) = \prod_{k=1}^{\infty} b_{z_k}(z)$ is a meromorphic Blaschke product, then*

$$\frac{B'(x)}{B(x)} = i\frac{d \arg B(x)}{dx} = 2i \sum_{k=1}^{\infty} \frac{\Im z_k}{|x - z_k|^2}$$

for each $x \in \mathbb{R}$. The series converges uniformly on compact subsets of $\mathbb{R}$.

Proof The sequence $B_K = \prod_{k=1}^{K} b_{z_k}$ converges uniformly to B on compact sets disjoint from $\{\bar{z}_k \ ; \ k \geq 1\}$. Since $\mathbb{R}$ is disjoint from the sets $\{\bar{z}_k \ ; \ k \geq 1\}$ and $\{z_k \ ; \ k \geq 1\}$, $\sum_{k=1}^{K} b'_{z_k}/b_{z_k}$ converges uniformly to B'/B on compact subsets of $\mathbb{R}$ [10, page 174]. ■

Corollary 4.6 *If $B(z) = \prod_{k=1}^{\infty} b_{z_k}(z)$ is a meromorphic Blaschke product, then*

$$\arg B(x) = \sum_{k=1}^{\infty} \arg b_{z_k}(x)$$

for each $x \in \mathbb{R}$. The series converges uniformly on every bounded interval.

Proof By Lemma 4.5 and the monotone convergence theorem

$$\begin{aligned} \arg B(x) &= \int_0^x \frac{d \arg B(t)}{dt}\, dt = \int_0^x \sum_{k=1}^{\infty} \frac{2\Im z_k}{|t - z_k|^2}\, dt \\ &= \sum_{k=1}^{\infty} \int_0^x \frac{2\Im z_k}{|t - z_k|^2}\, dt = \sum_{k=1}^{\infty} \arg b_{z_k}(x). \end{aligned}$$

■

4.6 de Branges' Phase Function

Let Θ be a meromorphic inner function. As we saw in Section 2.4, $\Theta(x) = \overline{E(x)}/E(x)$, $x \in \mathbb{R}$, where E is an entire function satisfying (2.3). Since E does not vanish on $\mathbb{C}_+ \cup \mathbb{R}$, it has a continuous $\arg E(x)$ which coincides with $-\varphi(x) + k\pi$ where φ is the so called *phase function* of E [6, page 54], and k is an integer. The phase function plays an outstanding role in the de Branges theory.

Now, a continuous argument of Θ, $\arg \Theta$, can be expressed as follows:

$$\arg \Theta(x) = -2 \arg E(x) = -2\varphi(x) + 2k\pi, \quad k \in \mathbb{Z}.$$

4.7 A Sufficient Condition for Admissibility

The condition

$$\Theta \exp(2i\tilde{\Omega}) = I \exp(2i\widetilde{\log m}).$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

in Theorem 4.4 is equivalent to

$$\operatorname{Arg}\Theta + 2\tilde{\Omega} = \operatorname{Arg} I + 2\widetilde{\log m} + 2\pi S, \tag{4.5}$$

where $\operatorname{Arg}\Theta$ and $\operatorname{Arg} I$ are the principal arguments of Θ and I and S is a measurable integer valued function on $\mathbb{R}$. Thus we arrive at the following *sufficient* condition for an ω to be in $\operatorname{Adm}\Theta$.

Theorem 4.7 *Suppose there exists an $m \in L^\infty(dt)$ with $m \geq 0$, $m\omega \in L^2(dt)$ and $\log m \in L^1(\frac{dt}{1+t^2})$, such that*

$$\arg\Theta + 2\tilde{\Omega} = 2\widetilde{\log m} + S,$$

where S is a step function with values all equal to integral multiples of 2π. Then $\omega \in \operatorname{Adm}\Theta$.

Proof The identity $\arg\Theta + 2\tilde{\Omega} = 2\widetilde{\log m} + S$ implies 4.5. Now apply Theorem 4.4. ■

5 Θ is the Circular Part of an Outer Function

Here we prove Theorem 1.5 stated in the Introduction. Let Θ be an inner function in $\mathbb{C}_+$. Then there exist many outer functions O whose circular part is Θ. Indeed, take any bounded argument of Θ (say, the principal one, $\operatorname{Arg}\Theta$) and put $P = -\widetilde{\operatorname{Arg}\Theta}$. Then $P \in L^p\big(dt/(1+t^2)\big)$, $1 \leq p < \infty$, since $\operatorname{Arg}\Theta$ is bounded [14, page 114]. Put $h = \exp P$ and $O = O_h$. We have $O_h = \exp(P + i\operatorname{Arg}\Theta)$ and $O_h\bar{\Theta} = \exp P \geq 0$.

5.1 Θ as the Circular Part of an Outer Function in $H^1(\mathbb{R})$

We restate Theorem 1.5 in a slightly different form.

Theorem 5.1 *Suppose Θ is the circular part of an outer function $O \in H^1(\mathbb{R})$. Then $\sqrt{|O(x)|}$ is a minimal majorant for K_Θ. Moreover, $\sqrt{|O(x)|} \in |K_\Theta|$.*

Proof Put $h(x) = \sqrt{|O(x)|}$, so that $O = O_{h^2}$. Then

$$h^2(x)\Theta(x) = |O(x)|\Theta(x) = O_h(x) \in H^1(\mathbb{R}),$$

and thus by Lemma 4.2 $h \in |K_\Theta|$. Hence $h \in \operatorname{Adm}\Theta$.

Suppose $\omega \in \operatorname{Adm}\Theta$ and $\omega \prec h$. Hence ω/h is a non-negative bounded function. Therefore, the following more general result (Theorem 5.2) implies $h \prec \omega$. ■

Theorem 5.2 *Let O be an arbitrary outer function (not necessarily in $H^1(\mathbb{R})$). Suppose Θ is its circular part. Put $h(x) = \sqrt{|O(x)|}$. If $\omega \in \operatorname{Adm}\Theta$ and*

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{h(x)} \cdot \frac{dx}{1+x^2} < \infty, \tag{5.1}$$

then $h \prec \omega$.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

Proof Put $\alpha = \omega/h$. Then $\log\alpha \in L^1\big(dx/(1+x^2)\big)$, (since $\log\omega$ and $\log h \in L^1\big(dx/(1+x^2)\big)$). The inclusion $\omega \in \operatorname{Adm}\Theta$ means $m\omega \in |K_\Theta|$ for an $m \in L^\infty(dt)$, $0 \le m \le 1$, and, by Lemma 4.2, $m^2\omega^2\Theta = m^2\alpha^2h^2\Theta \in H^1(\mathbb{R})$. Thus $m^2\alpha^2\omega^2\Theta = O_{m^2\alpha^2h^2}I$ where I is inner. But $\log(m\alpha) \in L^1\big(dx/(1+x^2)\big)$, since $\log(m\alpha h)$ and $\log h$ are in $L^1\big(dx/(1+x^2)\big)$, whence $O_{m^2\alpha^2}$ makes sense, and

$$O_{m^2\alpha^2} = \frac{O_{m^2\alpha^2h^2}}{O_{h^2}} = \frac{m^2\alpha^2h^2\Theta\bar{I}}{h^2\Theta} = m^2\alpha^2\bar{I}$$

almost everywhere on $\mathbb{R}$, so that $m^2\alpha^2 = O_{m^2\alpha^2}I$; (5.1) means $\alpha^2 \in L^{1/2}(dx/(1+x^2))$, whence $m^2\alpha^2 \in L^{1/2}\big(dx/(1+x^2)\big)$, and thus $(O_{m^2\alpha^2}\circ\gamma)(I\circ\gamma) \in H^{1/2}(\mathbb{D})$ where γ is a conformal mapping of the unit disc onto $\mathbb{C}_+$. But an element of $H^{1/2}(\mathbb{D})$ with non-negative boundary values almost everywhere on $\mathbb{T} = \{|z| = 1\}$ is constant [27]. We see that $m\alpha = \text{Const} > 0$, and $m \le 1$ implies $\alpha \ge c$ for a positive constant c. ■

Now, the hypothesis of Theorem 1.5 means there exists an argument $\arg\Theta$ of Θ (*i.e.*, $\operatorname{Arg}\Theta + 2\pi S$ where S is an integer valued Lebesgue measurable function on $\mathbb{R}$) satisfying

$$\arg\Theta = \tilde{P}, \tag{5.2}$$

where P is a real element of $L^1\big(dx/(1+x^2)\big)$ such that $\exp P \in L^1(dt)$; actually, $P = \log|O|$. A condition sufficient for the existence of such P is this:

$$\arg\Theta \quad\text{and}\quad \widetilde{\arg\Theta} \in L^1\big(dx/(1+x^2)\big) \quad\text{and}\quad \exp(-\widetilde{\arg\Theta}) \in L^1(dt). \tag{5.3}$$

In this case, we just put $P = -\widetilde{\arg\Theta}$ and (5.2) follows.

5.2 Another Interpretation of $\sum_k \log|z_k|/|z_k| < \infty$

We can now illustrate these facts by our Theorem 1.4. Under the assumptions of Theorem 1.4, (5.2) is fulfilled with $P(x) = -2\log|E(x)|$. Indeed,

$$\arg\Theta = -2\arg E = -2\widetilde{\log|E|},$$

since E is an outer function in the upper half plane. Moreover, $\exp P(x) = 1/|E(x)|^2 \in L^1(dt)$, since $1/|E(x)| \in L^2(dt)$. The condition (5.3) is fulfilled if and only if $\sum_{k=1}^\infty \log b_k/b_k < \infty$. Since

$$\arg B(x) = 2\sum_{k=1}^\infty \arctan\Big(\frac{x}{b_k}\Big),$$

we have

$$\int_{-\infty}^{\infty} \frac{|\arg B(x)|}{1+x^2}\,dx = 4\sum_{k=1}^\infty \int_0^\infty \frac{\arctan(x/b_k)}{1+x^2}\,dx.$$

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

But

$$\int_0^\infty \frac{\arctan(x/b_k)}{1+x^2}\,dx \asymp \frac{1}{b_k} + \int_1^\infty \frac{\arctan(x/b_k)}{x^2}\,dx \asymp \frac{1}{b_k} + \int_{1/b_k}^\infty \frac{\arctan t}{t^2}\,dt \asymp \frac{\log b_k}{b_k}.$$

Therefore,

$$\int_{-\infty}^\infty \frac{|\arg B(x)|}{1+x^2}\,dx \asymp \sum_{k=1}^\infty \frac{\log b_k}{b_k}.$$

Acknowledgements We express our gratitude to Paul Koosis for his advice, friendly assistance and numerous fruitful discussions. His suggestions enormously influenced and improved our exposition in an essential way. In particular, he suggested the proof of Theorem 3.3 in Section 3.1.

We are also indebted to M. Sodin for interesting comments and to A. Baranov for helpful discussions on de Branges' spaces and on asymptotic behavior of E. We thank the referee for many valuable remarks.

References

[1] P. Ahern and D. Clark, *On Functions Orthogonal to Invariant Subspaces*. Acta Math. **124**(1970), 191–204.

[2] A. Alexandrov, *Invariant subspaces of the backward shift operator in the space H^p, $p \in (0,1)$*. Investigations on linear operators and the theory of functions, IX, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI) **92**(1979), 7–29.

[3] R. Boas, *Entire Functions*. Academic Press, 1954.

[4] A. Beurling, *On two problems concerning linear transformations in Hilbert space*. Acta Math. **81**(1949), 239–255.

[5] A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support*. Acta Math. **107**(1962), 291–309.

[6] L. de Branges, *Hilbert Spaces of Entire Functions*. Prentice Hall, 1968.

[7] ———, *Some Hilbert spaces of entire functions*. Trans. Amer. Math. Soc. **96**(1960), 259–295.

[8] J. Cima and W. Ross, *The Backward Shift on the Hardy Space*. Math. Surveys Monogr. **79**, Amer. Math. Soc., 2000.

[9] W. Cohn, *Unitary equivalence of restricted shifts*. J. Operator Theory (1) **5**(1981), 17–28.

[10] J. Conway, *Functions of One Complex Variable*. Second Edition, Springer Verlag, 1978.

[11] R. Douglas, H. Shapiro and A. Shields, *Cyclic vectors and invariant subspaces for backward shift operator*. Ann. Inst. Fourier (Grenoble) **20**(1970), 37–76.

[12] P. Duren, *Theory of H^p Spaces*. Academic Press, 1970.

[13] K. Dyakonov, *On the moduli and arguments of analytic functions of H^p that are invariant for the backward shift operator*. Sibirsk. Mat. Zh. **31**(1990), 64–79.

[14] J. Garnett, *Bounded Analytic Functions*. Academic Press, 1981.

[15] P. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*. Second Edition, Chelsea Publishing Company, 1957.

[16] ———, *A Hilbert Space Problem Book*. Second Edition, Springer Verlag, 1982.

[17] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*. Springer-Verlag, 1994.

[18] V. Havin and Javad Mashreghi, *Admissible majorants for model subspaces of H^2, Part II: fast winding of the generating inner function*. Canad. J. Math. **55**(2003), 1264–1301.

[19] H. Helson, *Lectures on Invariant Subspaces*. Academic Press, 1964.

[20] P. Koosis, *A relation between two results about entire functions of exponential type*. Mat. Fiz. Anal. Geom. **5**(1995), 212–231.

[21] ———, *A result on polynomials and its relation to another, concerning entire functions of exponential type*. Mat. Fiz. Anal. Geom. **5**(1998), 68–86.

[22] ———, *Introduction to H^p Spaces*. Second Edition, Cambridge Tracts in Math. **115**, 1998.

[23] ———, *The Logarithmic Integral I*. Cambridge Stud. Adv. Math. **12**, 1988.

[24] ———, *The Logarithmic Integral II*. Cambridge Stud. Adv. Math. **21**, 1992.

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press

[25] ———, *Leçons sur le Théorème de Beurling et Malliavin*. Les Publications CRM, Montréal, 1996.
[26] B. Levin, *Distribution of zeros of Entire Functions*. Transl. Math. Monogr. **5**, 1980, Amer. Math. Soc.
[27] J. Neuwirth and D. Newman, *Positive $H^{1/2}$ functions are constant*. Proc. Amer. Math. Soc. (5) **18**(1967), 958.
[28] B. Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*. North-Holland, 1970.
[29] N. Nikolski, *Treatise on the Shift Operator*. Springer 1986.
[30] I. Privalov, *Intégral de Cauchy*. Bulletin de l'Université Saratov, 1918.
[31] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*. Chelsea Publishing Company, 1962.
[32] A. Volberg, *Thin and thick families of rational functions*. Lecture Notes in Math. **864**(1981), 440–480.
[33] A. Volberg and S. Treil, *Embedding theorems for invariant subspaces of the inverse shift operator*. J. Soviet Math. (2) **42**(1988), 1562–1572.
[34] H. Woracek, *de Branges spaces of entire functions closed under forming difference quotients*. Integral Equations Operator Theory (2) **37**(2000), 238–249.
[35] A. Zygmund, *Trigonometric Series*. Vol I, Cambridge University Press, 1968.

Department of Mathematics
and Statistics
McGill University
Montreal, Quebec
H3A 2K6

Département de mathématiques
et de statistique
Université Laval
Laval, Québec
G1K 7P4
e-mail: Javad.Mashreghi@mat.ulaval.ca

Department of Mathematics
and Mechanics
St. Petersburg State University
Russia 198904
e-mail: havin@havin.usr.pu.ru

https://doi.org/10.4153/CJM-2003-048-8 Published online by Cambridge University Press