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THE DESCENDING CHAIN CONDITION ON SOLUTION SETS FOR SYSTEMS OF EQUATIONS IN GROUPS

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The Ehrenfeucht Conjecture **[5]** states that if $\mathbb{M}$ is a finitely generated free monoid with nonempty subset S, then there is a finite subset $T\subset S$ (a "test set") such that given two endomorphisms f and g on $\mathbb{M}$, f and g agree on S if and only if they agree on T. In **[4]**, the authors prove that the above conjecture is equivalent to the following conjecture: a system of equations in a finite number of unknowns in $\mathbb{M}$ is equivalent to a finite subsystem. Since a finitely generated free monoid embeds naturally into the free group with the same number of generators, it is natural to ask whether a free group of finite rank has the above property on systems of equations. A restatement of the question motivates the following.

Definition. A group G will be said to satisfy the descending chain conditions on solution sets for equations in k variables (denoted $DCC(k)$) if for all sequences of group words on a fixed set of k variables, say $w_1=w_1(x_1,\dots,x_k)$, $w_2=w_2(x_1,\dots,x_k),\dots$ there exists only finitely many l such that the solution set of the system $w_1=1$, $w_2=1,\dots,$ $w_l=1$ is strictly larger than the solution set of the system $w_1=1,\dots,w_{l+1}=1$.

Our question now becomes: does a non-Abelian free group satisfy $DCC(k)$ for all positive integers k? An affirmative answer to this question would yield an affirmative answer to the Ehrenfeucht Conjecture while a negative answer to this question would suggest a negative answer to the Ehrenfeucht Conjecture. See **[10]** for an interesting relation between systems of equations in the free monoid and systems of equations in the free group.

Clearly if H is a subgroup of G and G satisfies $DCC(k)$, then H satisfies $DCC(k)$.

In this paper we look at the $DCC(k)$ property in various groups and obtain some results for the free group which partially answer the above question.

Theorem 1. *Let $\mathbb{V}$ be a variety of groups and let $\mathbb{F}_k$ denote the k-generated (relatively) free group in $\mathbb{V}$. The following are equivalent.*

(1) *The ascending chain condition on normal subgroups holds in $\mathbb{F}_k$.*

(2) *All groups in $\mathbb{V}$ satisfy $DCC(k)$.*

Proof. We first show that (1) implies (2).

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Let $\mathbb{A}_k$ denote the k-generated free group and suppose that we have a group $G \in \mathbb{V}$ and a sequence of elements of $\mathbb{A}_k$, say $w_1 = w_1(x_1, \ldots, x_k)$, $w_2 = w_2(x_1, \ldots, x_k), \ldots$ such that we do not have the descending chain condition on this system. Suppose in particular that we have $(a_{i1}, a_{i2}, \ldots, a_{ik}) \in G$ which is a solution to $w_j = 1$ for $j = 1, 2, \ldots, l_i$ but is not a solution to $w_{l_i+1} = 1$, $i = 1, 2, \ldots$. The map α_i from $\mathbb{A}_k$ to G sending x_j to a_{ij} factors naturally through $\mathbb{F}_k$ with $\beta: \mathbb{A}_k \to \mathbb{F}_k$ and $\gamma_i = \mathbb{F}_k \to G$. Now $\beta(w_j(x_1, \ldots, x_k))$, $j = 1, 2, \ldots, l_i$ are all in the kernel of γ_i whereas $\beta(w_{l_i+1}(x_1, \ldots, x_k))$ is not. It follows that the normal subgroup of $\mathbb{F}_k$ generated by $\{\beta(w_j(x_1, \ldots, x_k))\}$, $j = 1, 2, \ldots, l_i$ does not contain $\beta(w_{l_i+1}(x_1, \ldots, x_k))$; hence, we have an infinite ascending chain of normal subgroups of $\mathbb{F}_k$.

We now prove that (2) implies (1). Let β denote the natural map from the k-generated free group $\mathbb{A}_k$ to $\mathbb{F}_k$ sending generators to generators. Suppose that $N_1 \subseteq N_2 \subseteq \ldots$ is an infinite ascending chain of normal subgroups of $\mathbb{F}_k$. For each positive integer l choose $w_l \in \mathbb{F}_k$ such that $\beta(w_l) \in N_{l+1} \backslash N_l$. Let $G = \prod_{l=1}^{\infty} \mathbb{F}_k / N_l$. By construction the system of equations $\{w_l = 1\}_{l=1}^{\infty}$ is not equivalent to a finite subsystem. This completes the proof of the theorem.

It should be noted that in the above theorem, if $\mathbb{F}_k$ does not satisfy the ascending chain condition on normal subgroups, then we can find a countable group in the variety which does not satisfy $DCC(k)$.

Corollary 2. *A nilpotent group satisfies $DCC(k)$ for all positive integers k.*

Proof. A finitely generated nilpotent group satisfies the ascending chain condition on subgroups [2, p. 1].

For the (relatively) free groups in a variety we have a "test set condition" equivalent to the descending chain condition on solution sets.

Theorem 3. *Let $\mathbb{F}_k$ denote the k-generated free group in a variety. Suppose that each nonempty subset of $\mathbb{F}_k$ has a finite test subset. Then $DCC(k)$ holds in $\mathbb{F}_k$. Conversely, if $DCC(2k)$ holds in $\mathbb{F}_k$, then the above test set condition also holds.*

Proof. Let $\mathbb{A}_k$ denote the k-generated free group generated by $x_1, \ldots, x_k$ and let $a_1, \ldots, a_k$ be a set of generators of $\mathbb{F}_k$. Let β denote the natural map from $\mathbb{A}_k$ to $\mathbb{F}_k$. Let $w_1, w_2, \ldots$ be a sequence of elements of $\mathbb{A}_k$. The subset of S of $\mathbb{F}_k$ will be the image of $\{w_i\}_{i=1}^{\infty}$ under β. By hypothesis, the set S has a finite test subset $T = \{\beta(w_1), \ldots, \beta(w_t)\}$. We claim that a solution to the equations $w_1 = 1, \ldots, w_t = 1$ is a solution to $w_i = 1$ for all i. If not, suppose that $(b_1, \ldots, b_k)$ is a solution to the first t equations but not a solution to $w_i = 1$ for some i. Define $\delta: \mathbb{F}_k \to \mathbb{F}_k$ by $a_j \to b_j$ and let $\gamma: \mathbb{F}_k \to \mathbb{F}_k$ be the map sending all elements to 1. Now $\gamma|_T = \delta|_T$, but $\gamma(\beta(w_i)) \neq \delta(\beta(w_i))$, a contradiction. This completes the first part of the proof.

Suppose that $\mathbb{F}_k$ satisfies $DCC(2k)$. Let $S \subset \mathbb{F}_k$ and let $U \subset \mathbb{A}_k$ denote the preimage of S under β. Given $w_j \in U$, let $w_j(y_1, \ldots, y_k) = w_j(z_1, \ldots, z_k)$ be a group equation, $j = 1, 2, \ldots$. By hypothesis, this system of equations has a finite equivalent subsystem, say the first l equations. We claim that $T = \{\beta(w_1), \ldots, \beta(w_l)\}$ is a test set for S. Suppose that f and g are endomorphisms of $\mathbb{F}_k$ which agree on T. If f sends a_j to b_j and g sends a_j to c_j, then

$\beta(w_i)(b_1,\dots,b_k)=\beta(w_i)(c_1,\dots,c_k)$, $i=1,2,\dots,l$. Thus $\beta(w_i)(b_1,\dots,b_k)=\beta(w_i)(c_1,\dots,c_k)$ for all $w_i \in U$; hence, f and g agree on S.

In contrast to nilpotent groups there exist solvable groups of solvable length 3 which do not satisfy $DCC(2)$ In [3] there is an example of a 2-generated, solvable (length 3) non-Hopfian group. Such a group cannot satisfy the ascending chain condition on normal subgroups and so neither can the 2-generated free group in the variety generated by the group. This provides us with the example.

We now strengthen Corollary 2 for certain Abelian groups.

Theorem 4. *The following conditions are equivalent for an Abelian group G.*

(1) *For each positive integer k there is a positive integer $\gamma(k)$ such that a strictly descending chain of solution sets of equations in k variables has length at most $\gamma(k)$.*

(2) *G is isomorphic to a direct product of a torsion-free Abelian group and direct product of finite cyclic groups each of order less than N for some fixed integer N.*

Proof. In fact, as we will see, the existence of $\gamma(1)$ is enough to ensure (2). By considering the equations $x^{n!}=1, x^{(n-1)!}=1,\dots,x=1$, we can see that if $\gamma(1)$ exists, then G is a group of bounded order. We use [9, Theorem 6 and Theorem 8] to complete the proof.

If G has the property described in (2), we look at the torsion-free and torsion part separately. A torsion-free Abelian group can be embedded into a rational vector space and a descending chain of solution sets has length at most k. On the other hand, the torsion subgroup of G can be embedded into a direct product of copies of a finite Abelian group. Since a solution set of the direct product is the direct product of a solution set in the finite group, we have the desired result.

We now give several results for free groups.

Theorem 5. *A free group satisfies $DCC(2)$. The maximal length of a strictly descending chain of solution sets of equations in 2 variables is 3.*

Proof. Let $\mathbb{A}_2$ be the free group generated by x and y and let A be any free group. If $w(x,y)$ is a non-trivial word in $\mathbb{A}_2$ and (a,b) is a solution in $\mathbb{A}$, then the image of $\mathbb{A}_2$ under the map sending x to a and y to b has rank at most 1 [8, Proposition 2.12]. We are now in an infinite cyclic group, and here a descending chain of solution sets has length at most 2. This completes the proof of the theorem.

In a non-Abelian free group, the system of equations $1=1$, $xyx^{-1}y^{-1}=1$, $xy^{-1}=1$, $x=1$ has a strictly descending chain of solutions sets and this shows that the length 3 can actually occur.

Theorem 6. *A non-Abelian free group satisfies $DCC(k)$ if and only if the k-generated free group, $\mathbb{A}_k$, satisfies the ascending chain condition on normal subgroups N such that $\mathbb{A}/N$ is residually free.*

Proof. Suppose $\mathbb{A}_k$ has normal subgroups $N_1 \subsetneqq N_2 \subsetneqq N_3 \subsetneqq \dots$ such that $\mathbb{A}_k/N_l$ is residually free for all l. Choose $w_i(x_1,\dots,x_k) \in N_i - N_{i-1}$ and $\gamma_i : \mathbb{A}_k/N_{i-1} \to \mathbb{A}$ ($\mathbb{A}$ non-Abelian free such that $\gamma_i(w_i N_{i-1}) \neq 1$ (since any non-Abelian free group is embedded in any other, we may take a common codomain for all the γ_i). Suppose $\gamma_i(x_j N_{i-1}) = a_j$ $1 \leqq j \leqq k$. Then, $w_i(a_1,\dots,a_k) \neq 1$, but $w_1(a_1,\dots,a_k) = \cdots = w_{i-1}(a_1,\dots,a_k) = 1$. Thus a non-Abelian free group does not satisfy $DCC(k)$.

Suppose a non-Abelian free group $\mathbb{A}$ does not satisfy $DCC(k)$ and let $w_1, w_2, \dots$ be a sequence of polynomials with a strictly descending chain of solutions sets. Let $\Phi_l = \{\phi : \mathbb{A}_k \to \mathbb{A};\ \phi(w_i) = 1\ i = 1,\dots,l\}$. Let

$$N_l = \bigcap_{\phi \in \Phi_l} \operatorname{Ker} \phi.$$

Then, $w_1, w_2, \dots, w_l \in N_l$, $w_{l+1} \notin N_l$, so the N_l form a strictly increasing chain of normal subgroups. Moreover, $\mathbb{A}_k/N_l$ is a subdirect product of the $\phi(\mathbb{A}_k) \subseteq \mathbb{A}$, $\phi \in \Phi_l$, and as subgroups of free groups are free this shows $\mathbb{A}_k/N_l$ is residually free. Thus the proof is complete.

A group G is said to be fully residually free [**1**] if for each finite subset S of G-$\{1\}$, there is a normal subgroup N of G such that $N \cap S = \emptyset$ and G/N is free. The group $\langle a,b,c,d : a^2b^2c^2d^2 = 1\rangle$ is an example of a fully residually free group that is not free [**1**, p. 414]. The direct product of two free groups is an example of a residually free group that is not fully residually free [**1**, p. 404].

Theorem 7. *A fully residually free group that is not free cannot be embedded into a finite direct product of free groups.*

Proof. Suppose that to the contrary, G is a fully residually free group with normal subgroups $N_1, \dots, N_l$ such that G/N_i is free and $\bigcap_{i=1}^{l} N_i = \langle 1 \rangle$. We may suppose, without loss of generality, that $N = \bigcap_{i=1}^{l-1} N_i \neq \langle 1 \rangle$ and $N_l \neq \langle 1 \rangle$. As $N_l \cap N = \langle 1 \rangle$, each element in N_l commutes with each element in N. By [**1**, Theorem 1], N is Abelian; hence it is contained in the centre of G [**1**, Lemma 1]. By a second use of [**1**, Theorem 1], we see that this cannot happen in a non-Abelian fully residually free group. This contradiction completes the proof of the theorem.

Corollary 8. *The* 4*-generated free group,* $\mathbb{A}_4$*, does not satisfy the descending chain condition on normal subgroups* N *such that* $\mathbb{A}_4/N$ *is residually free.*

Proof. Let α be the natural map from $\mathbb{A}_4$ to $G = \langle a,b,c,d : a^2b^2c^2d^2 = 1\rangle$ and let β be the embedding of G into a direct product of free groups. If we let γ_i denote the projection from the direct product onto the direct product of the first i factors, then the kernels of the maps $\gamma_i \circ \beta \circ \alpha$ form, by Theorem 7, an infinite descending chain of normal subgroups of the desired kind.

We conclude this paper by constructing, for each positive integer l, an independent system in a non-Abelian free group, of l equations in 3 variables. By independent we mean that for each equation there is a non-solution in the free group which is a solution to the remaining $l-1$ equations.

Let $[a_1, a_2, \ldots, a_l] = [[\ldots[[a_1, a_2], a_3]\ldots]]$ denote the generalized commutator. Let $v_n = v_n(x, y, z) = z^{-n}x^n y^{-1} z^n$, $n = 1, 2, \ldots$. Define $w_1 = w_1(x, y, z) = [v_2, v_3, \ldots, v_l]$, $w_j = w_j(x, y, z) = [v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_l]$ for $2 \leqq j < l$, and $w_l = w_l(x, y, z) = [v_1, v_2, \ldots, v_{l-1}]$. Let b and c be among a set of generators of a non-Abelian free group $\mathbb{A}$. Then (b, b^j, c) is a solution to $w_i = 1$ if $i \neq j$ but it is not a solution to $w_j = 1$.

The above example shows that even if a non-Abelian free group satisfies $DCC(3)$ there can be no uniform bound on the length of a strictly descending chain of solution sets, as there is for systems of equations in 2 variables (Theorem 5).

Remark. Since this paper was submitted, the authors have succeeded in proving Ehrenfeucht's Conjecture. The proof makes use of Theorem 1 and some basic properties of metabelian groups. The proof will appear in Theoretical Computer Science. Further results on systems of equations in free and nilpotent groups will appear in a forthcoming paper.

REFERENCES

1. B. Baumslag, Residually free groups, *Proc. London Math. Soc.* (3) **17** (1967), 402–418.

2. G. Baumslag, Lecture Notes on Nilpotent Groups (CBMS Series Number 2, Amer. Math. Soc. 1969).

3. G. Baumslag and D. Solitar, Some two generator one relator non-Hopfian groups, *Bull. Amer. Math. Soc.* **68** (1962), 199–201.

4. K. Culik II and J. Karhumaki, Systems of equations over a free monoid and Ehrenfeucht's Conjecture, *Discrete Math.* **43** (1983), 139–153.

5. K. Culik II and A. Salomaa, Test sets and checking words for homomorphism equivalence, *J. of Computers and System Sciences* **20** (1980), 379–395.

6. S. M. Gersten, On fixed points of automorphisms of finitely generated free groups, *Bull. Amer. Math. Soc.* **8** (1983), 451–454.

7. R. Lyndon, Equations in groups, *Bol. Soc. Brasil Mat.* **11** (1980), 79–102.

8. R. Lyndon and P. Schupp, *Combinatorial Group Theory* (Springer-Verlag, 1977).

9. I. Kaplansky, *Infinite Abelian groups* (U. of Michigan Press, Ann Arbor, 1969).

10. G. S. Makanin, Systems of equations in free groups, *Siberian Math. J.* **13** (1972), 402–408.

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