UNIQUENESS THEOREMS FOR A GENERAL CLASS OF FUNCTIONAL EQUATIONS

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In a previous paper [1] J. Aczél has shown the following

THEOREM 1. *If in the (closed, half-closed or open, finite or infinite) interval* $\langle A, B\rangle$

$$f[F(x, y)] = H[f(x), f(y), x, y] \tag{$*$}$$

and there f, F *are continuous,* F *intern (the value* $F(x, y)$ *lies strictly between* x *and* y*) and* $u \to H(u, v, x, y)$ *or* $v \to H(u, v, x, y)$ *are injective (i.e.*

$$H(u_1, v, x, y) = H(u_2, v, x, y) \Rightarrow u_1 = u_2$$

or

$$H(u, v_1, x, y) = H(u, v_2, x, y) \Rightarrow v_1 = v_2),$$

then the functional equation $(*)$ *with the initial conditions*

$$f(a) = c, f(b) = d \qquad (a, b \in \langle A, B\rangle)$$

has at most one solution.

The above result has been established for functions with real variables. In the sequel we extend the notion of internness to vector spaces and derive results in topological vector spaces.

DEFINITIONS, NOTATIONS AND PRELIMINARIES: For two distinct points x and y of a vector space (v.s.) E over the real field $\mathbf{R}$, we denote the line joining x and y by

$$L\langle x, y\rangle = \{y + t(x - y) : t \in \mathbf{R}\}$$

and the (open) line segment joining x and y by

$$L(x, y) = \{y + t(x - y) : t \in (0, 1)\}.$$

A mapping F defined on some subset S of $E \times E$ into E is said to be *intern* if

$$F(x, y) \in L(x, y)$$

whenever $(x, y) \in S$ with $x \neq y$.

Given a vector space E over the real field $\mathbf{R}$ and a topology $\mathfrak{T}$ on E, the pair

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$(E, \mathfrak{T})$ is called a topological vector space (t.v.s.) over $\mathbf{R}$ if these two axioms are satisfied:

$$(LT)_1 \ (x, y) \to x+y \text{ is continuous on } E \times E \text{ into } E,$$

$$(LT)_2 \ (t, x) \to tx \text{ is continuous on } \mathbf{R} \times E \text{ into } E.$$

The field $\mathbf{R}$ is always considered to be endowed with its usual absolute value, under which it is a non-discrete valuated field. In addition it is always considered under its usual order ([6], page 11-12). Every t.v.s. E considered here is over the real field $\mathbf{R}$ if the field is not mentioned.

If E is a Hausdorff topological vector space over $\mathbf{R}$ and $\mathfrak{T}$ is the relative topology on $L\langle x, y\rangle$, the injective mapping Φ_x^y of $L\langle x, y\rangle$ onto $\mathbf{R}$ defined by

$$\Phi_x^y(y+t(x-y)) = t \qquad \text{for all } t \in \mathbf{R}$$

is a homeomorphism ([6], page 21).

If E and N are topological spaces with N Hausdorff and $f_1, f_2 : E \to N$ are continuous mappings, then the set

$$S = \{x : x \in E, f_1(x) = f_2(x)\}$$

is closed in E.

THEOREM 2. *Let E_1 be a closed subset of a Hausdorff t.v.s. E and let $F : E_1 \times E_1 \to E_1$ be intern, continuous in both variables. Let N be a set and $f_1, f_2 : E_1 \to N$ be mappings satisfying the functional equation*

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y]$$

where the mapping $H : N \times N \times E_1 \times E_1 \to N$ is injective either in its first variable or in its second variable. If f_1 and f_2 are identical on some E_1-neighbourhood V of a point $a \in E_1$, then f_1 and f_2 are identical on the entire domain E_1.

PROOF. Without loss of generality we may suppose H is injective in its first variable. Let $x_0 \in E_1 \setminus \{a\}$ be given. Define

$$F^0(x, a) = x$$

and

$$F^{n+1}(x, a) = F(F^n(x, a), a)$$

recursively for $x \in E_1$ and $n \in \omega = \{0, 1, 2, \cdots\}$. It follows from the internness of F that $\langle \Phi_{x_0}^a(F^n(x_0, a))\rangle_{n \in \omega}$ is a decreasing sequence of positive real numbers and therefore converges to a definite limit, say

$$\lim_{n \to \infty} \Phi_{x_0}^a(F^n(x_0, a)) = \alpha \geqq 0.$$

We now claim that the sequence $\langle F^n(x_0, a)\rangle_{n \in \omega}$ in $L\langle x_0, a\rangle$ converges to a. For $\Phi_{x_0}^a$ is a homeomorphism so that

$$\lim_{n\to\infty} F^n(x_0, a) = \Phi_{x_0}^{a-1}(\alpha) \in \bar{E}_1 = E_1$$

and the fact that F is continuous in its first variable yields

$$\begin{aligned} F(\Phi_{x_0}^{a-1}(\alpha), a) &= F(\lim_{n\to\infty} F^n(x_0, a), a) \\ &= \lim_{n\to\infty} F(F^n(x_0, a), a) \\ &= \lim_{n\to\infty} F^{n+1}(x_0, a) \\ &= \Phi_{x_0}^{a-1}(\alpha). \end{aligned}$$

In view of the internness of F, we must have $\Phi_{x_0}^{a-1}(\alpha) = a$ and convergence of $\langle F^n(x_0, a)\rangle_{n\in\omega}$ to its limit a follows. Since V is an E_1-neighbourhood of a, there exists $N \in \omega\setminus\{0\}$ such that

$$F^N(x_0, a) \in V$$

and consequently

$$f_1(F^N(x_0, a)) = f_2(F^N(x_0, a)).$$

By (*) we have

$$\begin{aligned} &H[f_1(F^{N-1}(x_0, a)), f_1(a), F^{N-1}(x_0, a), a] \\ &= f_1(F^N(x_0, a)) \\ &= f_2(F^N(x_0, a)) \\ &= H[f_2(F^{N-1}(x_0, a)), f_2(a), F^{N-1}(x_0, a), a] \end{aligned}$$

so that

$$f_1(F^{N-1}(x_0, a)) = f_2(F^{N-1}(x_0, a))$$

follows from the injectivity assumption on H in its first variable. Proceeding recursively we obtain

$$f_1(F^0(x_0, a)) = f_2(F^0(x_0, a))$$

or simply

$$f_1(x_0) = f_2(x_0).$$

The point $x_0 \in E_1\setminus\{a\}$ being arbitrary, f_1 and f_2 are identical on E_1.

THEOREM 3. *Let $F: E_1 \times E_1 \to E_1$ be an intern function defined on a closed subset E_1 of a Hausdorff t.v.s. E over $\mathbf{R}$, and let N be a Hausdorff space. Suppose $f_1, f_2: E_1 \to N$ are continuous mappings satisfying*

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y]$$

where H is a mapping from $N \times N \times E_1 \times E_1$ into N. Then the set

$$S = \{x : x \in E_1, f_1(x) = f_2(x)\}$$

is closed and convex.

PROOF. Since E_1 is closed in E and S is closed in E_1, S is also closed in E. For any two distinct points x and y of S, consider the line $L\langle x, y\rangle$ and its relative topology $\mathfrak{T}$. The map $\Phi_x^y: L\langle x, y\rangle \to \mathbf{R}$ is a homeomorphism. In view of this homeomorphism $L(x, y)\setminus S$ is open and can be represented as a countable union of disjoint open components (or equivalently, open intervals, open segments). Suppose, if possible, that $L(x, y)\setminus S$ is non-empty and there exists an open component of it, say $L(x_1, y_1)$. We must have $x_1, y_1 \in S$ and $L(x_1, y_1) \cap S = \emptyset$. However by (*)

$$\begin{aligned} f_1(F(x_1, y_1)) &= H[f_1(x_1), f_1(y_1), x_1, y_1] \\ &= H[f_2(x_1), f_2(y_1), x_1, y_1] \\ &= f_2[F(x_1, y_1)] \end{aligned}$$

we have $F(x_1, y_1) \in S$. Also by the interness of F, $F(x_1, y_1) \in L(x_1, y_1)$. Now $F(x_1, y_1) \in L(x_1, y_1) \cap S$ leads to a contradiction. So $L(x, y)\setminus S = \emptyset$ and $L(x, y) \subseteq S$. The points x, y being arbitrary, S is convex.

We are now able to state our final result which is an immediate consequence of the previous two theorems.

THEOREM 4. *Let E_1 be a closed subset of a Hausdorff t.v.s. E, and let $F: E_1 \times E_1 \to E_1$ be intern, continuous in both variables. Let N be a Hausdorff space. Suppose $f_1, f_2 : E_1 \to N$ are continuous mappings satisfying the functional equation*

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y]$$

where the mapping $H: N \times N \times E_1 \times E_1 \to N$ is either

injective in its first variable

$$(1) \qquad \text{i.e. } H(u_1, v, x, y) = H(u_2, v, x, y) \Rightarrow u_1 = u_2$$

or

injective in its second variable

$$(2) \qquad \text{i.e. } H(u, v_1, x, y) = H(u, v_2, x, y) \Rightarrow v_1 = v_2$$

If f_1 and f_2 are identical on some subset A of E_1 whose closed convex hull $\overline{\Gamma A}$ has non-empty interior in E_1 (interior taken in E_1), then f_1 and f_2 are identical on the entire E_1.

COROLLARY. *If in Theorem 4, E is locally convex Hausdorff of dimension n and $A = \{a_i : i = 1, 2, \cdots, n+1\}$ is such that $\{a_i - a_1 : i = 2, 3, \cdots, n+1\}$ is linearly independent, then there exists at most one continuous solution of (*) satisfying the $n+1$ initial conditions*

$$f(a_i) = b_i \qquad i = 1, 2, \cdots, n+1.$$

REMARKS. In case $n = 1$, this gives Theorem 1 for closed $\langle A, B\rangle$. However the assumption that $\langle A, B\rangle$ is closed may be omitted as any interval $\langle A, B\rangle$

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with $a, b \in \langle A, B\rangle$ can be represented as a union of closed intervals containing a and b. For Fréchet spaces, A may be chosen as a subset whose closed convex hull $\overline{\Gamma A}$, after a translation, contains a barrel.

Theorems 2, 3 and 4 are still valid if we do not assume E to be Hausdorff. In this case the map Φ_x^y may fail to be a homeomorphism, and this happens only when $\mathfrak{T}$ is indiscrete (on $L\langle x, y\rangle$). Extra discussions lead to the same conclusions.

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