ON THE WATSON AND LAPLACE TRANSFORMATIONS

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Let k be the kernel of a Watson transformation; that is $k(x)/x \in L_2(0,\infty)$, and if x and y are positive,

$$(1) \qquad \int_0^\infty k(xu)\,\overline{k(yu)}\,u^{-2}du = \min(x,y)\,.$$

Then if g is the transform of $F \in L_2(0,\infty)$, that is if

$$(2) \qquad g(x) = \frac{d}{dx}\int_0^\infty F(y)\,\overline{k(xy)}\,dy/y\,,$$

it is known that $g \in L_2(0,\infty)$, that

$$(3) \qquad F(x) = \frac{d}{dx}\int_0^\infty g(y)\,k(xy)\,dy/y,$$

and that

$$(4) \qquad \int_0^\infty |g(x)|^2 dx = \int_0^\infty |F(x)|^2 dx\,.$$

For these results see [1; theorem 79]. Also it follows from (4) that if $F_i \in L_2(0,\infty)$, $i = 1,2$, and if the transform of F_i is g_i, then

$$(5) \qquad \int_0^\infty g_1(x)\,\overline{g_2(x)}\,dx = \int_0^\infty F_1(x)\,\overline{F_2(x)}\,dx\,.$$

Occasionally it happens that it is difficult to use (1) when one wants to test whether a particular function is a Watson kernel, since the integration required in (1) may be very difficult to perform. An alternative condition to (1) is known in terms of the L_2-Mellin transform of k, but this later transform may be hard to compute. Hence it seems worthwhile to find other conditions alternative to (1). This we shall do here in terms of the Laplace transform of k, a transform usually much easier to compute than the Mellin transform.

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To this end we define

$$(6) \qquad K(s) = s \int_0^\infty e^{-st} k(t)dt, \quad \text{Re } s > 0 ,$$

and in theorem 1, we shall find necessary and sufficient conditions, in terms of K, that k be a Watson kernel. Also, we shall find necessary and sufficient conditions that K be represented in the form (6), with k a Watson kernel; this forms the content of theorem 2.

Two remarks are in order at this point. The first is that in [5], we used K as defined by (6) to find a number of inversion formulae for Watson transformations. The second remark is that if k is absolutely continuous on $[0,R]$ for every $R > 0$, with $k(0) = 0$, and if $e^{-sx} k'(x) \in L(0,\infty)$ for each $s > 0$, then

$$(7) \qquad K(s) = \int_0^\infty e^{-st} k'(t)dt .$$

This follows either from (6) on integration by parts, or from the operational rules for the Laplace transform.

THEOREM 1. A necessary and sufficient condition that a function k, with $k(x)/x \in L_2(0,\infty)$, be a Watson kernel is that if K is defined by (6), then for positive u and v,

$$(8) \qquad \int_0^\infty K(ux)\,\overline{K(vx)}\,dx = \frac{1}{u+v} .$$

Proof of necessity. Suppose k is a Watson kernel.

Since by [2; chapter 3, §2, theorem 1] we can differentiate a Laplace integral as often as we like under the integral sign, it follows that if x and s are positive,

$$x^{-1}\,\overline{K(s/x)} = (s/x^2) \int_0^\infty e^{-sy/x}\,\overline{k(y)}\,dy$$

$$= \frac{d}{dx} \int_0^\infty e^{-sy/x}\,\overline{k(y)}\,dy/y$$

$$= \frac{d}{dx} \int_0^\infty e^{-sy}\,\overline{k(xy)}\,dy/y ;$$

that is, if

$$F_s(x) = e^{-sx}, \text{ and } g_s(x) = x^{-1}\,\overline{K(s/x)} ,$$

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then g_s is the transform of F_s . But then from (5), if u and v are positive,

$$\int_0^\infty K(ux)\,\overline{K(vx)}\,dx = \int_0^\infty K(u/x)\,\overline{K(v/x)}\,dx/x^2$$

$$= \int_0^\infty \overline{g_u(x)}\,g_v(x)\,dx = \int_0^\infty \overline{F_u(x)}\,F_v(x)\,dx$$

$$= \int_0^\infty e^{-(u+v)x}\,dx = \frac{1}{u+v} .$$

Proof of sufficiency. Suppose K is given by (6), where $k(x)/x \in L_2(0,\infty)$, and K satisfies (8). Then from (8), if u and v are positive

$$(9) \qquad \int_0^\infty K(u/x)\,\overline{K(v/x)}\,dx/x^2 = \frac{1}{(u+v)} ,$$

and in particular, setting $v = u$,

$$(10) \qquad \int_0^\infty |K(u/x)|^2\,dx/x^2 = \frac{1}{2u} .$$

Now if $u > 0$,

$$(11) \quad K(u/x) = (u/x) \int_0^\infty e^{-ut/x}\,k(t)dt = u \int_0^\infty e^{-ut}k(xt)dt ,$$

and hence substituting (11) in (9) and dividing by u we obtain

$$(12) \quad \int_0^\infty \overline{K(v/x)}\,dx/x^2 \int_0^\infty e^{-ut}k(xt)dt = \frac{1}{u(u+v)}$$

$$= \int_0^\infty e^{-ut}((1-e^{-vt})/v)\,dt.$$

But then if we may interchange the order of integrations in (12) we obtain for each $v > 0$ and all $u > 0$,

$$\int_0^\infty e^{-ut} \left\{ \int_0^\infty \overline{K(v/x)}\,k(tx)\,dx/x^2 - ((1-e^{-vt})/v) \right\} dt = 0 ,$$

and hence from [2; chapter 2, § 9, theorem 2], if v and t are positive

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$$(13)\qquad \int_0^\infty \overline{K(v/x)}\, k(tx)\, dx/x^2 = (1 - e^{-vt})/v .$$

To justify the interchange of the order of integrations, it suffices to show that if $u > 0$, then

$$I = \int_0^\infty e^{-ut}dt \int_0^\infty \left| \overline{K(v/x)}\, k(tx) \right| dx/x^2 < \infty .$$

But from (10) and Schwartz's inequality, the inner integral in I is smaller than

$$\left\{ \int_0^\infty |K(v/x)|^2 \, dx/x^2 \right\}^{\frac{1}{2}} \left\{ \int_0^\infty |k(tx)|^2 \, dx/x^2 \right\}^{\frac{1}{2}}$$

$$= (t/2v)^{\frac{1}{2}} \left\{ \int_0^\infty |k(x)|^2 \, dx/x^2 \right\}^{\frac{1}{2}} = Mt^{\frac{1}{2}} ,$$

and thus

$$I \leq M \int_0^\infty e^{-ut}\, t^{\frac{1}{2}}\, dt < \infty ,$$

and (13) is valid.

But then if we change u to v in (11) and take conjugates, and then substitute this expression for $\overline{K(v/x)}$ in (13), we obtain after dividing by v, if $v > 0$

$$\int_0^\infty k(tx)\, dx/x^2 \int_0^\infty e^{-vs}\, \overline{k(xs)} ds = (1 - e^{-vt})/v^2$$

$$(14)\qquad = \int_0^\infty e^{-vs} \min(s,t) ds ,$$

and then if we may interchange the order of integrations in (14) we deduce that for all $v > 0$,

$$\int_0^\infty e^{-vs} \left\{ \int_0^\infty k(tx)\, \overline{k(sx)}\, dx/x^2 - \min(s,t) \right\} ds = 0 ,$$

and hence from [2; chapter 2, § 9, theorem 2], if s and t are positive,

$$(15)\qquad \int_0^\infty k(tx)\, \overline{k(sx)}\, dx/x^2 = \min(t,s) .$$

To justify the interchange of the order of integrations in (14), it suffices to show that if v and t are positive, then

$$J = \int_0^\infty e^{-vs} ds \int_0^\infty |k(tx)\overline{k(sx)}| \; dx/x^2 < \infty .$$

But from Schwartz's inequality, the inner integral in J is smaller than

$$\left\{ \int_0^\infty |k(tx)|^2 dx/x^2 \right\}^{\frac{1}{2}} \left\{ \int_0^\infty |k(sx)|^2 dx/x^2 \right\}^{\frac{1}{2}}$$

$$= (st)^{\frac{1}{2}} \int_0^\infty |k(x)|^2 dx/x^2 = Ns^{\frac{1}{2}} ,$$

and thus

$$J \le N \int_0^\infty e^{-vs} s^{\frac{1}{2}} ds < \infty ,$$

and (15) is valid. But (15) is just (1) with x and y replaced respectively by t and s. Hence (1) is valid, and k is a Watson kernel.

As an example of the use of (8), suppose

$$k(x) = (2/\pi)^{\frac{1}{2}} \sin x .$$

Then since $k(x)/x = O(x^{-1})$ as $x \to \infty$, $k(x)/x \in L_2(0, \infty)$. From [3; § 4.7(1)], $K(s) = (2/\pi)^{\frac{1}{2}} s/(s^2 + 1)$, and if u and v are positive

$$\int_0^\infty K(ux)\overline{K(vx)} dx = \frac{2}{\pi} uv \int_0^\infty \frac{x^2 dx}{(u^2x^2 + 1)(v^2x^2 + 1)}$$

$$= \frac{2}{\pi} \frac{uv}{u^2 - v^2} \int_0^\infty \left\{ \frac{1}{v^2x^2 + 1} - \frac{1}{u^2x^2 + 1} \right\} dx$$

$$= \frac{uv}{u^2 - v^2} \left\{ \frac{1}{v} - \frac{1}{u} \right\} = \frac{1}{u + v} ,$$

and k is the kernel of a Watson transformation - the transformation being the Fourier cosine transformation in this case.

Now if K is represented in the form (6), then $K(s)/s$ is the Laplace transform of a function of the form $t\,\phi(t)$, with $\phi \in L_2(0, \infty)$. Now from [4; theorem 3], a necessary and sufficient condition that a function F be the Laplace transform

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of a function of the form $t\,\phi(t)$ with $\phi \in L_2(0, \infty)$ is that

$$\int_0^\infty x\,dx \int_{-\infty}^\infty |F(x+iy)|^2 dy < \infty .$$

Applying this to $K(s)/s$, we arrive at our next theorem.

THEOREM 2. Necessary and sufficient conditions that an analytic function $K(s)$, regular for $\operatorname{Re} s > 0$, be represented in the form (6), where k is a Watson kernel are that

$$(16) \qquad \int_0^\infty x\,dx \int_{-\infty}^\infty \frac{|K(x+iy)|^2}{x^2+y^2}\, dy < \infty ,$$

and that (8) hold.

Theorem 2 can be used to "discover" kernels. For example

$$\frac{1}{u+v} = \int_0^\infty e^{-(u+v)x} dx ,$$

so that if $K(s) = e^{-s}$, K satisfies (8). But it also satisfies (16), for

$$\int_0^\infty x\,dx \int_{-\infty}^\infty \frac{|K(x+iy)|^2}{(x^2+y^2)}\, dy = \int_0^\infty xe^{-x} dx \int_{-\infty}^\infty \frac{dy}{x^2+y^2} = \pi .$$

Hence e^{-s}/s is the Laplace transform of a Watson kernel k.

From [3; 5.5(1)],

$$k(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & x > 1 \end{cases},$$

and from (2), the transformation is just the so-called "elementary" Watson transformation

$$g(x) = \frac{1}{x} F\left(\frac{1}{x}\right) .$$

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