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PAPER

A short proof of the Frobenius property for generic fibrations

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Abstract
We give a simple diagrammatic proof of the Frobenius property for generic fibrations that does not depend on any additional structure on the interval object such as connections.

Keywords: Category theory; homotopy type theory; generic fibrations; right proper model structures

1. Introduction

Let $\mathscr{C}$ be a locally cartesian closed category equipped with a class of morphisms called fibrations. The *Frobenius property* for $\mathscr{C}$ says that if $f : X \to Y$ and $p : Y \to Y'$ are fibrations of $\mathscr{C}$, then so is the pushforward $p_* f : X' \to Y'$. This condition arises when modeling Pi types in intensional type theory, because a type-in-context $\Gamma \vdash A$ type is interpreted as a fibration $f : A \to \Gamma$. If the fibrations are part of a suitable model structure on $\mathscr{C}$, then the Frobenius property is equivalent to the condition that the model category $\mathscr{C}$ is right proper (Gambino and Sattler 2017).

The Frobenius property can also serve as an intermediate step toward establishing the existence of a model category structure on $\mathscr{C}$ with the given class of fibrations. This is part of a broader strategy of using notions originating in type theory to construct model category structures, as explained in Awodey 2023). A particular class of fibrations often used in this context is the *generic* (or *unbiased*) fibrations with respect to a given "interval" object I of $\mathscr{C}$. These fibrations can be defined in terms of a lifting property involving the "generic point" $\delta : 1 \to \mathrm{I}$ obtained by passing to the slice category $\mathscr{C}/\mathrm{I}$, though we will find it convenient to use a more direct description (see Definition 1).

The purpose of this note is to give a simple, diagrammatic proof of the Frobenius property for generic fibrations that applies in wide generality (Corollary 8). In particular, it applies to cartesian cubical sets, and so it can be used to prove Corollary 73 of Awodey 2023). To explain the relationship between this proof and existing proofs in the literature, we briefly outline our strategy.

In contexts where one either already has a model category structure or is in the process of constructing one, the fibrations are the right class of a weak factorization system on $\mathscr{C}$, whose left class we refer to as trivial cofibrations. By standard adjunction arguments, the following two statements are then equivalent:

(1) The pushforward of a fibration along a fibration is a fibration (the Frobenius property).
(2) The pullback of a trivial cofibration along a fibration is a trivial cofibration.

One could therefore either try to prove (1) directly or instead try to prove (2).

Statement (1) is closer to the original type-theoretic motivation (the existence of Pi types). However, directly proving (1) involves a lot of reasoning about pushforwards, which is difficult to fit into the usual diagrammatic style of category theory (Awodey 2023, Section 5). Indeed, the original proofs of the Frobenius property were formulated as explicit type-theoretic constructions, as in Coquand (2014), Cohen et al. (2018), and Angiuli et al. (2021). Hazratpour and Riehl (2024), whose main theorem is closely related to ours, introduce a 2-categorical calculus of pasting diagrams and mates in order to systematize the required verifications.

Statement (2) appears more amenable to ordinary category-theoretic methods. The general approach to proving such a statement is well-known: reduce to the case of pulling back a generating trivial cofibration u along a fibration p, and then try to express the pullback p^*u as the retract of another generating trivial cofibration v, via a diagram obtained using the lifting property of p. However, it is trickier than one might expect to write down the correct lifting problem and retraction diagram. When the interval object I is equipped with extra structure such as connections, this task becomes a bit easier. Gambino and Sattler (2017) give a diagrammatic proof of statement (2) in a setting where the interval has connections. (They use a different definition of fibrations than the one considered here, but the two definitions become equivalent in the presence of connections.) In the category of cartesian cubical sets, however, the interval object (the 1-cube) does not have connections, so a different proof is required. The contribution of this work is to show that connections are not required in order to give a simple diagrammatic proof of statement (2) for the class of generic fibrations.

2. Generic Fibrations

In this section, we briefly review the definition of generic fibrations. Our terminology and notation mostly follow Awodey (2023).

For this section and the next one, we fix a category $\mathscr{C}$ and a class of morphisms of $\mathscr{C}$ called *cofibrations*, subject to the following standing hypotheses:

(H1) $\mathscr{C}$ has finite limits and finite colimits, and for any morphism $f : X' \to X$ of $\mathscr{C}$, the pullback functor $f^* : \mathscr{C}/X \to \mathscr{C}/X'$ preserves finite colimits.

(H2) The cofibrations are closed under pullback.

(H3) Any morphism whose domain is the initial object of $\mathscr{C}$ is a cofibration.

For instance, these hypotheses are satisfied whenever $\mathscr{C}$ is a finitely cocomplete, locally cartesian closed category (such as a topos) and the cofibrations of $\mathscr{C}$ satisfy conditions (H2) and (H3). In particular, they hold when $\mathscr{C}$ is the category of cartesian cubical sets and the cofibrations satisfy the axioms of Definition 9 of Awodey (2023). Note that we do not assume that every cofibration is a monomorphism.

Next, fix an "interval" object I of $\mathscr{C}$. In homotopy theory, we would traditionally ask that I also be equipped with "endpoint inclusions" δ_0, $\delta_1 : 1 \to \mathrm{I}$, and we would construct generating trivial cofibrations by forming the pushout product of a cofibration $c : C \to Z$ with an endpoint inclusion $\delta_\varepsilon : 1 \to \mathrm{I}$, $\varepsilon = 0$ or 1. The result is an "open box inclusion" $c \otimes \delta_\varepsilon : Z \amalg_C C \times \mathrm{I} \to Z \times \mathrm{I}$, which includes either the bottom or the top face of the box according to whether ε equals 0 or 1. To define "generic" (or "unbiased") fibrations, however, we consider a more general class of open box inclusions in which, informally, the bottom or top face of the box is replaced by a "cross-section," the graph of an arbitrary morphism $i : Z \to \mathrm{I}$.

Definition 1 (Awodey 2023, Definition 36). Given a cofibration $c : C \to Z$ and a morphism $i : Z \to \mathrm{I}$, we write $c \otimes_i \delta : Z \amalg_C C \times \mathrm{I} \to Z \times \mathrm{I}$ for the "cogap map" of the square below.

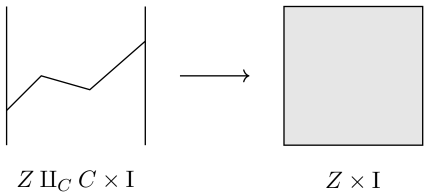


Figure 1. A typical generating trivial cofibration $c \otimes_i \delta : Z \amalg_C C \times \mathrm{I} \to Z \times \mathrm{I}$. Here $c : C \to Z$ is the inclusion of the endpoints of an interval, and $i : Z \to \mathrm{I}$ is a general morphism, represented here as a "piecewise linear" function.

$$\begin{array}{ccc} C & \xrightarrow{\langle 1, ic \rangle} & C \times \mathrm{I} \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c \times 1} \\ Z & \xrightarrow{\langle 1, i \rangle} & Z \times \mathrm{I} \end{array}$$

(The symbol δ is a fixed piece of notation from Awodey 2023), where its meaning is explained.)

Morphisms of this form $c \otimes_i \delta$ are called generating trivial cofibrations. A morphism of $\mathscr{C}$ is a fibration if it has the right lifting property with respect to all generating trivial cofibrations (Figure 1).

Remark 2. The terms "generating" and "trivial" notwithstanding, we do not assume a priori that the generating trivial cofibrations actually generate a weak factorization system, nor that they are related to a model structure on $\mathscr{C}$. Note that the generating trivial cofibrations typically form a proper class, so that even when $\mathscr{C}$ is locally presentable, we cannot use Quillen's small object argument to construct a weak factorization system whose right class is the class of fibrations.

Lemma 3 (Awodey 2023, Remark 31). *For any object X of $\mathscr{C}$ and morphism $i : X \to \mathrm{I}$, the graph $\langle 1, i \rangle : X \to X \times \mathrm{I}$ is isomorphic to a generating trivial cofibration.*

Proof. By (H1), the functor $- \times \mathrm{I}$ preserves the initial object 0 of $\mathscr{C}$, and by (H3), the unique morphism $c : 0 \to X$ is a cofibration. Therefore, $\langle 1, i \rangle : X \to X \times \mathrm{I}$ is isomorphic to the generating trivial cofibration $c \otimes_i \delta$. □

Note that this morphism $\langle 1, i \rangle : X \to X \times \mathrm{I}$ is automatically a monomorphism (even if not every cofibration is a monomorphism) since it admits the retraction $\mathrm{pr}_1 : X \times \mathrm{I} \to X$.

Lemma 4. *For any cofibration $c : C \to Z$ and morphism $i : Z \to \mathrm{I}$, the square appearing in Definition 1 is a pullback square, and the morphism $c \times 1 : C \times \mathrm{I} \to Z \times \mathrm{I}$ is also a cofibration.*

Proof. These statements follow from applying the pullback cancellation property repeatedly in the following diagram, whose top-left square is the square in question, and using (H2).

$$\begin{array}{ccccc} C & \xrightarrow{\langle 1, ic \rangle} & C \times \mathrm{I} & \xrightarrow{\mathrm{pr}_1} & C \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c \times 1} & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow{\langle 1, i \rangle} & Z \times \mathrm{I} & \xrightarrow{\mathrm{pr}_1} & Z \\ & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ & & \mathrm{I} & \longrightarrow & 1 \end{array}$$

□

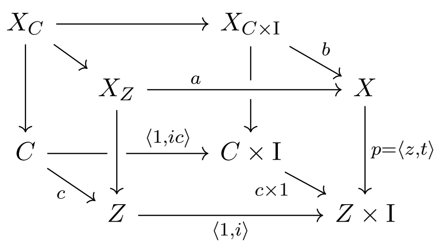
Example 5. Let $\mathscr{C}$ be the category of simplicial sets with the monomorphisms as its cofibrations and $\mathrm{I} = \Delta^1$ as the interval object. Then the fibrations in the sense of Definition 1 agree with the fibrations of the Kan–Quillen model structure, that is, the usual Kan fibrations of simplicial sets. To see this, note that any “open prism inclusion” $j_{n,\varepsilon} : \Delta^n \times \{\varepsilon\} \cup \partial\Delta^n \times \Delta^1 \to \Delta^n \times \Delta^1$ can be obtained as a generating trivial cofibration (in the sense of Definition 1) by taking c to be the boundary inclusion $c : \partial\Delta^n \to \Delta^n$ and i to be the constant morphism $i : \Delta^n \to \Delta^0 \xrightarrow{\varepsilon} \Delta^1$ at the vertex of Δ^1 specified by ε. It is well-known (Gabriel and Zisman 1967) that the morphisms $j_{n,\varepsilon}$ generate the class of anodyne extensions, in the sense that the Kan fibrations (usually instead defined using horn inclusions) are precisely the morphisms of simplicial sets that have the right lifting property with respect to all of the $j_{n,\varepsilon}$. Conversely, any generating trivial cofibration $c \otimes_i \delta : Z \amalg_C C \times \Delta^1 \to Z \times \Delta^1$ is an anodyne extension, that is, an acyclic cofibration in the Kan–Quillen model structure. This follows from left properness and the two-out-of-three property, since the horizontal maps in the diagram of Definition 1 are one-sided inverses to weak equivalences $\mathrm{pr}_1 : C \times \Delta^1 \to C$, $\mathrm{pr}_1 : Z \times \Delta^1 \to Z$.

3. The Frobenius Property

Proposition 6. *The pullback of a generating trivial cofibration along a fibration is a retract of a generating trivial cofibration.*

Proof. A generating trivial cofibration u has the form $u = c \otimes_i \delta : D \to Z \times \mathrm{I}$ for a cofibration $c : C \to Z$ and a morphism $i : Z \to \mathrm{I}$, where we write D for $Z \amalg_C C \times \mathrm{I}$. Let $p : X \to Z \times \mathrm{I}$ be a fibration, and write $p = \langle z, t\rangle$, with $z : X \to Z$ and $t : X \to \mathrm{I}$. Note that given this data, we can construct two (generally different) morphisms from X to I, namely iz and t.

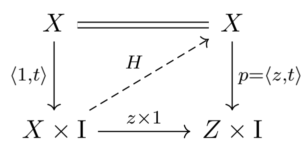
In the diagram below, the bottom face is the square appearing in Definition 1. By Lemma 4, this square is a pullback. We obtain the top square of the diagram by pulling back this square along the morphism $p : X \to Z \times \mathrm{I}$, producing a cube in which all faces are pullback squares, and in particular morphisms $a : X_Z \to X$, $b : X_{C\times \mathrm{I}} \to X$.



(∗)

Not shown in the above diagram is the original generating cofibration $u = c \otimes_i \delta : D \to Z \times \mathrm{I}$, the cogap map of the bottom face. We write $p^*u : X_D \to X$ for its pullback along p. By (H1), we can identify X_D with the pushout $X_Z \amalg_{X_C} X_{C\times\mathrm{I}}$ and p^*u with the cogap map of the top face of (∗).

By assumption, $p : X \to Z \times \mathrm{I}$ is a fibration, so the square



admits a lift $H : X \times \mathrm{I} \to X$, by Lemma 3. This produces a retraction diagram

$$X \xrightarrow{\langle 1,t\rangle} X \times \mathrm{I} \xrightarrow{H} X \qquad (\dagger 0)$$

and we also have the equations

$$zH = z \circ \mathrm{pr}_1 : X \times \mathrm{I} \to Z, \qquad tH = \mathrm{pr}_2 : X \times \mathrm{I} \to \mathrm{I}.$$

We are to show that p^*u is a retract of a generating trivial cofibration. Specifically, we will show that it is a retract of the generating trivial cofibration $v = b \otimes_{iz} \delta$. (Note that $b : X_{C\times \mathrm{I}} \to X$ is a pullback of $c \times 1 : C \times \mathrm{I} \to Z \times \mathrm{I}$, hence a cofibration by Lemma 4 and (H2).) We will do this by expressing the top face of $(*)$ as a retract of the square

$$\begin{array}{ccc} X_{C\times \mathrm{I}} & \xrightarrow{\langle 1, izb\rangle} & X_{C\times \mathrm{I}} \times \mathrm{I} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b\times 1} \\ X & \xrightarrow{\langle 1, iz\rangle} & X \times \mathrm{I} \end{array} \qquad (**)$$

in the category of commutative squares of $\mathscr{C}$. By functoriality of the pushout, it will follow that p^*u, the cogap map of the top face of $(*)$, is a retract of v, the cogap map of $(**)$.

In the lower right corner of this retraction diagram, we will use (†0). It is then enough to construct two retraction diagrams

$$\begin{array}{ccccc} X_Z & \dashrightarrow & X & \dashrightarrow & X_Z \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \langle 1, iz\rangle} & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{\langle 1, t\rangle} & X \times \mathrm{I} & \xrightarrow{H} & X \end{array} \qquad (\dagger 1)$$

and

$$\begin{array}{ccccc} X_{C\times \mathrm{I}} & \dashrightarrow & X_{C\times \mathrm{I}} \times \mathrm{I} & \dashrightarrow & X_{C\times \mathrm{I}} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b\times 1} & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{\langle 1, t\rangle} & X \times \mathrm{I} & \xrightarrow{H} & X \end{array} \qquad (\dagger 2)$$

since both the top face of $(*)$ and the square $(**)$ are pullback squares.

To produce diagram (†1), note that a is a monomorphism, being a pullback of $\langle 1, i\rangle : Z \to Z \times \mathrm{I}$, so it is enough to construct dotted morphisms making the two squares commute individually. Because the front face of $(*)$ is a pullback square, the diagram

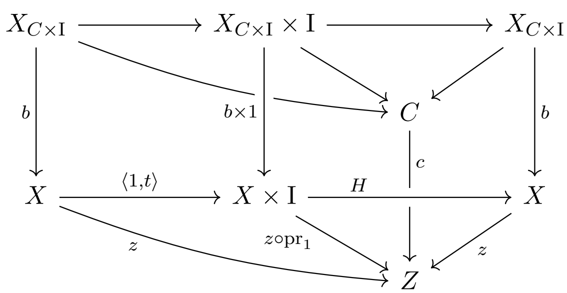
$$X_Z \xrightarrow{a} X \overset{iz}{\underset{t}{\rightrightarrows}} \mathrm{I}$$

is an equalizer. Hence, for the left dotted arrow in (†1), we may take the morphism $a : X_Z \to X$, while to obtain the right dotted arrow, it suffices to show that the compositions

$$X \xrightarrow{\langle 1, iz\rangle} X \times \mathrm{I} \xrightarrow{H} X \overset{iz}{\underset{t}{\rightrightarrows}} \mathrm{I}$$

agree. We have $izH = iz \circ \mathrm{pr}_1$ while $tH = \mathrm{pr}_2$, so both compositions equal iz.

To produce (†2), we simply pull back (†0) along the morphism $c : C \to Z$.



The resulting objects and vertical morphisms are the correct ones because of the equation $zH = z \circ \mathrm{pr}_1$ and the pullback squares below, in which the middle square is the right face of ($*$).

$$\begin{array}{ccccccc} X_{C\times \mathrm{I}} \times \mathrm{I} & \xrightarrow{\mathrm{pr}_1} & X_{C\times\mathrm{I}} & \longrightarrow & C \times \mathrm{I} & \xrightarrow{\mathrm{pr}_1} & C \\ \downarrow{\scriptstyle b\times 1} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c\times 1} & & \downarrow{\scriptstyle c} \\ X \times \mathrm{I} & \xrightarrow{\mathrm{pr}_1} & X & \xrightarrow{p} & Z \times \mathrm{I} & \xrightarrow{\mathrm{pr}_1} & Z \end{array}$$ □

Remark 7. We give a more informal account of the constructions involved in this proof. For simplicity, let us assume that the cofibration $c : C \to Z$ is a monomorphism and that $\mathscr{C}$ is a topos, so that the pushout appearing in the definition of a generating trivial cofibration is the union of subobjects. We also write as though an object X of $\mathscr{C}$ has actual elements $x : X$.

The fibration $p : X \to Z \times \mathrm{I}$ equips each $x : X$ with “coordinates” $z(x) : Z$ and $t(x) : \mathrm{I}$. Inside $Z \times \mathrm{I}$, the original generating trivial cofibration $u : D \to Z \times \mathrm{I}$ cuts out the subobject consisting of those pairs (z, t) such that either $t = i(z)$, or z belongs to the subobject $C \subseteq Z$. Hence the pullback $p^*u : X_D \to X$ cuts out those $x : X$ such that either $t(x) = iz(x)$, or $z(x)$ belongs to $C \subseteq Z$.

For $x : X$ and $t' : \mathrm{I}$, we think of $H(x, t') : X$ as “transporting” x to have t-coordinate t', while leaving its z-coordinate unchanged. The commutativity of the upper triangle in the lifting problem used to construct H says that if $t(x) = t'$, so that the old and new t-coordinates are the same, then $H(x, t')$ is the original point x. This is where we use the fact that we work with generic fibrations.

The cofibration v appearing in the proof cuts out those points $(x, t') : X \times \mathrm{I}$ such that either $t' = iz(x)$, or $z(x)$ belongs to $C \subseteq Z$. Call this subobject $E \subseteq X \times \mathrm{I}$. We claim that the morphisms

$$X \xrightarrow{\langle 1,t\rangle} X \times \mathrm{I} \xrightarrow{H} X$$

carry $X_D \subseteq X$ into $E \subseteq X \times \mathrm{I}$ and vice versa. For instance, if $(x, t') : X \times \mathrm{I}$ satisfies $t' = iz(x)$, then $t(H(x, t')) = t' = iz(x) = iz(H(x, t'))$, so $H(x, t') \in X_D$. The other cases are similar but easier.

Deducing the Frobenius property is now a standard matter of manipulating lifting conditions and adjunctions. We call a morphism of $\mathscr{C}$ a *trivial cofibration* if it has the left lifting property with respect to all fibrations. Then, for any object Y of $\mathscr{C}$, call a morphism $u : A \to B$ of the slice category $\mathscr{C}/Y$ a (generating) trivial cofibration whenever its underlying morphism of $\mathscr{C}$ is one. Using this terminology, we then observe the following:

- The fibrations of $\mathscr{C}$ are closed under pullback and the trivial cofibrations of $\mathscr{C}$ are closed under retracts, since these classes are defined by lifting properties (see, e.g., Hirschhorn (2019)).
- For a morphism $f : X \to Y$ of $\mathscr{C}$, the following conditions are equivalent:
  (1) As a morphism of $\mathscr{C}$, f is a fibration.
  (2) Viewing X as an object of $\mathscr{C}/Y$ via f, for every generating trivial cofibration $u : A \to B$ of $\mathscr{C}/Y$, the function $(- \circ u) : \mathrm{Hom}_{\mathscr{C}/Y}(B, X) \to \mathrm{Hom}_{\mathscr{C}/Y}(A, X)$ is surjective.
  (3) The same condition as (2), but with the word “generating” removed.
  Indeed, unpacking statements (2) and (3) shows that they say precisely that f has the right lifting property with respect to every (generating) trivial cofibration of $\mathscr{C}$.
- For a fibration $p : Y' \to Y$, the pullback functor $p^* : \mathscr{C}/Y \to \mathscr{C}/Y'$ takes generating trivial cofibrations to trivial cofibrations.
  Indeed, suppose $u : A \to B$ is a generating trivial cofibration of $\mathscr{C}/Y$. The underlying morphisms of u and p^*u fit in a diagram as shown below, in which both squares are pullbacks.

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle p^*u}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \longrightarrow & B \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{p} & Y \end{array}$$

Above, the morphism $B' \to B$ of $\mathscr{C}$ is a pullback of p, hence a fibration. So by Proposition 6, the morphism p^*u is a retract of a generating trivial cofibration, hence a trivial cofibration.

Corollary 8. *Under hypotheses (H1)–(H3) of Section 2, suppose $f : X \to Y$ and $p : Y \to Y'$ are fibrations such that the pushforward $p_*f : X' \to Y'$ exists. Then p_*f is also a fibration. In particular, if $\mathscr{C}$ is locally cartesian closed, then its fibrations satisfy the Frobenius property.*

Proof. We regard X' (via p_*f) as an object of the slice category $\mathscr{C}/Y'$. It comes equipped with an isomorphism $\mathrm{Hom}_{\mathscr{C}/Y}(p^*A, X) \cong \mathrm{Hom}_{\mathscr{C}/Y'}(A, X')$ natural in $A \in \mathscr{C}/Y'$.

We must show that if $u : A \to B$ is any morphism of $\mathscr{C}/Y'$ whose underlying morphism in $\mathscr{C}$ is a generating trivial cofibration, then the function $(- \circ u) : \mathrm{Hom}_{\mathscr{C}/Y'}(B, X') \to \mathrm{Hom}_{\mathscr{C}/Y'}(A, X')$ is surjective. Using the above isomorphism, this is equivalent to the statement that the function $(- \circ p^*u) : \mathrm{Hom}_{\mathscr{C}/Y}(p^*B, X) \to \mathrm{Hom}_{\mathscr{C}/Y}(p^*A, X)$ is surjective, which is true because p^*u is a trivial cofibration of $\mathscr{C}/Y$. □

By a similar adjunction argument, we deduce that if $\mathscr{C}$ is locally cartesian closed and $p : Y' \to Y$ is a fibration, then the pullback functor $p^* : \mathscr{C}/Y \to \mathscr{C}/Y'$ preserves all trivial cofibrations. Note that these arguments do not actually require the existence of trivial cofibration–fibration factorizations, nor that a general trivial cofibration can be presented as a retract of a transfinite composition of pushouts of generating trivial cofibrations.

Example 9. Continuing Example 5, we see that in the Kan–Quillen model category structure on simplicial sets, the pullback of an acyclic cofibration along a fibration is again an acyclic cofibration. Because the pullback of an acyclic fibration is always an acyclic fibration, we deduce that the pullback of any weak equivalence along a fibration is again a weak equivalence; that is, the model category of simplicial sets is right proper. A similar proof is given in Gambino and Sattler (2017), using the fact that the interval object Δ^1 has connections. We have shown that the connections are not really needed for such an argument.

Remark 10. Suppose the interval object I is equipped with a chosen point $p : 1 \to \mathrm{I}$. Then we may define a different class of fibrations, the p-biased fibrations, as those with the right lifting property with respect to the pushout products $c \otimes p : Z \amalg_C C \times \mathrm{I} \to Z \times \mathrm{I}$ of all cofibrations $c : C \to Z$ of $\mathscr{C}$ with the fixed morphism p. In general, the p-biased fibrations need not have the Frobenius property; that is, the analogue of Corollary 8 for p-biased fibrations does not hold.

Specifically, take $\mathscr{C}$ to be the category of simplicial sets with all monomorphisms as cofibrations, I to be Δ^1, and $p : 1 \to \Delta^1$ to be the morphism selecting the 0th vertex. Then the p-biased fibrations are the left fibrations of Joyal (2008), by Proposition 2.1.2.6 of Lurie (2009). We claim the left fibrations do not satisfy the Frobenius property. By adjunction, this is equivalent to the claim that the morphisms with the left lifting property with respect to left fibrations, namely, the left anodyne extensions, are not stable under pullback along left fibrations. For example, the morphism $p : 1 \to \Delta^1$ is itself a left anodyne extension, while the inclusion $q : 1 \to \Delta^1$ of the other vertex is a left fibration. (This can be checked directly or by using Proposition 2.1.1.3 of op.cit.)

The pullback $q^*p : 1 \times_{\Delta^1} 1 \to 1$ has empty domain, so it is not a left anodyne extension, because left anodyne extensions are in particular weak equivalences.

The correct statement in this situation is that the pushforward of a *right* fibration along a left fibration is again a *right* fibration, and vice versa. See Section 21 of Joyal (2008) or Section 4.1.2 of Lurie (2009).

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