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Abstract

We exhibit a strong link between the Hall algebra $\mathbf{H}_X$ of an elliptic curve X defined over a finite field $\mathbb{F}_l$ (or, more precisely, its spherical subalgebra $\mathbf{U}_X^+$) and Cherednik's double affine Hecke algebras $\ddot{\mathbf{H}}_n$ of type GL_n, for all n. This allows us to obtain a geometric construction of the Macdonald polynomials $P_\lambda(q, t^{-1})$ in terms of certain functions (Eisenstein series) on the moduli space of semistable vector bundles on the elliptic curve X.

Introduction

The spherical affine Hecke algebra $\mathbf{S}\dot{\mathbf{H}}_G$ of a reductive algebraic group G is the convolution algebra of $G(\mathcal{O})$-invariant functions on the affine Grassmannian $\widehat{\mathrm{Gr}} = G(\mathcal{K})/G(\mathcal{O})$, where $\mathcal{K} = \mathbb{F}_l((z))$ and $\mathcal{O} = \mathbb{F}_l[[z]]$; see [IM65]. The Satake isomorphism identifies $\mathbf{S}\dot{\mathbf{H}}_G$ with the representation ring $\mathrm{Rep}(G^L(\mathbb{C}))$ of the dual group of G. Now let us assume that $G = G^L = \mathrm{GL}_n$, so that the set of $\mathbb{F}_l$-points of $\widehat{\mathrm{Gr}}$ is equal to

$$\{L \subset \mathbb{F}_l^n((z)) : L \text{ is a free } \mathbb{F}_l[[z]]\text{-module of rank } n\}$$

and $\mathrm{Rep}(G) \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathfrak{S}_n}$. In [Lus81], the nilpotent cones $\mathcal{N}_k \subset \mathfrak{gl}_k$, $k \geqslant 1$, were embedded into the positive Schubert variety

$$\widehat{\mathrm{Gr}}^+ = \{L \subset \mathbb{F}_l^n[[z]] : L \text{ is a free } \mathbb{F}_l[[z]]\text{-module of rank } n\}$$

of $\widehat{\mathrm{Gr}}$. This yields a surjective algebra homomorphism

$$\Theta_n^+ : \mathbf{H}_{\mathrm{cl}} = \bigoplus_{k \geqslant 0} \mathbb{C}_{\mathrm{GL}_k}[\mathcal{N}_k] \to \mathbf{S}\dot{\mathbf{H}}_n^+ \simeq \mathbb{C}[v^{\pm 1}][x_1, \ldots, x_n]^{\mathfrak{S}_n}; \tag{0.1}$$

see [Mac95, ch. II] or [Lus81]. Here, $\mathbf{H}_{\mathrm{cl}}$ is the classical Hall algebra and $v = l^{-1/2}$. Since the dependence on v is polynomial, we may treat v as a formal parameter. Letting n tend to infinity in (0.1) yields an isomorphism in the stable limit,

$$\Theta_\infty^+ : \mathbf{H}_{\mathrm{cl}} \xrightarrow{\sim} \mathbf{S}\dot{\mathbf{H}}_\infty^+ = \mathbb{C}[v^{\pm 1}][x_1, x_2, \ldots]^{\mathfrak{S}_\infty}. \tag{0.2}$$

The first main result of this paper gives affine versions of (0.1) and (0.2). In [BS05] it was found that the Hall algebra $\mathbf{H}_X$ of the category of coherent sheaves on an elliptic curve X defined over $\mathbb{F}_l$ contains a natural 'spherical' subalgebra $\mathbf{U}_X^+$ which is a two-parameter deformation of the ring of diagonal invariants,

$$\mathbf{R}_n^+ = \mathbb{C}[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]^{\mathfrak{S}_n},$$

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where $\mathfrak{S}_n$ acts simultaneously on the x-variables and the y-variables. The two deformation parameters are the Frobenius eigenvalues σ and $\overline{\sigma}$ of $H^1(X, \overline{\mathbb{Q}}_p)$ (viewed as complex numbers). The dependence on σ and $\overline{\sigma}$ is polynomial, so we may treat these as formal variables.

Let $\ddot{\mathbf{H}}_n$ denote Cherednik's double affine Hecke algebra of type GL_n, and let $\mathbf{S}\ddot{\mathbf{H}}_n = S \cdot \ddot{\mathbf{H}}_n \cdot S$ stand for its spherical subalgebra. Here S is the complete idempotent associated to the *finite* Hecke algebra $\mathbf{H}_n \subset \ddot{\mathbf{H}}_n$. The algebra $\mathbf{S}\ddot{\mathbf{H}}_n$ is a deformation of the ring

$$\mathbf{R}_n = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]^{\mathfrak{S}_n}$$

that depends on two parameters t and q. Let $\mathbf{S}\ddot{\mathbf{H}}_n^+$ be the positive part of $\mathbf{S}\ddot{\mathbf{H}}_n$; see § 2.1. In Theorem 3.1 we prove the following result.

Theorem. *If $\sigma = q^{-1}$ and $\overline{\sigma} = t^{-1}$, then for any n there exists a surjective algebra homomorphism $\Phi_n^+ : \mathbf{U}_X^+ \twoheadrightarrow \mathbf{S}\ddot{\mathbf{H}}_n^+$. This map extends to a surjective algebra homomorphism*

$$\Phi_n : \mathbf{U}_X := \mathbf{DU}_X^+ \to \mathbf{S}\ddot{\mathbf{H}}_n.$$

Here $\mathbf{DU}_X^+$ is the Drinfeld double of $\mathbf{U}_X^+$. It is equipped with an action of $\mathrm{SL}(2, \mathbb{Z})$ that comes from the group of derived autoequivalences of $D^b(\mathrm{Coh}(X))$. Cherednik has defined an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbf{S}\ddot{\mathbf{H}}_n$; see [Che04, Ion03]. The map Φ_n is defined so as to intertwine these two actions. The maps Φ_n^+ behave well with respect to the stable limit, according to Theorem 4.6.

Theorem. *The maps Φ_n^+ induce an algebra isomorphism*

$$\Phi_\infty^+ : \mathbf{U}_X^+ \xrightarrow{\sim} \mathbf{S}\ddot{\mathbf{H}}_\infty^+ = \varprojlim \mathbf{S}\ddot{\mathbf{H}}_n^+.$$

One of the essential features of the construction of the spherical affine Hecke algebras as convolution algebras of functions (on the affine Grassmannian or on the nilpotent cones) is that it lifts to a tensor category of perverse sheaves (see, for example, [Gin95, MV00]). Such a geometric lift also makes sense here, and fits into Laumon's theory of automorphic sheaves. We refer to [Sch05] and § 4.3 for more details.

In the second part of this paper, we give an application of the above geometric construction of $\mathbf{S}\ddot{\mathbf{H}}_n$ to Macdonald polynomials.

The Hall algebra $\mathbf{H}_X^{\mathrm{vec}}$ of the category of vector bundles on X (or on any smooth projective curve) can be viewed as the algebra of (unramified) automorphic forms for GL_n, for all $n \geqslant 1$, over the function field of X. The product is given by the functor of parabolic induction; see [Kap97]. To obtain the whole Hall algebra $\mathbf{H}_X$, one needs to take into account the torsion sheaves as well. The Hall algebra $\mathbf{H}_X^{\mathrm{tor}}$ of the category of torsion sheaves on X acts on $\mathbf{H}_X^{\mathrm{vec}}$ by the adjoint action, and $\mathbf{H}_X$ is isomorphic to the semidirect product $\mathbf{H}_X^{\mathrm{vec}} \rtimes \mathbf{H}_X^{\mathrm{tor}}$. The actions of torsion sheaves can be interpreted in the language of automorphic forms as *Hecke operators*. For instance, the skyscraper sheaf $\mathcal{O}_x$ at a point $x \in X$ corresponds to the elementary modification at x.

Under the map Φ_∞^+, the element $\mathbf{1}_{(0,1)} \in \mathbf{U}_X^{\mathrm{tor}}$ responsible for the *Hecke operator* of rank one is sent to *Macdonald's element* $\Delta_1 = S \sum_i Y_i S \in \mathbf{S}\ddot{\mathbf{H}}_\infty^+$; see § 2. The importance of this element stems from the fact that in the polynomial representation of $\mathbf{S}\ddot{\mathbf{H}}_\infty^+$, the operator Δ_1 has distinct eigenvalues and the corresponding eigenvectors are the Macdonald polynomials $P_\lambda(q, t^{-1})$. Thus the map Φ_∞^+ allows us to relate Hecke eigenvectors on the Hall or automorphic side to Macdonald polynomials on the Hecke algebra side. In particular, we are naturally led to find Hecke eigenvectors in $\mathbf{U}_X^+$ whose eigenvalues match those of the $P_\lambda(q, t^{-1})$.

Eisenstein series yield a way of producing new Hecke eigenvectors from old ones via parabolic induction. In the present situation, it so happens that the simplest Eisenstein series, i.e. those induced from trivial characters of parabolic subgroups, already have the good eigenvalues under the Hecke operator. Unfortunately, we are unable to construct the polynomial representation of $\mathbf{S}\ddot{\mathbf{H}}_\infty^+$ in a geometric manner (see the remark in § 5.1), and thus we cannot obtain directly a geometric construction of $P_\lambda(q, t^{-1})$. To remedy this, we lift the Macdonald polynomials from the polynomial representation and view them inside the Hecke (or Hall) algebra itself. More precisely, it has been shown in [BS05] that the subalgebra of $\mathbf{U}_X^+$ consisting of functions supported on the set $\mathrm{Coh}(X)^{(0)}$ of semistable sheaves of zero slope is canonically isomorphic to the algebra

$$\Lambda^+_{(\sigma,\overline{\sigma})} = \mathbb{C}[\sigma^{\pm 1/2}, \overline{\sigma}^{\pm 1/2}][x_1, x_2, \ldots]^{\mathfrak{S}_\infty}.$$

In short, under the Fourier–Mukai transform the set $\mathrm{Coh}(X)^{(0)}$ is identified with the set of torsion sheaves on X, and any function on the set of torsion sheaves with a fixed punctual support in X can be viewed as an element of the classical Hall algebra; see [Pol03, Theorem 14.7] for details. If f is any function in $\mathbf{U}_X^+$, we let $f^{(0)}$ denote its restriction to $\mathrm{Coh}(X)^{(0)}$, viewed as an element of $\Lambda^+_{(\sigma,\overline{\sigma})}$. For any $l \in \mathbb{N}^+$, put

$$\mathbf{E}_l(z) = \sum_{d \in \mathbb{Z}} \mathbf{1}_{(l,d)} v^{d(l-1)} z^d \in \widehat{\mathbf{U}}_X^+[[z, z^{-1}]] \tag{0.3}$$

with $v = (\sigma\overline{\sigma})^{-1/2} = \#\mathbb{F}_l^{-1/2}$, where $\mathbf{1}_{(l,d)}$ denotes the characteristic function of the set of all coherent sheaves on X of rank l and degree d and $\widehat{\mathbf{U}}_X^+$ is a certain completion of $\mathbf{U}_X^+$. For $(l_1, \ldots, l_n) \in \mathbb{N}^n$, we form the Eisenstein series

$$\mathbf{E}_{l_1,\ldots,l_n}(z_1, \ldots, z_n) = \mathbf{E}_{l_1}(z_1) \cdot \mathbf{E}_{l_2}(z_2) \cdots \mathbf{E}_{l_n}(z_n) \in \widehat{\mathbf{U}}_X^+[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]. \tag{0.4}$$

By a theorem of Harder, this is a rational function in $z_1, \ldots, z_n$. Our second main result (Theorem 7.1) reads as follows.

Theorem. *Let $(l_1, \ldots, l_n) \in \mathbb{N}^n$. We have $\mathbf{E}_{l_1,\ldots,l_n}(z, \sigma z, \ldots, \sigma^{n-1} z) = 0$ unless $(l_1, \ldots, l_n)$ is dominant, i.e. unless $(l_1, \ldots, l_n)$ is a partition, in which case*

$$\mathbf{E}_{l_1,\ldots,l_n}(z, \sigma z, \ldots, \sigma^{n-1} z)^{(0)} = \omega P_\lambda(\sigma^{-1}, v^2)$$

where $\lambda = (l_1, \ldots, l_n)'$ is the conjugate partition and ω stands for the standard involution on symmetric functions.

We give a similar construction of skew Macdonald polynomials $P_{\lambda/\mu}(\sigma^{-1}, v^2)$. Note that the above Eisenstein series can be lifted to some constructible sheaves via the theory of *Eisenstein sheaves*; see [Lau90, Sch05]. Hence the Macdonald polynomials $P_\lambda(\sigma^{-1}, v^2)$ may be realized as Frobenius traces of certain canonical constructible sheaves on the moduli stack of semistable sheaves of zero slope on X. We hope to come back to this point in the future.

There is a well-known and important geometric approach to Macdonald polynomials, based on the equivariant K-theory of the Hilbert schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$ of points on $\mathbb{C}^2$. There the polynomials $P_\lambda(q, t^{-1})$ are realized as the classes of certain canonical coherent sheaves on $\mathrm{Hilb}_n(\mathbb{C}^2)$; see [Hai02]. In [SV09] we related this 'coherent sheaf' picture to our 'constructible functions' (or 'perverse sheaf') picture in the framework of Beilinson and Drinfeld's geometric Langlands duality for local systems on X in the formal neighborhood of the trivial local system.

The structure of this paper is as follows. Sections 1 and 2 recall some facts about the elliptic Hall algebras $\mathbf{H}_X$ and $\mathbf{U}_X^+$, taken from [BS05], and the Cherednik double affine Hecke algebras $\ddot{\mathbf{H}}_n$ and $\mathbf{S}\ddot{\mathbf{H}}_n$. In § 3 we construct the surjective algebra morphism $\Phi_n : \mathbf{D}\mathbf{U}_X^+ \to \mathbf{S}\ddot{\mathbf{H}}_n$, and in § 4 we study the stable limit $\mathbf{S}\ddot{\mathbf{H}}_\infty^+$ of the spherical Cherednik algebra and establish the

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isomorphism $\Phi^+_\infty : \mathbf{U}^+_X \xrightarrow{\sim} \mathbf{S}\ddot{\mathbf{H}}^+_\infty$; this is the first main result of the paper. A table that compares the classical Hall algebra with the 'finite' spherical affine Hecke algebra is given in § 4.3. Section 5 deals with Macdonald polynomials: we recall their definition and provide a characterization of the family of all (possibly skew) Macdonald polynomials that will be used later. In § 6 we introduce the Eisenstein series of relevance to this paper, and study some of their specializations. Finally, our second main theorem, which gives a geometric construction of (possibly skew) Macdonald polynomials from Eisenstein series, is given in § 7. Several of the proofs in this paper require lengthy computations; the details of these are presented in Appendix A to Appendix D.

Let us give a word of warning concerning notation. There is an unfortunate clash between the conventional notation used in the quantum group or Hall algebra literature and that used in the Macdonald polynomials literature: q generally denotes the size of the finite field in the former case, whereas it is the modular parameter in the latter case. We have opted to comply with the conventions of the Macdonald polynomials literature.

1. The elliptic Hall algebra

1.1 We will use the standard v-integers and v-factorials

$$[i] = [i]_v = \frac{v^i - v^{-i}}{v - v^{-1}} \quad \text{and} \quad [i]! = [2] \cdots [i]$$

as well as some positive and negative variants:

$$[i]^+ = \frac{v^{2i}-1}{v^2-1}, \quad [i]^+! = [2]^+ \cdots [i]^+, \quad [i]^- = \frac{v^{-2i}-1}{v^{-2}-1}, \quad [i]^-! = [2]^- \cdots [i]^-.$$

Let us denote by Λ^+_v Macdonald's ring of symmetric functions,

$$\Lambda^+_v = \mathbb{C}[v^{\pm 1}][x_1, x_2, \ldots]^{\mathfrak{S}_\infty},$$

defined over $\mathbb{C}[v^{\pm 1}]$; see [Mac95]. We will denote by e_λ, p_λ and m_λ the elementary, the power-sum and the monomial symmetric functions, respectively. The ring Λ^+_v is equipped with a natural bialgebra structure $\Delta : \Lambda^+_v \to \Lambda^+_v \otimes \Lambda^+_v$ defined by $\Delta(p_r) = p_r \otimes 1 + 1 \otimes p_r$ for $r \geqslant 1$.

We will often use notation relating to various subsets of $\mathbb{Z}^2$. We write $\mathbf{Z} = \mathbb{Z}^2$ and set

$$\mathbf{Z}^+ = \{(r,d) \in \mathbf{Z} \mid r > 0 \text{ or } r = 0, d > 0\}, \quad \mathbf{Z}^- = -\mathbf{Z}^+,$$
$$\mathbf{Z}^{++} = \{(r,d) \mid r \geqslant 0, d \geqslant 0\} \backslash \{(0,0)\}, \quad \mathbf{Z}^* = \mathbf{Z} \backslash \{(0,0)\}.$$

1.2 Let X be a smooth elliptic curve over some finite field $\mathbb{F}_l$, and let $\mathrm{Coh}(X)$ stand for the category of coherent sheaves on X. If F is a sheaf in $\mathrm{Coh}(X)$, we call the pair $\overline{F} = (\mathrm{rk}(F), \deg(F))$ the *class* of F. The set of all possible classes of sheaves in $\mathrm{Coh}(X)$ is equal to $\mathbf{Z}^+$. We briefly recall the definition of the Hall algebra of $\mathrm{Coh}(X)$; see [BS05] for further details.

Let $\mathcal{I}(X)$ stand for the set of isomorphism classes of objects in $\mathrm{Coh}(X)$. Following [Rin90], the $\mathbb{C}$-vector space

$$\mathbf{H}_X = \{f : \mathcal{I}(X) \to \mathbb{C} : |\mathrm{supp}(f)| < \infty\}$$

of finitely supported functions can be equipped with the convolution product

$$(f \cdot g)(M) = \sum_{N \subseteq M} v^{-\langle M/N, N \rangle} f(M/N) g(N),$$

where $v = l^{-1/2}$ and

$$\langle P, Q\rangle = \dim \mathrm{Hom}(P, Q) - \dim \mathrm{Ext}(P, Q)$$

is the Euler form. Here we write $\mathrm{Ext}(P, Q)$ for $\mathrm{Ext}^1(P, Q)$. The sum on the right-hand side is finite for any M because f and g have finite support and, for any $N, M \in \mathrm{Coh}(X)$, the group $\mathrm{Hom}(N, M)$ is finite. The above formula indeed defines an element in $\mathbf{H}_X$, as for any $P, Q \in \mathrm{Coh}(X)$ the group $\mathrm{Ext}(P, Q)$ is also finite. By the Riemann–Roch theorem, we have

$$\langle P, Q\rangle = \mathrm{rk}(P)\deg(Q) - \deg(P)\mathrm{rk}(Q). \tag{1.1}$$

By [Gre95], the algebra $\mathbf{H}_X$ also has the structure of a bialgebra, with coproduct

$$(\Delta(f))(P, Q) = \frac{\langle P, Q\rangle^{-1}}{|\mathrm{Ext}(P, Q)|} \sum_{\xi \in \mathrm{Ext}(P,Q)} f(M_\xi)$$

where M_ξ is the extension of P by Q corresponding to ξ. The product and coproduct are related by the pairing

$$\mathbf{H}_X \otimes \mathbf{H}_X \to \mathbb{C}, \quad \langle f, g\rangle_G = \sum_M \frac{f(M)g(M)}{|\mathrm{Aut}(M)|},$$

which is a Hopf pairing, i.e. it satisfies the identity $\langle fg, h\rangle_G = \langle f \otimes g, \Delta(h)\rangle_G$ for any f, g and h.

Remarks.

(i) In our situation, as opposed to [Gre95], it is not necessary to twist the product in $\mathbf{H}_X \otimes \mathbf{H}_X$ in order to obtain a bialgebra, because the Euler form $\langle\ ,\ \rangle$ is antisymmetric.

(ii) The coproduct Δ only takes values in a certain formal completion of $\mathbf{H}_X \otimes \mathbf{H}_X$; see [BS05, § 2.2] for details.

The characteristic functions $\{\mathbf{1}_M : M \in \mathcal{I}(X)\}$ form a basis for $\mathbf{H}_X$. Assigning to the element $\mathbf{1}_M$ the degree $(\mathrm{rk}(M), \deg(M))$ yields a $\mathbf{Z}$-grading on $\mathbf{H}_X$ that is compatible with the (co)multiplication.

1.3 Let $\mu(M) = \deg(M)/\mathrm{rk}(M) \in \mathbb{Q} \cup \{\infty\}$ be the slope of a sheaf $M \in \mathrm{Coh}(X)$, and for $\mu \in \mathbb{Q} \cup \{\infty\}$ let $\mathcal{C}_\mu$ stand for the category of semistable sheaves of slope μ. For instance, $\mathcal{C}_\infty$ is the category of torsion sheaves on X. The following fundamental result on the structure of $\mathrm{Coh}(X)$ is due to Atiyah.

Theorem 1.1 [Ati57].

(i) *For any μ and μ', there is an equivalence of abelian categories $\epsilon_{\mu,\mu'} : \mathcal{C}_{\mu'} \xrightarrow{\sim} \mathcal{C}_\mu$.*

(ii) *Any coherent sheaf $\mathcal{F}$ decomposes uniquely as a direct sum $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_s$ of semistable sheaves $\mathcal{F}_i \in \mathcal{C}_{\mu_i}$ with $\mu_1 < \cdots < \mu_s$.*

By a standard property of semistable sheaves, we have $\mathrm{Hom}(\mathcal{C}_\mu, \mathcal{C}_{\mu'}) = \{0\}$ for $\mu > \mu'$. By Serre duality, this implies that $\mathrm{Ext}(\mathcal{C}_{\mu'}, \mathcal{C}_\mu) = \{0\}$ whenever $\mu > \mu'$. Hence any extension $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ with $\mathcal{F} \in \mathcal{C}_\mu$ and $\mathcal{H} \in \mathcal{C}_{\mu'}$ is split. From the above two facts, it follows that in $\mathbf{H}_X$ we have

$$\mathbf{1}_\mathcal{H} \cdot \mathbf{1}_\mathcal{F} = v^{-\langle \mathcal{H}, \mathcal{F}\rangle} \mathbf{1}_{\mathcal{F} \oplus \mathcal{H}} \tag{1.2}$$

if $\mathcal{F} \in \mathcal{C}_\mu, \mathcal{H} \in \mathcal{C}_{\mu'}$ and $\mu > \mu'$.

For $\mu \in \mathbb{Q} \cup \{\infty\}$, let $\mathbf{H}_X^{(\mu)}$ stand for the subspace consisting of functions supported on the set of semistable sheaves of slope μ. Since $\mathcal{C}_\mu$ is stable under extensions, $\mathbf{H}_X^{(\mu)}$ is a subalgebra of $\mathbf{H}_X$. By Theorem 1.1(i), all these subalgebras are isomorphic. Let $\vec{\bigotimes}_\mu \mathbf{H}_X^{(\mu)}$ denote the ordered tensor product of spaces $\mathbf{H}_X^{(\mu)}$ with $\mu \in \mathbb{Q} \cup \{\infty\}$, i.e. the vector space spanned by elements of the form $a_{\mu_1} \otimes \cdots \otimes a_{\mu_r}$ with $a_{\mu_i} \in \mathbf{H}_X^{(\mu_i)}$ and $\mu_1 < \cdots < \mu_r$. From (1.2) and Theorem 1.1(ii) we deduce the following; see [BS05, Lemma 2.6].

Corollary 1.2. *The multiplication map is an isomorphism of vector spaces* $\vec{\bigotimes}_\mu \mathbf{H}_X^{(\mu)} \xrightarrow{\sim} \mathbf{H}_X$.

1.4 We will mainly be interested in a certain subalgebra $\mathbf{U}_X^+ \subset \mathbf{H}_X$, which we now define. For any class $\alpha \in \mathbf{Z}^+$ we set

$$\mathbf{1}_\alpha^{ss} = \sum_{\overline{F}=\alpha; F \in \mathcal{C}_{\mu(\alpha)}} \mathbf{1}_F \in \mathbf{H}_X.$$

The above sum is finite. Indeed, by Theorem 1.1(i), it is enough to check this for $\mu(\alpha) = \infty$, in which case finiteness of the sum follows from the fact that there are only finitely many closed points on X which are rational over a fixed finite extension of $\mathbb{F}_l$. Let $\mathbf{U}_X^+$ be the subalgebra generated by $\mathbf{1}_\alpha^{ss}$, $\alpha \in \mathbf{Z}^+$. It will be useful to consider a different set of generators T_α of $\mathbf{U}_X^+$, uniquely determined by the collection of formal relations

$$1 + \sum_{l \geqslant 1} \mathbf{1}_{l\alpha}^{ss} s^l = \exp\left(\sum_{l \geqslant 1} \frac{T_{l\alpha}}{[l} s^l\right) \tag{1.3}$$

for any $\alpha = (r, d)$ with r and d relatively prime.

To a slope $\mu \in \mathbb{Q} \cup \{\infty\}$ we can naturally associate the subalgebra $\mathbf{U}_X^{(\mu)} \subset \mathbf{U}_X^+$ generated by $\{\mathbf{1}_\alpha^{ss} : \mu(\alpha) = \mu\}$. Of course, $\mathbf{U}_X^{(\mu)} \subset \mathbf{H}_X^{(\mu)}$.

Proposition 1.3 [BS05, Theorem 4.5].

(i) *The multiplication map induces an isomorphism* $\vec{\bigotimes}_\mu \mathbf{U}_X^{(\mu)} \xrightarrow{\sim} \mathbf{U}_X^+$.

(ii) *For any* $\mathbf{x} = (r, d) \in \mathbf{Z}^+$ *with* r *and* d *relatively prime, the assignment* $T_{l\mathbf{x}}/[l] \mapsto p_l/l$ *extends to an isomorphism of algebras* $\mathbf{U}_X^{(\mu(\mathbf{x}))} \xrightarrow{\sim} (\Lambda_\nu^+)_{|\nu=v}$. *In particular,* $\mathbf{U}_X^{(\mu(\mathbf{x}))}$ *is a free commutative polynomial algebra in the generators* $\{T_{l\mathbf{x}} : l \geqslant 1\}$.

1.5 We now wish to give a presentation of $\mathbf{U}_X^+$ by generators and relations. In fact, we will give such a presentation for the *Drinfeld double* of $\mathbf{U}_X^+$, which is a more symmetric object. We will use Sweedler's notation for the coproduct of an element and write $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$. Recall that if H is a bialgebra equipped with a Hopf pairing $\langle\, , \rangle$, then its Drinfeld double is the algebra generated by two copies H^+ and H^- of H subject to the collection of relations

$$\sum_{i,j} (h^+)_i^{(1)} (g^-)_j^{(2)} \langle h_i^{(2)}, g_j^{(1)} \rangle = \sum_{i,j} (g^-)_j^{(1)} (h^+)_i^{(2)} \langle h_i^{(1)}, g_j^{(2)} \rangle. \tag{1.4}$$

Here g and h range over all elements of H, and we write h^+ and g^- for the corresponding elements of H^+ and H^-, respectively.

By [BS05, Theorem 4.5], the algebra $\mathbf{U}_X^+$ is a subbialgebra of $\mathbf{H}_X$, and we denote by $\mathbf{U}_X$ its Drinfeld double.

For $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$, let $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ stand for the triangle with vertices $\mathbf{o}, \mathbf{x}$ and $\mathbf{x}+\mathbf{y}$, where $\mathbf{o}=(0,0)$ denotes the origin in $\mathbf{Z}$. If $\mathbf{x}=(r,d)\in\mathbf{Z}^*$, we write $d(\mathbf{x})=\gcd(r,d)$. For a pair of non-collinear vectors $(\mathbf{x},\mathbf{y})\in\mathbf{Z}^*$, we set $\epsilon_{\mathbf{x},\mathbf{y}}$ to be $\operatorname{sign}(\det(\mathbf{x},\mathbf{y}))$.

Let $\mathbf{R}=\mathbb{C}[\sigma^{\pm 1/2},\overline{\sigma}^{\pm 1/2}]$ and $\mathbf{K}=\mathbb{C}(\sigma^{1/2},\overline{\sigma}^{1/2})$, where σ and $\overline{\sigma}$ are now treated as formal variables.

Definition. For $i\in\mathbb{N}$, put $\alpha_i=\alpha_i(\sigma,\overline{\sigma})=(1-\sigma^i)(1-\overline{\sigma}^i)(1-(\sigma\overline{\sigma})^{-i})/i\in\mathbf{R}$. Let $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ be the unital $\mathbf{K}$-algebra generated by elements $u_{\mathbf{x}}$, $\mathbf{x}\in\mathbf{Z}^*$, subject to the following set of relations.

(i) If $\mathbf{x}$ and $\mathbf{x}'$ belong to the same line in $\mathbf{Z}$, then $[u_{\mathbf{x}},u_{\mathbf{x}'}]=0$.

(ii) Assume that $\mathbf{x}$ and $\mathbf{y}$ are such that $d(\mathbf{x})=1$ and $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ has no interior lattice point; then

$$[u_{\mathbf{y}},u_{\mathbf{x}}]=\epsilon_{\mathbf{x},\mathbf{y}}\frac{\theta_{\mathbf{x}+\mathbf{y}}}{\alpha_1}$$

where the elements $\theta_{\mathbf{z}}$, $\mathbf{z}\in\mathbf{Z}$, are obtained by equating the Fourier coefficients of the collection of relations

$$\sum_i \theta_{i\mathbf{x}_0}s^i=\exp\bigg(\sum_{i\geqslant 1}\alpha_i u_{i\mathbf{x}_0}s^i\bigg),$$

for any $\mathbf{x}_0\in\mathbf{Z}$ such that $d(\mathbf{x}_0)=1$.

The algebra $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ is $\mathbf{Z}$-graded by $\deg(u_{\mathbf{x}})=\mathbf{x}$. Put

$$\tilde{u}_{\mathbf{x}}=\frac{1}{i}(\sigma^{-i/2}-\sigma^{i/2})(\overline{\sigma}^{-i/2}-\overline{\sigma}^{i/2})u_{\mathbf{x}}$$

and let $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ be the unital $\mathbf{R}$-subalgebra of $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ generated by $\{\tilde{u}_{\mathbf{x}}:\mathbf{x}\in\mathbf{Z}^*\}$. We will write $\boldsymbol{\mathcal{E}}_{\mathbf{R}}^{\pm}$ for the subalgebra of $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ generated by $\{\tilde{u}_{\mathbf{x}}:\mathbf{x}\in\pm\mathbf{Z}^+\}$. By [BS05, Proposition 5.1], the multiplication yields an isomorphism $\boldsymbol{\mathcal{E}}_{\mathbf{R}}^-\otimes_{\mathbf{R}}\boldsymbol{\mathcal{E}}_{\mathbf{R}}^+\simeq\boldsymbol{\mathcal{E}}_{\mathbf{R}}$. Let $\boldsymbol{\mathcal{E}}_{\mathbf{R}}^{++}$ be the subalgebra generated by $\{\tilde{u}_{\mathbf{x}}:\mathbf{x}\in\mathbf{Z}^{++}\}$. We have weight decompositions

$$\boldsymbol{\mathcal{E}}_{\mathbf{R}}^+=\bigoplus_{\mathbf{x}\in\mathbf{Z}^+}\boldsymbol{\mathcal{E}}_{\mathbf{R}}^+[\mathbf{x}],\quad \boldsymbol{\mathcal{E}}_{\mathbf{R}}^{++}=\bigoplus_{\mathbf{x}\in\mathbf{Z}^{++}}\boldsymbol{\mathcal{E}}_{\mathbf{R}}^{++}[\mathbf{x}].$$

The algebra $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ has an obvious symmetry: the group $\mathrm{SL}_2(\mathbb{Z})$ acts by automorphisms such that $g\cdot\tilde{u}_{\mathbf{x}}=\tilde{u}_{g(\mathbf{x})}$. To a slope $\mu\in\mathbb{Q}\cup\{\infty\}$ is naturally associated the subalgebra $\boldsymbol{\mathcal{E}}_{\mathbf{R}}^{(\mu)}\subset\boldsymbol{\mathcal{E}}_{\mathbf{R}}^+$ generated by $\{\tilde{u}_\alpha:\mu(\alpha)=\mu\}$. The group $\mathrm{SL}_2(\mathbb{Z})$ permutes these subalgebras.

Let σ and $\overline{\sigma}$ be the two eigenvalues of the Frobenius endomorphism acting on the vector space $H^1(X\otimes\overline{\mathbb{F}}_l,\overline{\mathbb{Q}}_p)$, with p prime to l. We shall fix once and for all a field isomorphism $\mathbb{C}\simeq\overline{\mathbb{Q}}_p$. This allows us to view σ and $\overline{\sigma}$ as complex numbers. Let $\boldsymbol{\mathcal{E}}_X$ stand for the specialization of $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ at these values of σ and $\overline{\sigma}$. Observe that

$$\sigma\overline{\sigma}=\#\mathbb{F}_l=v^{-2},\quad \#X(\mathbb{F}_{l^r})=(1-\sigma^r)(1-\overline{\sigma}^r). \tag{1.5}$$

Theorem 1.4 [BS05, Theorem 5.4]. *For $\mathbf{x}\in\mathbf{Z}^*$, the assignment $\tilde{u}_{\mathbf{x}}\mapsto T_{\mathbf{x}}/[\deg(\mathbf{x})]$ extends to an isomorphism $\boldsymbol{\mathcal{E}}_X\xrightarrow{\sim}\mathbf{U}_X$. It restricts to an isomorphism $\boldsymbol{\mathcal{E}}_X^+\xrightarrow{\sim}\mathbf{U}_X^+$.*

2. Double affine Hecke algebras

2.1 We set $\mathbf{R}'=\mathbb{C}[t^{\pm 1/2},q^{\pm 1/2}]$ and $\mathbf{K}'=\mathbb{C}(t^{1/2},q^{1/2})$. The double affine Hecke algebra $\ddot{\mathbf{H}}_n$ of GL_n, abbreviated DAHA, is the $\mathbf{K}'$-algebra generated by the elements $T_i^{\pm 1}$, $X_j^{\pm 1}$ and $Y_j^{\pm 1}$,

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with $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$, subject to the following relations:[1]

$$(T_i + t^{1/2})(T_i - t^{-1/2}) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{2.1}$$

$$T_i T_k = T_k T_i \quad \text{if } |i-k| > 1, \tag{2.2}$$

$$X_j X_k = X_k X_j, \quad Y_j Y_k = Y_k Y_j, \tag{2.3}$$

$$T_i X_i T_i = X_{i+1}, \quad T_i^{-1} Y_i T_i^{-1} = Y_{i+1}, \tag{2.4}$$

$$T_i X_k = X_k T_i \quad \text{and} \quad T_i Y_k = Y_k T_i \quad \text{if } |i-k| > 1, \tag{2.5}$$

$$Y_1 X_1 \cdots X_n = q X_1 \cdots X_n Y_1, \tag{2.6}$$

$$X_1^{-1} Y_2 = Y_2 X_1^{-1} T_1^{-2}. \tag{2.7}$$

The subalgebra $\mathbf{H}_n$ generated by $\{T_i\}$ is the usual Hecke algebra of the symmetric group $\mathfrak{S}_n$, while the subalgebra $\dot{\mathbf{H}}_{n,X}$, generated by $\mathbf{H}_n$ and $\{X_i^{\pm 1}\}$, and the subalgebra $\dot{\mathbf{H}}_{n,Y}$, generated by $\mathbf{H}_n$ and $\{Y_i^{\pm 1}\}$, are both isomorphic to the Hecke algebra of the affine Weyl group $\widehat{\mathfrak{S}}_n \simeq \mathfrak{S}_n \ltimes \mathbb{Z}^n$. We define a $\mathbf{Z}$-grading on $\ddot{\mathbf{H}}_n$ by giving T_i, X_i and Y_i degrees 0, $(1,0)$ and $(0,1)$, respectively. We will make use of the subalgebra $\ddot{\mathbf{H}}_n^{++}$ of $\ddot{\mathbf{H}}_n$ generated by the elements T_i, X_j and Y_j.

Let $s_i \in \mathfrak{S}_n$ denote the transposition $(i, i+1)$, and let $l : \mathfrak{S}_n \to \mathbb{N}$ be the standard length function. If $w = s_{i_1} \cdots s_{i_r}$ is a reduced decomposition of $w \in \mathfrak{S}_n$, we set $T_w = T_{i_1} \cdots T_{i_r}$ and put $\tilde{S} = \sum_{w \in \mathfrak{S}_n} t^{-l(w)/2} T_w$. We have $\tilde{S}^2 = [n]_t^- ! \tilde{S}$, so the element $S = \tilde{S}/[n]_t^-!$ is idempotent. Here,

$$[n]_t = \frac{t^{n/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}}, \quad [n]_t^+ = \frac{t^n - 1}{t - 1}, \quad [n]_t^- = \frac{1 - t^{-n}}{1 - t^{-1}}$$

and $[n]_t^{\pm}! = [1]_t^{\pm} \cdots [n]_t^{\pm}$. For any i we have $T_i S = S T_i = t^{-1/2} S$.

We will mainly be interested in the *spherical* DAHA of $\ddot{\mathbf{H}}_n$, which is $\mathbf{S}\ddot{\mathbf{H}}_n = S \ddot{\mathbf{H}}_n S$. We also write $\mathbf{S}\ddot{\mathbf{H}}_n^{++} = S \mathbf{H}_n^{++} S$. Before we can give bases for $\mathbf{S}\ddot{\mathbf{H}}_n$, we need some more notation. Let $\mathbf{R}_n$ denote the algebra

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots y_n^{\pm 1}]^{\mathfrak{S}_n}.$$

Here the symmetric group $\mathfrak{S}_n$ acts by simultaneous permutation on the x_i and y_i.

There is an action of the braid group B_3 on three strands by automorphisms on $\ddot{\mathbf{H}}_n$, which is explicitly given by the following operators:

$$\rho_1 : \begin{cases} T_i \mapsto T_i, \\ X_i \mapsto X_i Y_i (T_{i-1} \cdots T_i)(T_i \cdots T_{i-1}), \\ Y_i \mapsto Y_i, \end{cases}$$

$$\rho_2 : \begin{cases} T_i \mapsto T_i, \\ Y_i \mapsto Y_i X_i (T_{i-1}^{-1} \cdots T_i^{-1})(T_i^{-1} \cdots T_{i-1}^{-1}), \\ X_i \mapsto X_i. \end{cases}$$

These operators preserve $\mathbf{S}\ddot{\mathbf{H}}_n$, and the corresponding B_3-action factors through an $\mathrm{SL}_2(\mathbb{Z})$-action $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbf{S}\ddot{\mathbf{H}}_n)$ which satisfies $\rho(A_1) = \rho_1$ and $\rho(A_2) = \rho_2$, where $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

The following technical lemma will be used frequently.

[1] Note that the signs in our formulas differ from the standard conventions of [Che04] according to the relationship $t \mapsto t^{-1}$.

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Lemma 2.1. *For each $l \geqslant 2$ let*

$$\alpha_l = T_{l-1}^{-1} \cdots T_2^{-1} T_1^{-2} T_2 \cdots T_{l-1}.$$

Then the following relations hold:

$$X_l^{-1} Y_1 X_l = \alpha_l Y_1, \tag{2.8}$$

$$Y_l X_1 = X_1 Y_l + (t^{1/2} - t^{-1/2}) T_{l-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{l-1}^{-1} Y_1 X_1, \tag{2.9}$$

$$q X_1 Y_1 = T_1^{-1} \cdots T_{n-2}^{-1} T_{n-1}^{-2} T_{n-2}^{-1} \cdots T_1^{-1} Y_1 X_1, \tag{2.10}$$

$$\alpha_2 \cdots \alpha_l = T_1^{-1} \cdots T_{l-2}^{-1} T_{l-1}^{-2} T_{l-2}^{-1} \cdots T_1^{-1}. \tag{2.11}$$

Proof. By definition we have $T_1^{-2} Y_1 X_2^{-1} = X_2^{-1} Y_1$, which is relation (2.8) for $l = 2$. Multiplying on the left and on the right by T_2^{-1} and using the fact that $[T_2, Y_1] = 0$, we obtain

$$T_2^{-1} T_1^{-2} T_2 Y_1 \cdot T_2^{-1} X_2^{-1} T_2^{-1} = T_2^{-1} X_2^{-1} T_2^{-1} Y_1 = Y_1 X_3^{-1}.$$

Since $T_2^{-1} X_2^{-1} T_2^{-1} = X_3^{-1}$, we get

$$T_2^{-1} T_1^{-2} T_2 Y_1 X_3^{-1} = X_3^{-1} Y_1,$$

which is (2.8) for $l = 3$. Similar reasoning, using multiplication on the left and on the right by T_3^{-1}, yields (2.8) for $l = 4$, and so on.

We now prove (2.9). From the defining relations of $\ddot{\mathbf{H}}_n$ we have

$$\begin{aligned} Y_2 X_1 &= X_1 Y_2 X_1^{-1} T_1^{-2} X_1 = X_1 Y_2 + (t^{1/2} - t^{-1/2}) X_1 Y_2 X_1^{-1} T_1^{-1} X_1 \\ &= X_1 Y_2 + (t^{1/2} - t^{-1/2}) X_1 Y_2 X_1^{-1} T_1^{-1} X_1 T_1 T_1^{-1} \\ &= X_1 Y_2 + (t^{1/2} - t^{-1/2}) X_1 Y_2 X_1^{-1} T_1^{-2} X_2 T_1^{-1} = X_1 Y_2 + (t^{1/2} - t^{-1/2}) Y_2 X_2 T_1^{-1} \\ &= X_1 Y_2 + (t^{1/2} - t^{-1/2}) T_1^{-1} Y_1 X_1, \end{aligned}$$

which is (2.9) for $l = 2$. Now we multiply on the left and on the right by T_2^{-1} and use the fact that $[T_2, X_1] = [T_2, Y_1] = 0$ to get

$$T_2^{-1} Y_2 T_2^{-1} X_1 = X_1 T_2^{-1} Y_2 T_2^{-1} + (t^{1/2} - t^{-1/2}) T_2^{-1} T_1^{-1} T_2^{-1} Y_1 X_1,$$

which, by virtue of the relation $T_2^{-1} Y_2 T_2^{-1} = Y_3$, gives (2.9) for $l = 3$. To obtain (2.9) for $l = 4, 5$ etc., we successively multiply on the left and on the right by T_3^{-1}, T_4^{-1} etc.

Next, we turn to (2.10). Recall that, by definition,

$$(X_n^{-1} \cdots X_2^{-1} Y_1 X_2 \cdots X_n) X_1 = q X_1 Y_1.$$

By (2.8) above we have $X_2^{-1} Y_1 X_2 = \alpha_2 Y_1$. Since $[\alpha_l, X_k] = 0$ if $k > l$, upon conjugation by X_3 we obtain

$$X_3^{-1} X_2^{-1} Y_1 X_2 X_3 = \alpha_2 X_3^{-1} Y_1 X_3 = \alpha_2 \alpha_3 Y_1.$$

Continuing in this manner yields, eventually,

$$X_n^{-1} \cdots X_2^{-1} Y_1 X_2 \cdots X_n = \alpha_2 \cdots \alpha_n Y_1.$$

Thus (2.10) will follow from (2.11), which we now prove. It is easy to check (2.11) for $l = 2$ and $l = 3$. Let us argue by induction on l. Fix l and assume that

$$\alpha_2 \cdots \alpha_l = T_1^{-1} \cdots T_{l-2}^{-1} T_{l-1}^{-2} T_{l-2}^{-1} \cdots T_1^{-1}.$$

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Then

$$\begin{aligned}\alpha_2\cdots\alpha_{l+1} &= T_1^{-1}\cdots T_{l-2}^{-1}T_{l-1}^{-2}T_{l-2}^{-1}\cdots T_1^{-1}\cdot T_l^{-1}\cdots T_2^{-1}T_1^{-2}T_2\cdots T_l\\ &= T_1^{-1}\cdots T_{l-2}^{-1}T_{l-1}^{-2}T_{l-2}^{-1}\cdots T_2^{-1}\cdot T_l^{-1}\cdots T_3^{-1}T_1^{-1}T_2^{-1}T_1^{-2}T_2T_3\cdots T_l\\ &= T_1^{-1}\cdots T_{l-2}^{-1}T_{l-1}^{-2}T_{l-2}^{-1}\cdots T_2^{-1}\cdot T_l^{-1}\cdots T_3^{-1}T_2^{-1}T_1^{-1}T_2^{-1}T_1^{-1}T_2T_3\cdots T_l\\ &= T_1^{-1}\cdots T_{l-2}^{-1}T_{l-1}^{-2}T_{l-2}^{-1}\cdots T_2^{-1}\cdot T_l^{-1}\cdots T_3^{-1}T_2^{-2}T_1^{-1}T_3\cdots T_l\\ &= T_1^{-1}\cdots T_{l-2}^{-1}T_{l-1}^{-2}T_{l-2}^{-1}\cdots T_2^{-1}\cdot T_l^{-1}\cdots T_3^{-1}T_2^{-2}T_3\cdots T_lT_1^{-1}.\end{aligned}$$

The last expression above is of the form $T_1^{-1}ZT_1^{-1}$ where, by the induction hypothesis, Z is equal to $\alpha_2\cdots\alpha_l$ *for the subalgebra of* $\mathbf{H}_n$ *generated by* $T_2,\ldots,T_l$. In particular, T_1 is not involved in Z. Using our induction hypothesis again, we deduce that

$$Z = T_2^{-1}\cdots T_{l-1}^{-1}T_l^{-2}T_{l-1}^{-1}\cdots T_2^{-1},$$

from which

$$\alpha_2\cdots\alpha_{l+1} = T_1^{-1}\cdots T_{l-1}^{-1}T_l^{-2}T_{l-1}^{-1}\cdots T_1^{-1}$$

follows. Thus the lemma is proved. □

2.2 For $e>0$ we set $P^n_{(0,e)} = S\sum_i Y_i^e S$. More generally, if $(r,d)=g\cdot(0,e)$, we put $P^n_{(r,d)} = \rho(g)P^n_{(0,e)}$. If the element $g'\in \mathrm{SL}_2(\mathbb{Z})$ fixes the pair $(0,e)$, then $\rho(g')=\rho_1^l$ for some l and hence $\rho(g')P^n_{(0,e)} = P^n_{(0,e)}$. Therefore the above definition makes sense, and for each $\mathbf{x}\in\mathbf{Z}^*$ it yields an element $P^n_{\mathbf{x}}\in \mathbf{S\ddot{H}}_n$ such that $\rho(g)P^n_{\mathbf{x}} = P^n_{g(\mathbf{x})}$ for any $g\in\mathrm{SL}_2(\mathbb{Z})$.

As an illustration, let us give the expressions for certain elements $P^n_{(r,d)}$ where r and d are relatively prime. To simplify the notation, we will drop the exponent n from $P^n_{(r,d)}$.

Lemma 2.2. *For any $l\in\mathbb{Z}$ we have*

$$P_{(l,1)} = [n]_t^+ SY_1X_1^lS, \tag{2.12}$$

$$P_{(1,l)} = q[n]_t^- SX_1Y_1^lS, \tag{2.13}$$

$$P_{(0,-1)} = q[n]_t^- SY_1^{-1}S, \tag{2.14}$$

$$P_{(l,-1)} = q[n]_t^- SX_1^lY_1^{-1}S. \tag{2.15}$$

Proof. Observe that since

$$SY_{i+1}S = ST_i^{-1}Y_iT_i^{-1}S = tSY_iS,$$

we have

$$P_{(0,1)} = S\sum_i Y_iS = [n]_t^+SY_1S.$$

Equation (2.12) follows from this and an application of ρ_2^l. In particular, we have $P_{(1,1)} = [n]_t^+SY_1X_1S$. By (2.10) in Lemma 2.1, we have

$$P_{(1,1)} = qt^{1-n}[n]_t^+SX_1Y_1S = q[n]_t^-SX_1Y_1S.$$

Applying ρ_1^l yields (2.13). The equalities (2.14) and (2.15) are proved using similar techniques. □

The value of $P_{(r,d)}$ when r and d are not relatively prime is usually harder to compute. Here we give a few examples, which will be important for our purposes later on.

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Lemma 2.3. *For any $l \geqslant 1$ we have*

$$P_{(-l,0)} = S \sum_i X_i^{-l} S, \tag{2.16}$$

$$P_{(0,-l)} = q^l S \sum_i Y_i^{-l} S, \tag{2.17}$$

$$P_{(l,0)} = q^l S \sum_i X_i^{l} S. \tag{2.18}$$

Proof. Set $A_3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $A_4 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$. Then

$$\rho(A_3)Y_1 = \rho_1\rho_2^{-1} \cdot Y_1 = X_1^{-1}$$

and hence $\rho(A_3)Y_i = X_i^{-1}$ for all i. The first relation (2.16) immediately follows. The proofs of the second and third relations are identical, so we treat only (2.17). By (2.10) in Lemma 2.1, we have

$$\begin{aligned}\rho(A_4)Y_1 &= \rho_2^{-1}\rho_1^2\rho_2^{-1} \cdot Y_1 = X_1^{-1}Y_1^{-1}X_1 \\ &= qY_1^{-1}T_1 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_1 \\ &= qT_1^{-1} \cdots T_{n-1}^{-1}Y_n^{-1}T_{n-1} \cdots T_1,\end{aligned}$$

where for the last equality we have used the relations $Y_i^{-1}T_i = T_i^{-1}Y_{i+1}$. It follows that

$$\rho(A_4)(Y_1^l) = q^l T_1^{-1} \cdots T_{n-1}^{-1} Y_n^{-l} T_{n-1} \cdots T_1 \tag{2.19}$$

for any l. Hence $\rho(A_4)(SY_1^lS) = q^l SY_n^{-l}S$. It is easy to show (and is a well-known fact) that the elements SY_1^lS, $l = 1, \ldots, n$, freely generate the ring $S\mathbb{C}[Y_1, \ldots, Y_n]S$. Let

$$\theta : \mathbb{C}[SY_1S, \ldots, SY_1^nS] \xrightarrow{\sim} S\mathbb{C}[Y_1, \ldots, Y_n]S = S\mathbb{C}[Y_1, \ldots, Y_n]^{\mathfrak{S}_n}S$$

and

$$\theta' : \mathbb{C}[SY_n^{-1}S, \ldots, SY_n^{-n}S] \xrightarrow{\sim} S\mathbb{C}[Y_1^{-1}, \ldots, Y_n^{-1}]S = S\mathbb{C}[Y_1^{-1}, \ldots, Y_n^{-1}]^{\mathfrak{S}_n}S$$

be defined in a similar fashion. Equation (2.17) is then a consequence of the following result.

Sublemma 2.4. *The composition $u = \theta' \circ \rho(A_4) \circ \theta^{-1}$ satisfies*

$$u(SP(Y_1, \ldots, Y_n)S) = q^l SP(Y_1^{-1}, \ldots, Y_n^{-1})S$$

for any symmetric polynomial $P(t_1, \ldots, t_n)$.

Proof of sublemma. Let $\dot{\mathbf{H}}^+_{n,Y}$ (respectively, $\dot{\mathbf{H}}^-_{n,Y}$) be the subalgebra generated by $\mathbf{H}_n$ and the elements $Y_1, \ldots, Y_n$ (respectively, $\mathbf{H}_n$ and the elements $Y_1^{-1}, \ldots, Y_n^{-1}$). The assignment $T_i \mapsto T_{n-i}$, $Y_i \mapsto Y_{n+1-i}^{-1}$ gives rise to an isomorphism of algebras $\Theta : \dot{\mathbf{H}}^+_{n,Y} \xrightarrow{\sim} \dot{\mathbf{H}}^-_{n,Y}$. It restricts to an isomorphism of spherical algebras $S\Theta : \mathbf{S}\dot{\mathbf{H}}^+_{n,Y} \xrightarrow{\sim} \mathbf{S}\dot{\mathbf{H}}^-_{n,Y}$. This last map clearly satisfies

$$S\Theta(SP(Y_1, \ldots, Y_n)S) = SP(Y_1^{-1}, \ldots, Y_n^{-1})S$$

for any symmetric polynomial $P(t_1, \ldots, t_n)$. It remains to observe that u coincides with $q^lS\Theta$ on the elements SY_1^lS, and that these elements generate $S\dot{\mathbf{H}}^+_{n,Y}S$ so that, in fact, $u = q^lS\Theta$. □

This establishes (2.17) and completes the proof of Lemma 2.3. □

Proposition 2.5. *The elements $\{P^n_{\mathbf{x}} : \mathbf{x} \in \mathbf{Z}^*\}$ generate $\mathbf{S}\ddot{\mathbf{H}}_n$ as a $\mathbf{K}'$-algebra.*

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Proof. Let R' be the localization of $\mathbf{R}'$ with respect to the multiplicative set generated by $[n]_t!$. Let $\ddot{\mathbf{H}}^{++}_{R',n}$ be the R'-subalgebra form of $\ddot{\mathbf{H}}_n$ generated by the elements T_i, X_j and Y_j for $i=1,\ldots,n-1$ and $j=1,\ldots,n$. We also set $\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}=S\ddot{\mathbf{H}}^{++}_{R',n}S$. For any $(r,d)\in\mathbf{Z}^{++}$, let $\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}[r,d]$ stand for the piece of $\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}$ of degree (r,d). We claim that $\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}[r,d]$ is free of finite rank as an R'-module. Indeed, it is known that $\ddot{\mathbf{H}}^{++}_{R',n}$ is free of finite rank over the subalgebra $R'[X_1,\ldots,X_n,Y_1,\ldots,Y_n]^{\mathfrak{S}_n\times\mathfrak{S}_n}$ (by the PBW and Pittie–Steinberg theorems). Since the spherical part $\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}$ is a direct summand of $\ddot{\mathbf{H}}^{++}_{R',n}$, the same holds for $\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}$. This proves our claim. Let $\pi:\mathbf{S}\ddot{\mathbf{H}}^{++}_{R',n}\to\mathbf{R}^{++}_n$ be the specialization at $q^{1/2}=t^{1/2}=1$, where $\mathbf{R}^{++}_n$ is the positive part of $\mathbf{R}_n$. One can check, by looking at the $\mathrm{SL}_2(\mathbb{Z})$ action defined above, that

$$\pi(P^n_{(r,d)})=\sum_i x_i^r y_i^d.$$

By Weyl's theorem (see [Wey49]), the elements $\{\pi(P^n_{(r,d)}):(r,d)\in\mathbf{Z}^{++}\}$ generate the ring $\mathbf{R}^{++}_n$. Applying Nakayama's lemma to each graded piece, we see that $\{P^n_{(r,d)}:(r,d)\in\mathbf{Z}^{++}\}$ generates the algebra $\mathbf{S}\ddot{\mathbf{H}}^{++}_n$ over the field $\mathbf{K}'$. To finish the proof of the proposition, we use the $\mathrm{SL}_2(\mathbb{Z})$-action once more. □

3. The projection map

3.1 Recall that $\mathbf{K}=\mathbb{C}(\sigma^{1/2},\bar{\sigma}^{1/2})$, $\mathbf{K}'=\mathbb{C}(t^{1/2},q^{1/2})$ and $\boldsymbol{\mathcal{E}}_{\mathbf{K}}=\boldsymbol{\mathcal{E}}_{\mathbf{R}}\otimes\mathbf{K}$. The first main result of this paper is the following.

Theorem 3.1. *For any $n>0$, the assignment $\sigma^{1/2}\mapsto q^{-1/2}$, $\bar{\sigma}^{1/2}\mapsto t^{-1/2}$ and*

$$u_{\mathbf{x}}\mapsto\frac{1}{q^{d(\mathbf{x})}-1}P^n_{\mathbf{x}}$$

for $\mathbf{x}\in\mathbf{Z}^*$ extends to a surjective $\mathbb{C}$-algebra homomorphism

$$\Phi_n:\boldsymbol{\mathcal{E}}_{\mathbf{K}}\to\mathbf{S}\ddot{\mathbf{H}}_n.$$

Proof. Fix an integer n and, for simplicity, drop the index n from the notation. We have to show that the elements $P^n_{\mathbf{x}}/(q^{d(\mathbf{x})}-1)$ satisfy relations (i) and (ii) of § 1.5. Relation (i) is clear for $\mathbf{x}=(0,r)$ and $\mathbf{x}'=(0,r')$ when r and r' are of the same sign, and it follows from (2.17) in Lemma 2.3 when r and r' are of different signs. By applying a suitable automorphism ρ_g with $g\in\mathrm{SL}_2(\mathbb{Z})$, we deduce relation (i) for any other line in $\mathbf{Z}$ through the origin.

The proof of relation (ii) is much more involved. We reduce it to establishing two sets of equalities, (3.3) and (3.9), which are dealt with in Appendix A and Appendix B, respectively. Note that the assignment Φ respects the $\mathrm{SL}_2(\mathbb{Z})$-action on both sides; hence it is enough to check relation (ii) for one pair in each orbit under this $\mathrm{SL}_2(\mathbb{Z})$-action. By the same argument as in [BS05, Theorem 5.4] (based on Pick's formula), we can constrain ourselves to the case where $\mathbf{x}=(1,0)$ and $\mathbf{y}=(0,l)$ with $l\in\mathbb{Z}^*$, or where $\mathbf{x}=(0,1)$ and $\mathbf{y}=(l,-1)$ with $l\geqslant 0$.

Case (A1). Assume that $\mathbf{x}=(1,0)$ and $\mathbf{y}=(0,l)$ with $l>0$. We have to show that

$$[\Phi(u_{(1,0)}),\Phi(u_{(0,l)})]=-\Phi(u_{(1,l)}),\tag{3.1}$$

which we may rewrite as

$$[P_{(1,0)},P_{(0,l)}]=(1-q^l)P_{(1,l)}.\tag{3.2}$$

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By Lemma 2.2, we have

$$P_{(1,0)} = q[n]_t^- SX_1S, \quad P_{(0,l)} = \sum_i SY_i^lS, \quad P_{(1,l)} = q[n]_t^- SX_1Y_1^lS;$$

hence verifying (3.2) is equivalent to proving the following proposition.

Proposition 3.2. *For any $l > 0$ we have*

$$\Big[SX_1S, \sum_i SY_i^lS\Big] = S\Big[X_1, \sum_i Y_i^l\Big]S = (1-q^l)SX_1Y_1^lS. \tag{3.3}$$

Case (A2). Now let us assume that $\mathbf{x} = (1,0)$ and $\mathbf{y} = (0,-l)$ with $l > 0$. We have to show that

$$[\Phi(u_{(1,0)}), \Phi(u_{(0,-l)})] = \Phi(u_{(1,-l)}), \tag{3.4}$$

which, after using the definitions and Lemmas 2.2 and 2.3, reduces to

$$\Big[SX_1S, \sum_i SY_i^{-l}S\Big] = (1-q^{-l})SX_1Y_1^{-l}S. \tag{3.5}$$

Consider the $\mathbb{C}$-algebra isomorphism $\sigma : \ddot{\mathbf{H}}_n \to \ddot{\mathbf{H}}_n$ given by

$$T_i \mapsto T_i^{-1}, \quad X_i \mapsto Y_i, \quad Y_i \mapsto X_i, \quad t^{1/2} \mapsto t^{-1/2}, \quad q^{1/2} \mapsto q^{-1/2};$$

see [Che04]. Applying σ to (3.5) gives the equation

$$\Big[SY_1S, \sum_i SX_i^{-l}S\Big] = (1-q^l)SY_1X_1^{-l}S,$$

which, once transformed by the automorphism $\rho(A_5)$, $A_5 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, is none other than (3.3). Thus this case also follows from Proposition 3.2.

Case (B). The final case to consider is that of $\mathbf{x} = (0,1)$ and $\mathbf{y} = (l,-1)$ with $l > 0$. Here we have to show that

$$[\Phi(u_{(0,1)}), \Phi(u_{(l,-1)})] = \frac{\Phi(\theta_{(l,0)})}{(1-t^{-1})(1-q^{-1})(1-qt)}, \tag{3.6}$$

which reduces to

$$\frac{(1-t^{-n})(1-qt)}{q-1}\Big[\sum_i SY_iS, SX_1^lY_1^{-1}S\Big] = \Phi(\theta_{(l,0)}). \tag{3.7}$$

By forming a generating series, we can write this as

$$1 + \sum_{l\geqslant 1} \frac{(1-t^{-n})(1-qt)}{q-1} S\Big[\sum_i Y_i, X_1^lY_1^{-1}\Big]Ss^l = 1 + \sum_{l\geqslant 1} \Phi(\theta_{(l,0)})s^l. \tag{3.8}$$

Given the definition of $\theta_{(l,0)}$, we obtain that establishing (3.6) for all $l > 0$ is equivalent to proving the following assertion.

Proposition 3.3. *The following holds:*

$$\exp\Big(\sum_{l\geqslant 1} \frac{(1-t^{-l})(1-q^lt^l)}{l} \sum_i SX_i^lSs^l\Big) = 1 + \sum_{l\geqslant 1} \frac{(1-qt)(1-t^{-n})}{q-1} S\Big[\sum_i Y_i, X_1Y_1^{-1}\Big]Ss^l. \tag{3.9}$$

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Thus, once Propositions 3.2 and 3.3 have been proved (see Appendix A and Appendix B), the proof of Theorem 3.1 will be complete. □

3.2 We may twist the map Φ_n by any of the automorphisms of $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ defined in [BS05, § 6.3]. Observe that the defining relations for $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ are invariant under any permutation of $\{\sigma, \overline{\sigma}, (\sigma\overline{\sigma})^{-1}\}$. We therefore have the following result.

Corollary 3.4. *For any $\gamma, \gamma' \in \{\sigma, \overline{\sigma}, (\sigma\overline{\sigma})^{-1}\}$ with $\gamma \neq \gamma'$, there is a surjective algebra morphism $\Phi_n^{\gamma,\gamma'} : \boldsymbol{\mathcal{E}}_{\mathbf{K}} \to \mathbf{S}\ddot{\mathbf{H}}_n$ such that $\gamma \mapsto q^{-1}, \gamma' \mapsto t^{-1}$ and $u_{\mathbf{x}} \mapsto P_{\mathbf{x}}^n/(q^{d(\mathbf{x})} - 1)$.*

4. Stable limits of DAHAs

4.1 In considering stable limits of DAHAs, we will be concerned with the graded subalgebras $\mathbf{S}\ddot{\mathbf{H}}_m^+$ and $\mathbf{S}\ddot{\mathbf{H}}_m^{++}$ of $\mathbf{S}\ddot{\mathbf{H}}_m$ generated by the elements $P_{\mathbf{x}}^m$ for $\mathbf{x} \in \mathbf{Z}^+$ and $\mathbf{x} \in \mathbf{Z}^{++}$, respectively. We have

$$\mathbf{S}\ddot{\mathbf{H}}_m^+ = \bigoplus_{\mathbf{x} \in \mathbf{Z}^+} \mathbf{S}\ddot{\mathbf{H}}_m^+[\mathbf{x}], \quad \mathbf{S}\ddot{\mathbf{H}}_m^{++} = \bigoplus_{\mathbf{x} \in \mathbf{Z}^{++}} \mathbf{S}\ddot{\mathbf{H}}_m^{++}[\mathbf{x}].$$

Proposition 4.1. *The assignment $P_{\mathbf{x}}^m \mapsto P_{\mathbf{x}}^{m-1}$ for each $\mathbf{x} \in \mathbf{Z}^+$ extends to a unique surjective $\mathbf{K}'$-algebra morphism $\Phi_m : \mathbf{S}\ddot{\mathbf{H}}_m^+ \to \mathbf{S}\ddot{\mathbf{H}}_{m-1}^+$. A similar statement holds for $\mathbf{S}\ddot{\mathbf{H}}_m^{++}$.*

Proof. The proof is based on the realization of double affine Hecke algebras as certain algebras of q-difference operators. Let $\mathbf{D}_m$ stand for the algebra $\mathbf{K}'[x_1^{\pm 1}, \partial_i^{\pm 1}, \ldots, x_m^{\pm 1}, \partial_m^{\pm 1}]$ with defining relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0 \quad \text{and} \quad \partial_i x_j = q^{\delta_{ij}} x_j \partial_i.$$

We also denote by $\mathbf{D}_{m,\mathrm{loc}}$ the localization of $\mathbf{D}_m$ with respect to the elements

$$\{x_i - t^{l/2} q^{n/2} x_j \mid l, n \in \mathbb{Z},\ i, j = 1, \ldots, m\}.$$

The symmetric group $\mathfrak{S}_m$ acts on $\mathbf{D}_{m,\mathrm{loc}}$ in an obvious fashion, and we may form the semidirect product $\mathbf{D}_{m,\mathrm{loc}} \rtimes \mathfrak{S}_m$. The following lemma is due to Cherednik.

Lemma 4.2 [Che04]. *Set $\omega = s_{m-1} \cdots s_1 \partial_1$. There is a unique embedding of algebras*

$$\varphi_m : \ddot{\mathbf{H}}_m \to \mathbf{D}_{m,\mathrm{loc}} \ltimes \mathfrak{S}_m$$

satisfying

$$\varphi_m(X_i) = x_i,$$
$$\varphi_m(T_i) = t^{-1/2} s_i + \frac{t^{-1/2} - t^{1/2}}{x_i/x_{i+1} - 1}(s_i - 1),$$
$$\varphi_m(Y_i) = \varphi_m(T_i) \cdots \varphi_m(T_{m-1}) \omega \varphi_m(T_1^{-1}) \cdots \varphi_m(T_{i-1}^{-1}).$$

It is known that $\varphi_m(\mathbf{S}\ddot{\mathbf{H}}_m) \subset \mathbf{D}_{m,\mathrm{loc}}^{\mathfrak{S}_m} \rtimes \mathfrak{S}_m$. Composing the restriction of φ_m to $\mathbf{S}\ddot{\mathbf{H}}_m$ with the projection

$$\mathbf{D}_{m,\mathrm{loc}}^{\mathfrak{S}_m} \rtimes \mathfrak{S}_m \to \mathbf{D}_{m,\mathrm{loc}}^{\mathfrak{S}_m}, \quad P(x_i^{\pm 1}, \partial_i^{\pm 1})\sigma \mapsto P(x_i^{\pm 1}, \partial_i^{\pm 1})$$

provides us with an embedding

$$\psi_m : \mathbf{S}\ddot{\mathbf{H}}_m \to \mathbf{D}_{m,\mathrm{loc}}^{\mathfrak{S}_m}.$$

We write $\mathbf{D}_{m,\mathrm{loc}}^{++}$ for $\mathbf{K}'[x_1, \partial_1, \ldots, x_m, \partial_m]_{\mathrm{loc}}$.

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Lemma 4.3. *We have* $\psi_m(\mathbf{S\ddot{H}}_m^{++}) \subset (\mathbf{D}_{m,\mathrm{loc}}^{++})^{\mathfrak{S}_m}$.

Proof. It is easy to see that $\boldsymbol{\mathcal{E}}_{\mathbf{K}}^{++}$ is generated by $\{u_{(0,l)}, u_{(l,0)} : l \geqslant 1\}$. Hence, by Theorem 3.1, $\mathbf{S\ddot{H}}_m^{++}$ is generated by $\{P_{(0,l)}^m, P_{(l,0)}^m : l \geqslant 1\}$; it therefore suffices to check the veracity of the lemma for these elements, which is obvious. □

We now consider the map

$$\pi_m : (\mathbf{D}_{m,\mathrm{loc}}^{++})^{\mathfrak{S}_m} \to (\mathbf{D}_{m-1,\mathrm{loc}})^{\mathfrak{S}_{m-1}}$$

that sends x_l, ∂_l to $x_l, t^{-1/2}\partial_l$ for $l < m$ and x_m, ∂_m to zero. This is a well-defined algebra homomorphism. We can summarize the situation in a diagram of algebra homomorphisms

$$\begin{array}{ccc} \mathbf{S\ddot{H}}_m^{++} & \xrightarrow{\psi_m} & (\mathbf{D}_{m,\mathrm{loc}}^{++})^{\mathfrak{S}_m} \\ & & \downarrow{\scriptstyle \pi_m} \\ \mathbf{S\ddot{H}}_{m-1}^{++} & \xrightarrow{\psi_{m-1}} & (\mathbf{D}_{m-1,\mathrm{loc}})^{\mathfrak{S}_{m-1}} \end{array}$$

where ψ_m and ψ_{m-1} are embeddings. Therefore, Proposition 4.1 will be proved for the algebra $\mathbf{S\ddot{H}}_m^{++}$ once we show that

$$\pi_m \circ \psi_m(P_{\mathbf{x}}^m) = \psi_{m-1}(P_{\mathbf{x}}^{m-1}) \quad \text{for } \mathbf{x} \in \mathbf{Z}^{++}. \tag{4.1}$$

Lemma 4.4. *For any* $\mathbf{x} \in \mathbf{Z}^{++}$*, there exists a polynomial* $Q_{\mathbf{x}} \in \mathbf{F}_\infty$ *such that the following formula holds in* $\mathbf{S\ddot{H}}_m$ *for any* m*:*

$$Q_{\mathbf{x}}(P_{(0,1)}^m, P_{(0,2)}^m, \ldots, P_{(1,0)}^m, P_{(2,0)}^m, \ldots) = P_{\mathbf{x}}^m.$$

Proof. Since $\boldsymbol{\mathcal{E}}_{\mathbf{K}}^{++}$ is generated by the elements $u_{(0,l)}$ and $u_{(l,0)}$ with $l \geqslant 1$, for any $\mathbf{x} \in \mathbf{Z}^{++}$ there exists a polynomial $R_{\mathbf{x}}$ such that

$$R_{\mathbf{x}}(u_{(0,1)}, u_{(0,2)}, \ldots, u_{(1,0)}, u_{(2,0)}, \ldots) = u_{\mathbf{x}}.$$

By Theorem 3.1, we may take as $Q_{\mathbf{x}}$ the polynomial defined by

$$\begin{aligned} &Q_{\mathbf{x}}(z_{(0,1)}, z_{(0,2)}, \ldots, z_{(1,0)}, z_{(2,0)}, \ldots) \\ &\quad = \frac{1}{o(d(\mathbf{x}))} R_{\mathbf{x}}(o(1)z_{(0,1)}, o(2)z_{(0,2)}, \ldots, o(1)z_{(1,0)}, o(2)z_{(2,0)}, \ldots), \end{aligned}$$

where we have set $o(l) = 1/(q^l - 1)$. □

Lemma 4.4 implies that it is enough to show that (4.1) holds for $\mathbf{x} = (l, 0)$ or for $\mathbf{x} = (0, l)$. In the case of $\mathbf{x} = (l, 0)$, this is obvious by definition. We shall deal with the second case. Of course, it suffices to prove that

$$\pi_m \circ \psi_m(S f_r(Y_1, \ldots, Y_m) S) = \psi_{m-1}(S f_r(Y_1, \ldots, Y_{m-1}, 0) S)$$

for any family of symmetric polynomials $\{f_r\}$ that generates the ring $\mathbb{C}[Y_1, \ldots, Y_m]^{\mathfrak{S}_m}$. In particular, for f_r we can take the monomial symmetric function

$$m_{1^r}(Y_1, \ldots, Y_m) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant m} Y_{i_1} \cdots Y_{i_r}.$$

We now use the following explicit computation of $\psi_n(S m_{1^r}(Y_1, \ldots, Y_n) S)$; see [Mac95, ch. VI.5] for a proof.

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Lemma 4.5 (Macdonald). *For any n and any $l \geqslant 1$,*

$$\psi_n(Sm_r(Y_1,\ldots,Y_n)S)=\sum_{I\subset\{1,\ldots,n\}}A_I(x_1,\ldots,x_n)\prod_{i\in I}\partial_i$$

where I ranges over all subsets of $\{1,\ldots,n\}$ of size r and

$$A_I(x_1,\ldots,x_n)=\prod_{i\in I,j\notin I}\frac{t^{-1/2}x_i-t^{1/2}x_j}{x_i-x_j}.$$

By the above lemma, we have

$$\begin{aligned}\pi_m\circ\psi_m(Sm_{1^r}(Y_1,\ldots,Y_m)S)&=\sum_{I\subset\{1,\ldots,m-1\}}A_I(x_1,\ldots,x_{m-1},0)\prod_{i\in I}t^{1/2}\partial_i\\&=\sum_{I\subset\{1,\ldots,m-1\}}A_I(x_1,\ldots,x_{m-1})\prod_{i\in I}\partial_i\\&=\psi_{m-1}(Sm_{1^r}(Y_1,\ldots,Y_{m-1})S).\end{aligned}$$

We have thus proved that the assignment $P^m_{\mathbf{x}}\mapsto P^{m-1}_{\mathbf{x}}$ for each $\mathbf{x}\in\mathbf{Z}^{++}$ extends to a surjective $\mathbf{K}'$-algebra homomorphism $\Psi_m:\mathbf{S\ddot{H}}^{++}_m\to\mathbf{S\ddot{H}}^{++}_{m-1}$. By applying the operator $\rho(A_1^{-k})$ and using the fact that $\rho(g)\cdot P^n_{\mathbf{x}}=P^n_{g\cdot\mathbf{x}}$ for any $g\in\mathrm{SL}_2(\mathbb{Z})$ and $\mathbf{x}\in\mathbf{Z}$, we deduce that the map Ψ_m extends to a surjective algebra homomorphism

$$\Psi_m:\mathbf{S\ddot{H}}^{\geqslant -k}_m\to\mathbf{S\ddot{H}}^{\geqslant -k}_{m-1}.$$

Here, for any n, we have written $\mathbf{S\ddot{H}}^{\geqslant -k}_n$ for the subalgebra of $\mathbf{S\ddot{H}}^{+}_n$ generated by the elements $P^n_{\mathbf{x}}$ such that $\mathbf{x}=(r,d)\in\mathbf{Z}^+$ satisfies $d/r\geqslant -k$. Letting k tend to infinity, we finally obtain that the map Ψ_m extends to a surjective algebra homomorphism

$$\Psi_m:\mathbf{S\ddot{H}}^{+}_m\to\mathbf{S\ddot{H}}^{+}_{m-1}$$

such that $\Psi_m(P^m_{\mathbf{x}})=P^{m-1}_{\mathbf{x}}$ for all $\mathbf{x}\in\mathbf{Z}^+$. This completes the proof of Proposition 4.1. □

4.2 Proposition 4.1 allows us to define the projective limits

$$\varprojlim \mathbf{S\ddot{H}}^{+}_m\quad\text{and}\quad\varprojlim \mathbf{S\ddot{H}}^{++}_m.$$

By construction, the collection of generators $P^m_{\mathbf{x}}$, $m\geqslant 1$, gives rise to elements $P_{\mathbf{x}}$ of these projective limits. Let $\mathbf{S\ddot{H}}^{+}_\infty$ and $\mathbf{S\ddot{H}}^{++}_\infty$ stand for the subalgebras generated by $P_{\mathbf{x}}$ for $\mathbf{x}\in\mathbf{Z}^+$ and $\mathbf{x}\in\mathbf{Z}^{++}$, respectively; we shall call these the stable limits of the projective systems $(\mathbf{S\ddot{H}}^{+}_m)$ and $(\mathbf{S\ddot{H}}^{++}_m)$, respectively. We may view $\varprojlim \mathbf{S\ddot{H}}^{+}_m$ and $\varprojlim \mathbf{S\ddot{H}}^{++}_m$ as completions of $\mathbf{S\ddot{H}}^{+}_\infty$ and $\mathbf{S\ddot{H}}^{++}_\infty$.

By construction, the map $\Phi_m:\boldsymbol{\mathcal{E}}_{\mathbf{K}}\to\mathbf{S\ddot{H}}_m$ sends $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ and $\boldsymbol{\mathcal{E}}^{++}_{\mathbf{K}}$ onto $\mathbf{S\ddot{H}}^{+}_m$ and $\mathbf{S\ddot{H}}^{++}_m$, respectively. Let us call Φ^+_m and Φ^{++}_m the restrictions of Φ_m to $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ and $\boldsymbol{\mathcal{E}}^{++}_{\mathbf{K}}$, respectively. The collection of maps Φ^+_m and Φ^{++}_m gives rise, in the limit, to algebra homomorphisms $\Phi^+_\infty:\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}\to\mathbf{S\ddot{H}}^{+}_\infty$ and $\Phi^{++}_\infty:\boldsymbol{\mathcal{E}}^{++}_{\mathbf{K}}\to\mathbf{S\ddot{H}}^{++}_\infty$.

Theorem 4.6. *The maps Φ^+_∞ and Φ^{++}_∞ are algebra isomorphisms.*

Proof. Both Φ^+_∞ and Φ^{++}_∞ are surjective by construction; we have to show their injectivity. The subgroup

$$\mathbf{G}=\left\{\begin{pmatrix}1&n\\0&1\end{pmatrix}:n\in\mathbb{Z}\right\}\subset\mathrm{SL}_2(\mathbb{Z})$$

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preserves $\mathbf{Z}^+$, and for any $\mathbf{x} \in \mathbf{Z}^+$ there exists $g \in \mathbf{G}$ such that $g \cdot \mathbf{x} \in \mathbf{Z}^{++}$. Since the map Φ^+_∞ is clearly compatible with the action of $\mathbf{G}$ on $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ and $\mathbf{S\ddot{H}}^+_\infty$, it in fact suffices to prove the injectivity of Φ^{++}_∞.

Now fix $(r, d) \in \mathbf{Z}^{++}$. By [BS05, § 5], the dimension of the weight space $\boldsymbol{\mathcal{E}}^{++}_{\mathbf{K}}[r, d]$ is equal to the number of convex paths $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ with $\mathbf{x}_i \in \mathbf{Z}^{++}$ for all i and $\sum_i \mathbf{x}_i = (r, d)$. By the proof of Proposition 2.5, the dimension of the weight space $\mathbf{S\ddot{H}}^{++}_n[r, d]$ is equal to the dimension of the space of polynomial diagonal invariants

$$\mathbf{R}^{++}_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{\mathfrak{S}_n}$$

of x-degree r and y-degree d.

The latter dimension is equal to the number of orbits under $\mathfrak{S}_n$ of monomials $x_1^{g_1} \cdots x_n^{g_n} y_1^{h_1} \cdots y_n^{h_n}$ with $g_i, h_i \in \mathbb{N}$ satisfying $\sum_i g_i = r$ and $\sum_i h_i = d$; equivalently, it is equal to the number of n-tuples of pairs $\{(g_1, h_1), \ldots, (g_n, h_n)\}$ such that $\sum_i g_i = r$ and $\sum_i h_i = d$, or the number of convex paths $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ in $\mathbf{Z}^{++}$ of length $r \leqslant n$ which have $\sum_i \mathbf{x}_i = (r, d)$. It remains to observe that for any given (r, d), the length of the convex paths $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ in $\mathbf{Z}^{++}$ that have $\sum_i \mathbf{x}_i = (r, d)$ is bounded above, say by $n(r, d)$. Hence

$$\dim \boldsymbol{\mathcal{E}}^{++}_{\mathbf{K}}[r, d] = \dim \mathbf{S\ddot{H}}^{++}_n[r, d]$$

whenever $n \geqslant n(r, d)$, and so, finally,

$$\dim \boldsymbol{\mathcal{E}}^{++}_{\mathbf{K}}[r, d] = \dim \mathbf{S\ddot{H}}^{++}_\infty[r, d].$$

The injectivity of the map Φ^{++}_∞ follows, and Theorem 4.6 is proved. □

Of course, Theorem 4.6 holds for the stable limits of the twisted versions $\Phi^{+,\gamma,\gamma'}_n$ and $\Phi^{++,\gamma,\gamma'}_n$ as well (see § 3.2).

Remarks. Theorem 4.6 allows us to transform the PBW basis $\{\beta_{\mathbf{p}} : \mathbf{p} \in \mathbf{Conv}^+\}$ and the canonical basis $\{\mathbf{b}_{\mathbf{p}} : \mathbf{p} \in \mathbf{Conv}^+\}$ of $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ defined in [Sch05, § 2.3] to bases $\{\gamma_{\mathbf{p}} : \mathbf{p} \in \mathbf{Conv}^+\}$ and $\{\mathbf{c}_{\mathbf{p}} : \mathbf{p} \in \mathbf{Conv}^+\}$ of $\mathbf{S\ddot{H}}^+_\infty$ such that $\gamma_{\mathbf{p}} = \Phi^+_\infty(\beta_{\mathbf{p}})$ and $\mathbf{c}_{\mathbf{p}} = \Phi^+_\infty(\mathbf{b}_{\mathbf{p}})$. The element $\mathbf{c}_{\mathbf{p}}$ belongs to the completion $\widehat{\mathbf{SH}}^+_\infty$ of $\mathbf{S\ddot{H}}^+_\infty$, which is equal to the sum $\bigoplus_{(r,d)} \widehat{\mathbf{SH}}^+_\infty[r, d]$ over all pairs $(r, d) \in \mathbf{Z}^+$ of the vector spaces

$$\widehat{\mathbf{SH}}^+_\infty[r, d] = \prod_{\mathbf{p}} \mathbf{K}' \gamma_{\mathbf{p}}$$

where $\mathbf{p}$ runs over all paths $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ in $\mathbf{Conv}^+$ satisfying $\sum_i \mathbf{x}_i = (r, d)$.

4.3 Theorem 4.6 could be viewed within the context of the theory of the classical Hall algebra $\mathbf{H}_{\mathrm{cl}}$ of a discrete valuation ring $\mathcal{O}$; see [Mac95, ch. II]. Recall that $\mathbf{H}_{\mathrm{cl}}$ is canonically isomorphic to the algebra Λ^+_v and that this isomorphism naturally fits into a chain

$$\mathbf{H}_{\mathrm{cl}} \simeq \mathbf{S\dot{H}}^+_\infty \simeq \Lambda^+_v \tag{4.2}$$

where $\mathbf{S\dot{H}}^+_\infty$ is the stable limit of the positive spherical affine Hecke algebra of type GL_n as n tends to infinity. Hence Theorem 4.6 may be interpreted as an affine version of (4.2). Observe that $\mathbf{S\dot{H}}^+_\infty$ is a *trivial* one-parameter deformation Λ^+_v of Λ^+, while $\mathbf{S\ddot{H}}^+_\infty$ is a *non-trivial* two-parameter deformation of the ring

$$\mathbf{R}^+ = \mathbb{C}[x_1, x_2, \ldots, y_1^{\pm 1}, y_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty}.$$

The analogy can be summarized in Table 1.

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Table 1.

Classical Hall algebra $\mathbf{H}_{\mathrm{cl}}$	Elliptic Hall algebra $\boldsymbol{\mathcal{E}}^+$
$\mathcal{O}$-Mod	$\mathrm{Coh}(X)$
$\Lambda^+ = \mathbb{C}[x_1, x_2, \ldots]^{\mathfrak{S}_\infty}$	$\mathbf{R}^+ = \mathbb{C}[x_1, x_2, \ldots, y_1^{\pm 1}, y_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty}$
$\Theta_\infty^+ : \mathbf{H}_{\mathrm{cl}} \xrightarrow{\sim} \mathbf{S\dot{H}}_\infty^+$	$\Phi_\infty^+ : \boldsymbol{\mathcal{E}}^+ \xrightarrow{\sim} \mathbf{S\ddot{H}}_\infty^+$
$\Pi = \bigsqcup_n (\mathbb{Z}^+)^n/\mathfrak{S}_n$	$\mathbf{Conv}^+ = \bigsqcup_n (\mathbf{Z}^+)^n/\mathfrak{S}_n$
$\mathbf{1}_{\mathcal{O}_\lambda} = v^{-2n(\lambda)} P_\lambda(v^2)$	PBW basis $\beta_{\mathbf{p}}$
$\mathcal{N}_n,\ n \in \mathbb{N}$	$\underline{\mathrm{Coh}}^{r,d}(X),\ (r,d) \in \mathbf{Z}^+$
$\bigsqcup_n \mathcal{P}_{\mathrm{GL}_n}(\mathcal{N}_n)$	$\bigsqcup_{r,d} \mathcal{Q}^{r,d}$
$IC(\mathcal{O}_\lambda),\ \lambda \in \Pi$	$\mathbb{P}_{\mathbf{p}},\ \mathbf{p} \in \mathbf{Conv}^+$
$\Theta_\infty^+(\mathrm{tr}(IC(\mathcal{O}_\lambda)) = s_\lambda$	$\Phi_\infty^+(\mathrm{tr}(\mathbb{P}_{\mathbf{p}})) = \mathbf{c}_{\mathbf{p}}$
$K_{\lambda,\mu}(v) \in \mathbb{N}[v]$	$\urcorner_{\mathbf{p},\mathbf{q}} \in \mathbb{N}[v, -\sigma v]$
Affine Grassmannian $\widehat{\mathrm{Gr}}$	??
Geometric Satake isomorphism	??
$\bigsqcup_n \mathcal{P}_{\mathrm{GL}_n}(\mathcal{N}_n) \simeq \mathrm{Rep}^+ \mathrm{GL}_\infty$	

Here P_λ is the Hall–Littlewood polynomial, s_λ is the Schur polynomial, and $K_{\lambda,\mu}$ is the Kostka polynomial. The middle portion of the table is based on the geometric version of the elliptic Hall algebra, which involves the theory of *automorphic sheaves* as defined in [Lau90] and studied in detail in [Sch05] for an elliptic curve. Here $\mathcal{Q}^{r,d}$ is a certain category of semisimple perverse sheaves over the moduli stack $\underline{\mathrm{Coh}}^{r,d}(X)$ of coherent sheaves of rank r and degree d over an elliptic curve X, while the $\mathbb{P}_{\mathbf{p}}$, $\mathbf{p} \in \mathbf{Conv}^+$, are the simple perverse sheaves in $\bigsqcup_{r,d} \mathcal{Q}^{r,d}$. The basis elements $\mathbf{c}_{\mathbf{p}}$, defined as the traces of the $\mathbb{P}_{\mathbf{p}}$, are analogues of the Kazhdan–Lusztig basis elements of $\mathbf{S\dot{H}}_\infty^+$. Finally, the coefficients $\urcorner_{\mathbf{p},\mathbf{q}}$ are the entries of the transition matrices between the $\mathbf{c}_{\mathbf{p}}$ and the PBW basis elements $\beta_{\mathbf{q}}$ (essentially, the Poincaré polynomials of the stalks of the $\mathbb{P}_{\mathbf{p}}$ over the stratas of $\underline{\mathrm{Coh}}^{r,d}(X)$); we refer the reader to [Sch05] for more details.

In the bottom part of the table, we point out two important features of the classical picture for which we do not know of any analogue in the elliptic Hall algebra setting: one is that functions on the nilpotent cone $\mathcal{N}_n$ may be lifted to functions on some Schubert variety of the affine Grassmannian $\widehat{\mathrm{Gr}}$ of type GL_n; the other is that the category of perverse sheaves $\bigsqcup_n \mathcal{P}_{\mathrm{GL}_n}(\mathcal{N}_n)$ is

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equivalent to the category $\mathrm{Rep}^+(\mathrm{GL}_\infty)$ of finite-dimensional *polynomial* representations of GL_∞ (see [Gin95, MV00]).

5. Macdonald polynomials

5.1 Macdonald discovered in [Mac88] a remarkable family of symmetric polynomials $P_\lambda(q,t)$ that depend on two parameters and from which many of the classical symmetric functions can be obtained via specializations. Because of our sign conventions (see the footnote in § 2.1), we will actually be working with $P_\lambda(q,t^{-1})$ rather than $P_\lambda(q,t)$.

The Macdonald polynomials are defined as eigenfunctions of certain difference operators acting on the spaces of symmetric functions

$$\Lambda^m_{(q,t)} = \mathbf{K}'[x_1,\dots,x_m]^{\mathfrak{S}_m}.$$

Recall the embedding $\psi_m : \mathbf{S\ddot{H}}^+_m \to \mathbf{D}^{\mathfrak{S}_m}_{m,\mathrm{loc}}$, which gives rise to an action ρ_m of $\mathbf{S\ddot{H}}^+_m$ on $\Lambda^m_{(q,t)}$. Consider the following linear operator on $\Lambda^m_{(q,t)}$:

$$D_m = \rho_m(S(Y_1+\cdots+Y_m)S) = \sum_{i=1}^m \bigg(\prod_{j\neq i}\frac{t^{-1/2}x_i - t^{1/2}x_j}{x_i - x_j}\bigg)\partial_i.$$

By [Mac95, ch. VI, (3.10)], the operator D_m is upper triangular with respect to the basis $\{m_\lambda\}$ of monomial symmetric functions and has distinct eigenvalues.

We are interested in the stable limit as m goes to infinity of the corresponding eigenfunctions. Let $\theta_m : \Lambda^m_{(q,t)} \to \Lambda^{m-1}_{(q,t)}$ be the specialization $x_m = 0$. It is not true that $\theta_m \circ D_m = D_{m-1}\circ\theta_m$; however, the operator $E_m = t^{(m-1)/2}(D_m - [m])$ does satisfy $\theta_m\circ E_m = E_{m-1}\circ\theta_m$. Recall that the space

$$\Lambda^+_{(q,t)} = \mathbf{K}'[x_1,x_2,\dots]^{\mathfrak{S}_\infty}$$

of symmetric functions is the projective limit of $(\Lambda^m_{(q,t)},\theta_m)$ in the category of graded rings; see [Mac95, ch. I, Remark 1.2.1]. Hence the operators E_m, $m\geqslant 1$, give rise to a linear operator E on the space $\Lambda^+_{(q,t)}$. This operator is still upper triangular with respect to the basis $\{m_\lambda\}$, and it has distinct eigenvalues $\{\alpha_\lambda\}$ given by

$$\alpha_\lambda = \sum_{i\geqslant 1}(q^{\lambda_i}-1)t^{i-1}. \tag{5.1}$$

The Macdonald polynomial is defined to be the unique α_λ-eigenvector of E such that

$$P_\lambda(q,t^{-1}) \in m_\lambda \oplus \bigoplus_{\mu<\lambda}\mathbf{K}' m_\mu.$$

For a pair of partitions $\mu\subset\lambda$, the skew Macdonald polynomial $P_{\lambda/\mu}(q,t^{-1})$ is determined by the coproduct formula

$$\Delta(P_\lambda(q,t^{-1})) = \sum_{\mu\subset\lambda} P_\mu(q,t^{-1})\otimes P_{\lambda/\mu}(q,t^{-1}).$$

Here $\Delta : \Lambda^+_{(q,t)} \to \Lambda^+_{(q,t)}\otimes\Lambda^+_{(q,t)}$ is the standard coproduct on the ring $\Lambda^+_{(q,t)}$ (satisfying, for example, $\Delta(p_l) = p_l\otimes 1 + 1\otimes p_l$ for power-sum functions $p_l = \sum_i x_i^l$).

Examples.

(i) We have $P_{(1^r)}(q,t^{-1}) = e_r$.

(ii) We have

$$P_{(r)}(q,t^{-1}) = \prod_{l=0}^{r-1} \frac{1-q^{l+1}}{1-t^{-1}q^l} \cdot \sum_{\lambda \vdash r} z_\lambda(q,t^{-1})^{-1} p_\lambda,$$

where

$$z_\lambda(q,t^{-1}) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{-\lambda_i}}, \quad z_{(1^{m_1}2^{m_2}\ldots)} = \prod_i i^{m_i} m_i!.$$

In particular,

$$P_{(2)}(q,t^{-1}) = \frac{(1-q)(1+t^{-1})}{2(1-qt^{-1})} p_2 + \frac{(1+q)(1-t^{-1})}{2(1-qt^{-1})} p_1^2.$$

Remark. The representations $\rho_m : \mathbf{S}\ddot{\mathbf{H}}_m^+ \to \mathrm{End}(\Lambda^m_{(q,t)})$ lift, after a suitable renormalization, to a stable limit representation $\rho_\infty : \mathbf{S}\ddot{\mathbf{H}}_\infty \to \mathrm{End}(\Lambda^+_{(q,t)})$ in which $P_{(0,1)} = S(\sum_i Y_i)S$ acts as the operator E. By composing with the isomorphism Φ^+_∞, we obtain a representation of the Hall algebra $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ on $\Lambda_{(q,t)}$ in which the element $u_{(0,1)}$, i.e. the so-called *Hecke operator*, acts as Macdonald's operator $E/(q-1)$. We will not need this representation here (but see [SV09, § 4.3]).

5.2 There are many different characterizations of Macdonald polynomials; see [Hai02], for instance. The characterization that best fits our needs treats the polynomials $P_\lambda(q,t^{-1})$ and $P_{\lambda/\mu}(q,t^{-1})$ at the same time. First, let us recall some standard notation from [Mac95].

Let $\mu \subset \lambda$ be two partitions, and put $|\lambda/\mu| = |\lambda| - |\mu|$. The skew partition λ/μ is said to be a *vertical strip* if $\lambda_i - \mu_i \leqslant 1$ for all i, i.e. if the corresponding diagram contains at most one box per row. A skew partition λ/μ is a *horizontal strip* if its conjugate λ'/μ' is a vertical strip. If λ/μ is a horizontal strip, we put

$$\psi_{\lambda/\mu}(q,t^{-1}) = \prod \frac{(1-t^{\mu'_j-\mu'_i}q^{j-i-1})(1-t^{1+\mu'_j-\mu'_i}q^{j-i+1})}{(1-t^{\mu'_j-\mu'_i}q^{j-i})(1-t^{1+\mu'_j-\mu'_i}q^{j-i})},$$

where the sum ranges over all pairs (i,j) with $i<j$ such that $\mu'_i = \lambda'_i$ but $\mu'_i = \lambda'_i - 1$. In particular, we have $\psi_{\lambda/\mu}(q,t^{-1}) = 1$ if λ/μ is a horizontal strip containing no empty columns.

Proposition 5.1. *The family $\{P_{\lambda/\mu}(q,t^{-1}) : \mu \subset \lambda\}$ is uniquely determined by the following set of properties.*

(i) *The polynomial $P_{\lambda/\mu}(q,t^{-1})$ is homogeneous of degree $|\lambda/\mu|$.*

(ii) *We have*

$$\Delta(P_{\lambda/\mu}(q,t^{-1})) = \sum_{\mu \subseteq \nu \subseteq \lambda} P_{\nu/\mu}(q,t^{-1}) \otimes P_{\lambda/\nu}(q,t^{-1}).$$

(iii) *If λ/μ is not a horizontal strip, then*

$$P_{\lambda/\mu}(q,t^{-1}) \in \bigoplus_{\nu<(r)} \mathbf{K}' m_\nu \quad \text{where } r = |\lambda/\mu|.$$

(iv) *If λ/μ is a horizontal strip, then*

$$P_{\lambda/\mu}(q,t^{-1}) \in \psi_{\lambda/\mu}(q,t^{-1}) m_r \oplus \bigoplus_{\nu<(r)} \mathbf{K}' m_\nu \quad \text{where } r = |\lambda/\mu|.$$

Proof. Properties (i) through (iv) are all known to hold for Macdonald polynomials: statement (ii) follows from [Mac95, ch. VI, § 7, (7.9')], while statements (iii) and (iv) are consequences of [Mac95, ch. VI, § 7, (7.13')]). We now prove the uniqueness of polynomials satisfying properties (i) through (iv). Let $Q_{\lambda/\mu}(q,t^{-1})$ be such a family. When $|\lambda/\mu|=1$, by (iv) we have

$$Q_{\lambda/\mu}(q,t^{-1})=\psi_{\lambda/\mu}(q,t^{-1})m_1=P_{\lambda/\mu}(q,t^{-1}).$$

Let $r>1$ and assume that $Q_{\eta/\nu}(q,t^{-1})=P_{\eta/\nu}(q,t^{-1})$ for all η/ν satisfying $|\eta/\nu|<r$. Let λ/μ be a skew partition with $|\lambda/\mu|=r$. By (ii) and the induction hypothesis,

$$\begin{aligned}\Delta(Q_{\lambda/\mu}(q,t^{-1}))&=Q_{\lambda/\mu}(q,t^{-1})\otimes 1+1\otimes Q_{\lambda/\mu}(q,t^{-1})+\sum_{\mu\subset\nu\subset\lambda}Q_{\nu/\mu}(q,t^{-1})\otimes Q_{\lambda/\nu}(q,t^{-1})\\&=Q_{\lambda/\mu}(q,t^{-1})\otimes 1+1\otimes Q_{\lambda/\mu}(q,t^{-1})+\sum_{\mu\subset\nu\subset\lambda}P_{\nu/\mu}(q,t^{-1})\otimes P_{\lambda/\nu}(q,t^{-1}).\end{aligned}$$

It follows that $Q_{\lambda/\mu}(q,t^{-1})-P_{\lambda/\mu}(q,t^{-1})$ is contained in

$$\mathrm{Ker}(\Delta-\mathrm{Id}\otimes 1-1\otimes\mathrm{Id})=\mathbf{K}'p_{|\lambda/\mu|}.$$

But then the coefficient of $p_{|\lambda/\mu|}$ in $Q_{\lambda/\mu}(q,t^{-1})$ is uniquely determined by (iii) or (iv), and

$$Q_{\lambda/\mu}(q,t^{-1})=P_{\lambda/\mu}(q,t^{-1}).$$ □

6. Eisenstein series

6.1 We return to the setting of § 1; that is, we assume that X is a smooth elliptic curve over $\mathbb{F}_l$, $\mathbf{H}_X$ is its Hall algebra, and $\mathbf{U}^+_X\subset\mathbf{H}_X$ is the spherical subalgebra introduced in § 1.4. Recall that $\mathbf{H}_X$ and $\mathbf{U}^+_X$ are $\mathbf{Z}$-graded in the following way:

$$\mathbf{H}_X=\bigoplus_{(r,d)}\mathbf{H}_X[r,d],\quad \mathbf{U}^+_X=\bigoplus_{(r,d)}\mathbf{U}^+_X[r,d].$$

The Eisenstein series that we will need to consider are certain elements of a completion of the Hall algebra, which we now define in detail. Let $\widehat{\mathbf{H}}_X[r,d]$ stand for the space of *all* functions $f:\mathcal{I}(X)_{r,d}\to\mathbb{C}$ on the set of coherent sheaves of rank r and degree d, and put $\widehat{\mathbf{H}}_X=\bigoplus_{(r,d)}\widehat{\mathbf{H}}_X[r,d]$. By [BS05, Proposition 2.2], the space $\widehat{\mathbf{H}}_X$ is still a bialgebra. Recall from § 1.3 that, as a vector space,

$$\mathbf{H}_X[r,d]=\bigoplus_{\alpha_1,\ldots,\alpha_n}\mathbf{H}_X^{(\mu(\alpha_1))}[\alpha_1]\otimes\cdots\otimes\mathbf{H}_X^{(\mu(\alpha_n))}[\alpha_n]$$

where the sum ranges over all tuples $(\alpha_1,\ldots,\alpha_n)$ of elements in $\mathbf{Z}^+$ that satisfy $\mu(\alpha_1)<\cdots<\mu(\alpha_n)$ and $\sum\alpha_i=(r,d)$. Then we have

$$\widehat{\mathbf{H}}_X[r,d]=\prod_{\alpha_1,\ldots,\alpha_n}\mathbf{H}_X^{(\mu(\alpha_1))}[\alpha_1]\otimes\cdots\otimes\mathbf{H}_X^{(\mu(\alpha_n))}[\alpha_n].$$

In a similar fashion, we define the subalgebra $\widehat{\mathbf{U}}^+_X$ of $\widehat{\mathbf{H}}_X$ by $\widehat{\mathbf{U}}^+_X=\bigoplus_{(r,d)}\widehat{\mathbf{U}^+}_X[r,d]$ where

$$\widehat{\mathbf{U}}^+_X[r,d]=\prod_{\alpha_1,\ldots,\alpha_n}\mathbf{U}_X^{(\mu(\alpha_1))}[\alpha_1]\otimes\cdots\otimes\mathbf{U}_X^{(\mu(\alpha_n))}[\alpha_n].$$

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For instance, for any (r, d) the element

$$\mathbf{1}_{(r,d)} = \sum_{\overline{\mathcal{F}}=(r,d)} \mathbf{1}_{\mathcal{F}}$$

is a function with infinite support that belongs to $\widehat{\mathbf{U}}_X^+[r, d]$, since it may be written as the infinite sum (see [BS05, (4.4)])

$$\mathbf{1}_{r,d} = \mathbf{1}^{ss}_{r,d} + \sum_{\substack{\mu(\alpha_1)<\cdots<\mu(\alpha_n)\\ \alpha_1+\cdots+\alpha_n=(r,d)}} v^{\sum_{i<j}\langle\alpha_i,\alpha_j\rangle} \mathbf{1}^{ss}_{\alpha_1}\cdots\mathbf{1}^{ss}_{\alpha_n}. \tag{6.1}$$

6.2 Consider the generating series

$$\mathbf{E}_0(z) = 1 + \sum_{d\geqslant 1} \mathbf{1}_{(0,d)} v^{-d} z^d$$

and, for $r \geqslant 1$,

$$\mathbf{E}_r(z) = \sum_{d\in\mathbb{Z}} \mathbf{1}_{(r,d)} v^{(r-1)d} z^d.$$

These take values in the space $\widehat{\mathbf{U}}_X^+[[z, z^{-1}]]$ of Laurent series that extend infinitely in both directions. We will be interested in products

$$\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n) = \mathbf{E}_{r_1}(z_1)\cdots\mathbf{E}_{r_n}(z_n) \in \widehat{\mathbf{U}}_X^+[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$$

where $r_1, \ldots, r_n$ are non-negative integers. The value of such a series at a coherent sheaf of rank $r = \sum r_i$ and degree d is equal to the infinite sum

$$\mathbf{E}_{r_1,\ldots,r_n}(\mathcal{F}) = v^{-(r+1)d} \sum_{\mathcal{F}_1\subset\cdots\subset\mathcal{F}_n=\mathcal{F}} v^{2\sum_i r_i \deg(\mathcal{F}_i)} z_1^{\deg(\mathcal{F}_1)} \cdots z_n^{\deg(\mathcal{F}_n/\mathcal{F}_{n-1})}, \tag{6.2}$$

where $\mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1}) = r_i$ for all i. The following fundamental result is due to Harder.

Theorem 6.1 [Har74]. *The series $\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)$ converges in the region $|z_1| \ll \cdots \ll |z_n|$ to a rational function in $\widehat{\mathbf{U}}_X^+(z_1, \ldots, z_n)$ with at most simple poles along the hyperplanes*

$$z_i/z_j \in \{1, v^2, \ldots, v^{2r}\} \quad \textit{where } r = \sum r_i.$$

In other words, for each $\mathcal{F}$ the series (6.2) is the expansion in the region $|z_1| \ll \cdots \ll |z_n|$ of some rational function in the variables $z_1, \ldots, z_n$. When $r_1 = \cdots = r_n = 1$, the series $\mathbf{E}_{1,\ldots,1}(z_1, \ldots, z_n)$ is the Eisenstein series attached to the cusp form of rank one corresponding to the trivial character $\mathrm{Pic}(X) \to \mathbb{C}^*$ taken n times; see [Kap97, § 2.4] for details. For other values of $r_1, \ldots, r_n$, the series $\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)$ is the Eisenstein series attached to the trivial character of the parabolic subgroup $\mathrm{GL}_{r_1}(\mathbf{k}_X) \times \cdots \times \mathrm{GL}_{r_n}(\mathbf{k}_X)$ of $\mathrm{GL}_r(\mathbf{k}_X)$, where $\mathbf{k}_X$ is the function field of X and $r = \sum r_i$.

The Eisenstein series behave well with respect to the coproduct.

Proposition 6.2. *For non-negative integers $r_1, \ldots, r_n$,*

$$\Delta(\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)) = \sum_{0\leqslant s_i\leqslant r_i} \mathbf{E}_{s_1,\ldots,s_n}(z_1, \ldots, z_n) \otimes \mathbf{E}_{r_1-s_1,\ldots,r_n-s_n}(v^{2s_1}z_1, \ldots, v^{2s_n}z_n).$$

In particular, we have

$$\Delta(\mathbf{E}_r(z)) = \sum_{s=0}^{r} \mathbf{E}_s(z) \otimes \mathbf{E}_{r-s}(v^{2s}z).$$

Proof. This result is a consequence of the fact that $\widehat{\mathbf{U}}_X^+$ is a bialgebra and that, by [BS05, (4.5)],

$$\Delta(\mathbf{1}_{(r,d)}) = \sum_{\substack{r_1+r_2=r \\ d_1+d_2=d}} v^{r_1d_2-r_2d_1} \mathbf{1}_{(r_1,d_1)} \otimes \mathbf{1}_{(r_2,d_2)}. \qquad \square$$

One of the properties of Eisenstein series most crucial for us is the fact that they are eigenvectors for the adjoint action of the element $T_{(0,1)} = \sum_{x \in X(\mathbb{F}_l)} \mathbf{1}_{\mathcal{O}_x}$ and, more generally, of the elements $T_{(0,d)}$ for $d \geqslant 1$. These are the so-called *Hecke operators* in the theory of automorphic forms on function fields. Let

$$\zeta(z) = \frac{(1-\sigma z)(1-\overline{\sigma} z)}{(1-z)(1-v^{-2}z)}$$

be the zeta function of X.

THEOREM 6.3. *For any $r \geqslant 0$, the following hold:*

$$[T_{(0,1)}, \mathbf{E}_r(z)] = v\#X(\mathbb{F}_l)\frac{v^{-2r}-1}{v^{-2}-1} z^{-1}\mathbf{E}_r(z), \tag{6.3}$$

$$\mathbf{E}_0(z_1)\mathbf{E}_r(z_2) = \prod_{i=0}^{r-1} \zeta\left(v^{-2i}\frac{z_1}{z_2}\right) \cdot \mathbf{E}_r(z_2)\mathbf{E}_0(z_1). \tag{6.4}$$

In particular, we have

$$\mathbf{E}_0(z_1)\mathbf{E}_1(z_2) = \zeta\left(\frac{z_1}{z_2}\right)\mathbf{E}_1(z_2)\mathbf{E}_0(z_1).$$

Proof. Both statements are well known (perhaps in a different form) in the theory of automorphic forms. For the reader's convenience, we have included in Appendix C a proof in the spirit of Hall algebras. $\square$

We finish with the so-called *functional equation* for rank-one Eisenstein series.

THEOREM 6.4 (Harder [Har74]). *The rational function $\mathbf{E}_{1,\ldots,1}(z_1, \ldots, z_n)$ is symmetric in the variables $z_1, \ldots, z_n$.*

Remarks. Strictly speaking, the Eisenstein series most often considered in the theory of automorphic forms are given by expressions like (6.2) in which, additionally, each factor $\mathcal{F}_i/\mathcal{F}_{i-1}$ is required to be a vector bundle. In other words, if one sets

$$\mathbf{1}^{\mathrm{vec}}_{(r,d)} = \sum_{\overline{\mathcal{V}}=(r,d)} \mathbf{1}_{\mathcal{V}} \quad \text{and} \quad \mathbf{E}^{\mathrm{vec}}_r(z) = \sum_{d \in \mathbb{Z}} \mathbf{1}^{\mathrm{vec}}_{(r,d)} v^{(r-1)d} z^d$$

where $\mathcal{V}$ runs over the set of vector bundles, the corresponding product would be

$$\mathbf{E}^{\mathrm{vec}}_{r_1,\ldots,r_n}(z_1, \ldots, z_n) = \mathbf{E}^{\mathrm{vec}}_{r_1}(z_1) \cdots \mathbf{E}^{\mathrm{vec}}_{r_n}(z_n).$$

The two series, when restricted to vector bundles, are related by a global rational factor, the so-called *L-factor*. Indeed, there is an obvious factorization

$$\mathbf{E}_r(z) = \mathbf{E}^{\mathrm{vec}}_r(z)\mathbf{E}_0(v^{2r}z).$$

Therefore, by Theorem 6.3 we have, upon restricting to the set of vector bundles,

$$\mathbf{E}_{r_1,\dots,r_n}(z_1,\dots,z_n) = L_{r_1,\dots,r_n}(z_1,\dots,z_n)\mathbf{E}^{\mathrm{vec}}_{r_1,\dots,r_n}(z_1,\dots,z_n),$$

where

$$L_{r_1,\dots,r_n}(z_1,\dots,z_n) = \prod_{i<j}\prod_{k=0}^{r_j-1}\zeta\bigg(v^{2(r_i-k)}\frac{z_i}{z_j}\bigg).$$

Example. To conclude this section, we present the series $\mathbf{E}_{1,1}(z_1, z_2)$ as an example. For simplicity, we will compute only the degree-zero component

$$\mathbf{E}_{1,1}(z_1, z_2)_0 = \sum_{d\in\mathbb{Z}}\bigg(\frac{z_1}{z_2}\bigg)^d \mathbf{1}_{(1,d)}\mathbf{1}_{(1,-d)}$$

and, then, only the values of $\mathbf{E}_{1,1}(z_1, z_2)_0$ on vector bundles. So, let $\mathcal{F}$ be a vector bundle of degree zero and rank two. Because any rank-one subsheaf of $\mathcal{F}$ is a line bundle and any non-zero map from a line bundle to $\mathcal{F}$ is injective, we have

$$\mathbf{E}_{1,1}(z_1, z_2)(\mathcal{F}) = \sum_{d\in\mathbb{Z}}\bigg(\frac{z_1}{z_2}\bigg)^d v^{2d}\sum_{\mathcal{L}_{-d}\in\mathrm{Pic}^{-d}(X)}\frac{\#\,\mathrm{Hom}(\mathcal{L}_{-d},\mathcal{F})-1}{v^{-2}-1}.$$

If $\mathcal{F}$ is a stable bundle, then

$$\mathrm{Hom}(\mathcal{L}_{-d},\mathcal{F}) = \begin{cases}\mathbb{F}_l^{2d} & \text{if } d>0,\\ \{0\} & \text{if } d\leqslant 0.\end{cases}$$

Hence

$$\begin{aligned}\mathbf{E}_{1,1}(z_1, z_2)(\mathcal{F}) &= \frac{\#X(\mathbb{F}_l)}{v^{-2}-1}\sum_{d>0}\bigg(\frac{z_1}{z_2}\bigg)^d v^{2d}(v^{-4d}-1)\\ &= \frac{z_1z_2(1+v^{-2})\#X(\mathbb{F}_l)}{(z_2-v^{-2}z_1)(v^{-2}z_2-z_1)}.\end{aligned}\tag{6.5}$$

If $\mathcal{F} = \mathcal{L}_0\oplus\mathcal{L}_0'$ is a direct sum of two distinct line bundles of degree zero, then

$$\mathrm{Hom}(\mathcal{L}_{-d},\mathcal{F}) = \begin{cases}\mathbb{F}_l^{2d} & \text{if } d>0,\\ \mathbb{F}_l & \text{if } d=0 \text{ and } \mathcal{L}_{-d}\in\{\mathcal{L}_0,\mathcal{L}_0'\},\\ \{0\} & \text{if } d=0 \text{ and } \mathcal{L}_{-d}\notin\{\mathcal{L}_0,\mathcal{L}_0'\},\\ \{0\} & \text{if } d<0.\end{cases}$$

Hence we get

$$\mathbf{E}_{1,1}(z_1, z_2)(\mathcal{F}) = \frac{z_1z_2(1+v^{-2})\#X(\mathbb{F}_l)}{(z_2-v^{-2}z_1)(v^{-2}z_2-z_1)} + 2.\tag{6.6}$$

From (6.5) and (6.6) we deduce that the semistable component of $\mathbf{E}_{1,1}(z_1, z_2)_0$ is equal to

$$\mathbf{E}_{1,1}(z_1, z_2)_{(0)} = \frac{z_1z_2(1+v^{-2})\#X(\mathbb{F}_l)}{(z_2-v^{-2}z_1)(v^{-2}z_2-z_1)}\bigg\{\frac{T_{(2,0)}}{[2]}+\frac{T_{(1,0)}^2}{2}\bigg\} + T_{(1,0)}^2.$$

Finally, to compute the unstable component of $\mathbf{E}_{1,1}(z_1, z_2)_0$ we use the coproduct. Observe that since $\mathrm{Ext}(\mathcal{L}_{-d},\mathcal{L}_d) = \{0\}$, the component of bidegree $(1,-d)$, $(1,d)$ of $\Delta(\mathbf{1}_{\mathcal{L}_{-d}\oplus\mathcal{L}_d})$ is equal to

$$v^{2d}\mathbf{1}_{\mathcal{L}_{-d}}\otimes\mathbf{1}_{\mathcal{L}_d},$$

and no other term can contribute to $\mathbf{1}_{\mathcal{L}_{-d}} \otimes \mathbf{1}_{\mathcal{L}_d}$. Hence

$$\mathbf{E}_{1,1}(z_1, z_2)(\mathcal{L}_{-d} \oplus \mathcal{L}_d) = v^{-2d} \Delta(\mathbf{E}_{1,1}(z_1, z_2))(\mathcal{L}_{-d}, \mathcal{L}_d).$$

By Proposition 6.2 and Theorem 6.3 we have

$$\begin{aligned}\Delta_{1,1}(\mathbf{E}_{1,1}(z_1, z_2)) &= \mathbf{E}_0(z_1)\mathbf{E}_1(z_2) \otimes \mathbf{E}_1(z_1)\mathbf{E}_0(v^2 z_2) + \mathbf{E}_1(z_1)\mathbf{E}_0(z_2) \otimes \mathbf{E}_0(v^2 z_1)\mathbf{E}_1(z_2)\\ &= \zeta\left(\frac{z_1}{z_2}\right)\mathbf{E}_1(z_2)\mathbf{E}_0(z_1) \otimes \mathbf{E}_1(z_1)\mathbf{E}_0(v^2 z_2)\\ &\quad + \zeta\left(\frac{z_2}{z_1}\right)\mathbf{E}_1(z_1)\mathbf{E}_0(z_2) \otimes \mathbf{E}_1(z_2)\mathbf{E}_0(v^2 z_1),\end{aligned}$$

from which we eventually obtain

$$\mathbf{E}_{1,1}(z_1, z_2)(\mathcal{L}_{-d} \oplus \mathcal{L}_d) = v^{-2d}\left[\zeta\left(\frac{z_1}{z_2}\right) z_1^d z_2^{-d} + \zeta\left(\frac{z_2}{z_1}\right) z_1^{-d} z_2^d\right].$$

6.3 So far, we have considered the Eisenstein series $\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)$ only for a *fixed* elliptic curve X. Recall from § 1.5 that there exists an algebra $\boldsymbol{\mathcal{E}}^+_{\mathbf{R}}$ defined over the ring $\mathbf{R} = \mathbb{C}[\sigma^{\pm 1/2}, \overline{\sigma}^{\pm 1/2}]$ whose specialization for any X is isomorphic to $\mathbf{U}^+_X$. Using the formulas (1.3) and (6.1), we see that the generating series $\mathbf{E}_r(z)$ and hence the Eisenstein series $\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)$ can naturally be lifted to elements

$${}_{\mathbf{R}}\mathbf{E}_r(z) \in \widehat{\boldsymbol{\mathcal{E}}}^+_{\mathbf{R}}[[z, z^{-1}]] \quad \text{and} \quad {}_{\mathbf{R}}\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n) \in \widehat{\boldsymbol{\mathcal{E}}}^+_{\mathbf{R}}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]].$$

Proposition 6.5. *The series ${}_{\mathbf{R}}\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)$ converges in the region $|z_1| \ll \cdots \ll |z_n|$ to a rational function in $\widehat{\boldsymbol{\mathcal{E}}}^+_{\mathbf{R}}(z_1, \ldots, z_n)$ with at most simple poles along the hyperplanes $z_i/z_j \in \{1, v^2, \ldots, v^{2r}\}$, where $r = \sum r_i$.*

Proof. The coefficient of ${}_{\mathbf{R}}\mathbf{E}_{r_1,\ldots,r_n}(z_1, \ldots, z_n)$ on any basis element of $\boldsymbol{\mathcal{E}}^+_{\mathbf{R}}$ is given by a Laurent series of the form

$$P(z_1, \ldots, z_n) \sum_{d_1,\ldots,d_n \geqslant 0} \alpha_{d_1,\ldots,d_n} \left(\frac{z_1}{z_2}\right)^{d_1} \cdots \left(\frac{z_{n-1}}{z_n}\right)^{d_n},$$

where $P(z_1, \ldots, z_n) \in \mathbf{R}[z_1^{\pm 1}, \ldots, z_n^{\pm n}]$ and $\alpha_{d_1,\ldots,d_n} \in \mathbf{R}$. By Harder's theorem, the specialization for any elliptic curve X of the expression

$$\left(\prod_{l=1}^{r} \prod_{i,j} (z_i - v^{-2l} z_j)\right) \cdot P(z_1, \ldots, z_n) \sum_{d_1,\ldots,d_n} \alpha_{d_1,\ldots,d_n} \left(\frac{z_1}{z_2}\right)^{d_1} \cdots \left(\frac{z_{n-1}}{z_n}\right)^{d_n}$$

is a Laurent polynomial (of fixed degree). This is equivalent to the vanishing of certain $\mathbf{R}$-linear combinations of the $\alpha_{d_1,\ldots,d_n}$. Of course, if such a linear combination vanishes when evaluated at all (i.e. infinitely many) elliptic curves X, then it must already vanish in $\mathbf{R}$. So we are done. □

6.4 Motivated by the analogy between the Hecke operator $T_{(0,1)}$ and Macdonald's operator (see § 5.1 and, in particular, the remark in that section), we introduce for every partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ the following specialization of Eisenstein series:

$$\mathbf{E}_\lambda(z) = {}_{\mathbf{R}}\mathbf{E}_{\lambda_1,\ldots,\lambda_n}(z, \sigma^1 z, \ldots, \sigma^{n-1} z).$$

By Proposition 6.5, the line $(z, \sigma z, \ldots, \sigma^{n-1} z)$ is not contained in the pole locus of the rational function ${}_{\mathbf{R}}\mathbf{E}_{\lambda_1,\ldots,\lambda_n}(z_1, \ldots, z_n)$, and hence $\mathbf{E}_\lambda(z)$ belongs to $\widehat{\boldsymbol{\mathcal{E}}}^+_{\mathbf{R}}(z)$. More generally, for any pair of partitions $\mu \subset \lambda$ we put

$$\mathbf{E}_{\lambda/\mu}(z) = {}_{\mathbf{R}}\mathbf{E}_{\lambda_1-\mu_1,\ldots,\lambda_n-\mu_n}(v^{2\mu_1}z, v^{2\mu_2}\sigma z, \ldots, v^{2\mu_n}\sigma^{n-1}z).$$

Observe that, by Theorem 6.3, the series $\mathbf{E}_\lambda(z)$ are eigenvectors for the adjoint action of the Hecke operator $T_{(0,1)}$, whose eigenvalues β_λ are (up to a global factor) the same as those of the Macdonald polynomials; specifically, we have

$$\beta_\lambda = z^{-1}\mathbf{c}_1(\sigma,\overline{\sigma})\sum_i \frac{v^{-2\lambda_i}-1}{v^{-2}-1}\sigma^{1-i} = z^{-1}\sigma\frac{\mathbf{c}_1(\sigma,\overline{\sigma})}{1-\sigma}\alpha_{\lambda'},$$

where $\mathbf{c}_1(\sigma,\overline{\sigma}) = (\sigma^{1/2}-\sigma^{-1/2})(\overline{\sigma}^{1/2}-\overline{\sigma}^{-1/2})$ and $\alpha_{\lambda'}$ is given by formula (5.1). Note that we have in mind the map $\Phi_\infty^{\sigma,(\sigma\overline{\sigma})^{-1}}$ to identify $\mathbf{U}_X^+ = \boldsymbol{\mathcal{E}}_X^+$ with $\mathbf{S}\ddot{\mathbf{H}}_\infty^+$. Hence the relevant specialization between variables in the Hall algebra and those in the Cherednik algebra or Macdonald polynomials is

$$\sigma \mapsto q^{-1}, \quad (\sigma\overline{\sigma})^{-1} = v^2 \mapsto t^{-1}. \tag{6.7}$$

It would seem natural to define, more generally, the specialization

$$\mathbf{E}_{\underline{l}}(z) = {}_{\mathbf{R}}\mathbf{E}_{l_1,\ldots,l_n}(z, \sigma z, \ldots, \sigma^{n-1}z)$$

for *any* sequence of non-negative integers $l_1, \ldots, l_n$. However, there is the following vanishing result.

Lemma 6.6. *If $\underline{l} = (l_1, \ldots, l_n)$ is not dominant, i.e. if $l_k > l_{k-1}$ for some k, then $\mathbf{E}_{\underline{l}}(z) = 0$.*

Proof. One can check that the L-factor $L_{l_1,\ldots,l_n}(z_1, \ldots, z_n)$ vanishes on the line $(z, \sigma z, \ldots, \sigma^{n-1}z)$ whenever $\underline{l}$ is not dominant. Hence Lemma 6.6 would follow from the fact that the unnormalized Eisenstein series $\mathbf{E}^{\mathrm{vec}}_{l_1,\ldots,l_n}(z_1, \ldots, z_n)$ is regular on that line. Instead of appealing to this fact, we provide a direct proof. To make the notation less cumbersome, we shall drop the subscript $\mathbf{R}$ throughout the proof. By Proposition 6.2 we have

$$\Delta_{(1,\ldots,1)}(\mathbf{E}_r(z)) = \mathbf{E}_1(z) \otimes \mathbf{E}_1(v^2z) \otimes \cdots \otimes \mathbf{E}_1(v^{2(r-1)}z),$$

and, more generally, given integers $\epsilon_k \in \{0,1\}$ such that $\epsilon_{i_1} = \cdots = \epsilon_{i_r} = 1$ while $\epsilon_k = 0$ if $k \notin \{i_1, \ldots, i_r\}$, we have

$$\Delta_{(\epsilon_1,\ldots,\epsilon_n)}(\mathbf{E}_r(z)) = \mathbf{E}_{\epsilon_1}(z) \otimes \cdots \otimes \mathbf{E}_{\epsilon_k}(v^{2s_k}z) \otimes \cdots \otimes \mathbf{E}_{\epsilon_n}(v^{2s_n}z) \tag{6.8}$$

where $s_k = \#\{l : i_l < k\}$. Now take $\underline{l} = (l_1, \ldots, l_n) \in \mathbb{N}^n$ and set $l = \sum l_i$. We can compute $\Delta_{(1,\ldots,1)}(\mathbf{E}_{\underline{l}}(z))$ using (6.8); it is equal to a sum, indexed by the set of maps $\phi : \{1, \ldots, l\} \to \{1, \ldots, n\}$, of terms

$$a_\phi = \Delta_{(\epsilon_1^1,\ldots,\epsilon_n^1)}(\mathbf{E}_{l_1}(z)) \cdots \Delta_{(\epsilon_1^k,\ldots,\epsilon_n^k)}(\mathbf{E}_{l_k}(\sigma^{k-1}z)) \cdots \Delta_{(\epsilon_1^n,\ldots,\epsilon_n^n)}(\mathbf{E}_{l_n}(\sigma^{n-1}z))$$

where $\epsilon_i^k \in \{0,1\}$ is defined by

$$\epsilon_i^k = \begin{cases} 0 & \text{if } \phi(i) \neq k, \\ 1 & \text{if } \phi(i) = k. \end{cases}$$

In other words, the map ϕ describes the way the coproducts (6.8) of the $\mathbf{E}_{l_k}(\sigma^{k-1}z)$ have been distributed among the l components of the tensor product. We claim that if $\underline{l}$ is not dominant, then each term a_ϕ vanishes. Indeed, suppose that $l_k > l_{k-1}$ for some k; then a_ϕ is divisible by a

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term of the form

$$\begin{aligned}\Delta_{(\epsilon_1,\dots,\epsilon_n)}(\mathbf{E}_{l_{k-1}}(\sigma^{k-2}z))\cdot\Delta_{(\epsilon'_1,\dots,\epsilon'_n)}(\mathbf{E}_{l_k}(\sigma^{k-1}z))\\ =\mathbf{E}_{\epsilon_1}(\sigma^{k-2}z)\mathbf{E}_{\epsilon'_1}(\sigma^{k-1}z)\otimes\cdots\otimes\mathbf{E}_{\epsilon_n}(v^{2s_n}\sigma^{k-2}z)\mathbf{E}_{\epsilon'_n}(v^{2s'_n}\sigma^{k-1}z).\end{aligned}\tag{6.9}$$

If $\epsilon_i=1$, then $\epsilon'_i=0$, and vice versa. Because $s_1=s'_1=0$ while

$$s_n\in\{l_{k-1},l_{k-1}-1\}\quad\text{and}\quad s'_n\in\{l_k,l_k-1\},$$

it is easy to see that there exists an index j for which $\epsilon_j=0$, $\epsilon'_j=1$ and $s_j=s'_j$. But then the jth component of (6.9) is equal to

$$\mathbf{E}_0(v^{2s_j}\sigma^{k-2}z)\mathbf{E}_1(v^{2s_j}\sigma^{k-1}z)=\zeta(\sigma^{-1})\mathbf{E}_1(v^{2s_j}\sigma^{k-1}z)\mathbf{E}_0(v^{2s_j}\sigma^{k-2}z)=0$$

since $\zeta(\sigma^{-1})=0$. Hence $a_\phi=0$ as claimed and $\Delta_{(1,\dots,1)}(\mathbf{E}_{\underline{l}}(z))=0$. It remains to show that the map

$$\Delta_{(1,\dots,1)}:\widehat{\mathbf{U}}^+[r,d]\to\prod_{d_1+\cdots+d_r=d}\widehat{\mathbf{U}}^+[1,d_1]\otimes\cdots\otimes\widehat{\mathbf{U}}^+[1,d_r]$$

is injective, but this follows from the fact that $\widehat{\mathbf{U}}^+$ is equipped with a non-degenerate Hopf pairing and that it is generated by elements of degree zero and one; see [BS05, Corollary 6.1]. □

Proposition 6.7. *For any partition λ we have*

$$\Delta(\mathbf{E}_\lambda(z))=\sum_{\mu\subseteq\lambda}\mathbf{E}_\mu(z)\otimes\mathbf{E}_{\lambda/\mu}(z).\tag{6.10}$$

More generally, for any skew partition λ/μ we have

$$\Delta(\mathbf{E}_{\lambda/\mu}(z))=\sum_{\mu\subseteq\nu\subseteq\lambda}\mathbf{E}_{\nu/\mu}(z)\otimes\mathbf{E}_{\lambda/\nu}(z).\tag{6.11}$$

Proof. We prove the first statement. From Proposition 6.2 it follows that

$$\begin{aligned}\Delta(\mathbf{E}_\lambda(z,\dots,\sigma^{n-1}z))=\sum_{\substack{s_1,\dots,s_n\\ 0\leqslant s_i\leqslant\lambda_i}}{}_{\mathbf{R}}\mathbf{E}_{s_1,\dots,s_n}(z,\dots,\sigma^{n-1}z)\\ \otimes\ {}_{\mathbf{R}}\mathbf{E}_{\lambda_1-s_1,\dots,\lambda_n-s_n}(v^{2s_1}z,\dots,v^{2s_n}\sigma^{n-1}z).\end{aligned}\tag{6.12}$$

By Lemma 6.6 we have

$${}_{\mathbf{R}}\mathbf{E}_{s_1,\dots,s_n}(z,\sigma z,\dots,\sigma^{n-1}z)=0$$

if $(s_1,\dots,s_n)$ is not a partition. Therefore the right-hand side of (6.12) reduces to (6.10). The proof of the second statement is similar. □

7. Geometric construction of Macdonald polynomials

In this section, we make explicit the link between Macdonald polynomials $P_\lambda(q,t^{-1})$ and the Eisenstein series $\mathbf{E}_\lambda(z)$.

7.1 For any skew partition λ/μ, we denote by $\mathbf{E}^{(0)}_{\lambda/\mu}$ the restriction of $\mathbf{E}_{\lambda/\mu}(z)$ to the set of *semistable* vector bundles of degree zero. Notice that, by homogeneity, this is independent of z; it is therefore an element of the subalgebra $\boldsymbol{\mathcal{E}}^{+,(0)}_{\mathbf{K}}$ of the universal Hall algebra $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ generated by elements $u_{(r,0)}$ for $r\geqslant 0$. See § 1.5 for details. By Proposition 1.3, this last subalgebra is canonically identified with the algebra of symmetric functions $\Lambda^+_{(\sigma,v^2)}$. Explicitly, the isomorphism is given by $\tilde{u}_{(r,0)} = p_r/r$. For instance, from the example in § 6.2 we see that

$$\begin{aligned}\mathbf{E}^{(0)}_{1,1} &= \frac{(1+v^{-2})(\sigma-v^{-2})(1-\sigma)}{(\sigma-v^{-2})(\sigma v^{-2}-1)}\left(\frac{p_2}{2}+\frac{p_1^2}{2}\right)+p_1^2\\ &= \frac{(1+v^{-2})(1-\sigma)}{\sigma v^{-2}-1}\frac{p_2}{2}+\frac{(v^{-2}-1)(1+\sigma)}{\sigma v^{-2}-1}\frac{p_1^2}{2}.\end{aligned}$$

Let ω stand for the standard involution on symmetric functions, defined by $\omega(p_r) = (-1)^{r-1}p_r$.

7.2 We are now ready to state the second main theorem of this paper.

Theorem 7.1. *For any partition λ we have*

$$\mathbf{E}^{(0)}_{\lambda} = \omega P_{\lambda'}(\sigma^{-1}, v^2),$$

and for any skew partition λ/μ we have

$$\mathbf{E}^{(0)}_{\lambda/\mu} = \omega P_{\lambda'/\mu'}(\sigma^{-1}, v^2).$$

The rest of this section is devoted to the proof of this theorem. We will use the characterization of the polynomials $P_{\lambda/\mu}(\sigma^{-1}, v^2)$ given in Proposition 5.1. It is clear from the definitions that $\omega(\mathbf{E}^{(0)}_{\lambda'/\mu'})$ is of degree $|\lambda/\mu|$. Property (ii) of Proposition 5.1 was shown for $\omega(\mathbf{E}^{(0)}_{\lambda'/\mu'})$ in Proposition 6.7. Thus it only remains to check that the coefficient of m_r in $\omega(\mathbf{E}^{(0)}_{\lambda'/\mu'})$ for $r = |\lambda/\mu|$ is given by properties (iii) and (iv) of Proposition 5.1.

To do this, we introduce the following family of elements in $\boldsymbol{\mathcal{E}}^{+,(0)}_{\mathbf{K}}$:

$$g_r = \sum_{\lambda\vdash r} z_\lambda^{-1}\prod_i (v^{-\lambda_i}-v^{\lambda_i})u_{(\lambda_i,0)} = \sum_{\lambda\vdash r} z_\lambda^{-1}\prod_i \frac{v^{-2\lambda_i}-1}{(1-\sigma^{\lambda_i})(1-\overline{\sigma}^{\lambda_i})}p_\lambda.$$

Alternatively, these elements can be defined through the formula

$$1+\sum_{r>0} g_r s^r = \exp\left(\sum_{r\geqslant 1}\frac{v^{-2r}-1}{(1-\sigma^r)(1-\overline{\sigma}^r)}\frac{p_r}{r}s^r\right).$$

Recall that $\boldsymbol{\mathcal{E}}^+_X$ is equipped with a non-degenerate Hopf scalar product, which, by [BS05, Lemma 4.10], satisfies

$$\langle T_{(r,0)}, T_{(s,0)}\rangle_G = \delta_{r,s}\frac{[r]^2\#X(\mathbb{F}_{l^r})}{r(v^{-2r}-1)}. \tag{7.1}$$

This scalar product lifts to a non-degenerate scalar product on $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$ such that

$$\langle u_{(r,0)}, u_{(s,0)}\rangle_G = \delta_{r,s}\frac{r}{(1-v^{2r})(1-\sigma^r)(1-\overline{\sigma}^r)}, \tag{7.2}$$

which, after identification with $\Lambda_{(\sigma,v^2)}$, reads

$$\langle p_r, p_s\rangle_G = \delta_{r,s} r\frac{(1-\sigma^r)(1-\overline{\sigma}^r)}{v^{-2r}-1}. \tag{7.3}$$

From [Mac95, ch. VI.2] we deduce that g_r is dual to m_r with respect to the basis $\{m_\lambda\}$; that is,

$$\begin{aligned}\langle g_r, m_r\rangle_G &= 1,\\ \langle g_r, m_\lambda\rangle_G &= 0 \quad \text{if } |\lambda| = r \text{ and } \lambda < (r).\end{aligned}$$

Therefore the proof of Theorem 7.1 will be complete once we have shown that

$$\langle g_r, \omega(\mathbf{E}^{(0)}_{\lambda'/\mu'})\rangle_G = 0 \tag{7.4}$$

if λ/μ is not a horizontal strip while

$$\langle g_r, \omega(\mathbf{E}^{(0)}_{\lambda'/\mu'})\rangle_G = \psi_{\lambda/\mu}(q, v^2) \tag{7.5}$$

if λ/μ is a horizontal strip. As we will see, verifying these equations essentially amounts to establishing certain relations between the factors $\psi_{\lambda/\mu}(q, v^2)$ and the L-factors appearing in the Eisenstein series.

Observe that since g_r is itself semistable of degree zero (i.e. $g_r \in \boldsymbol{\mathcal{E}}^{+,(0)}_{\mathbf{K}}$) and the subalgebras $\boldsymbol{\mathcal{E}}^{+,(\mu)}_{\mathbf{K}}$ are all mutually orthogonal, we may as well replace $\mathbf{E}^{(0)}_{\lambda'/\mu'}$ by $\mathbf{E}_{\lambda'/\mu'}(z)$ in equations (7.4) and (7.5). Note also that ω is an orthogonal involution for $\langle\,,\rangle$.

7.3 The basic idea is to find a factorization of g_r and use the Hopf property of the scalar product $\langle\,,\rangle_G$ to reduce (7.4) and (7.5) to a lower rank. For simplicity we shall drop the index G from the scalar product notation. Of course, since g_r is dual to m_r and m_r is primitive, factoring g_r directly within $\boldsymbol{\mathcal{E}}^{+,(0)}_{\mathbf{K}}$ is not feasible. However, this becomes possible as soon as we step out of the subalgebra $\boldsymbol{\mathcal{E}}^{+,(0)}_{\mathbf{K}}$. More precisely, let us put $g^{(1)}_r = [u_{(0,1)}, g_r]$ and $\omega g^{(1)}_r = [u_{(0,1)}, \omega g_r]$.

Lemma 7.2. *For any $r \geqslant 1$ we have*

$$\omega g^{(1)}_{r+1} = v[u_{(1,0)}, \omega g^{(1)}_r] + (v^{-1} - v)\omega g_r u_{(1,1)}.$$

Proof. An essentially direct computation, based on the relation $[u_{(s,1)}, u_{(t,0)}] = u_{(s+t,1)}$ for any s and t, yields

$$\omega g^{(1)}_r = (-1)^r \sum_{s=1}^{r} v^s(1 - v^{-2})(-1)^s \omega g_{r-s} u_{(s,1)}. \tag{7.6}$$

The recursion formula in the lemma is an easy consequence of (7.6). □

Lemma 7.3. *For any skew partition λ/μ we have*

$$\langle \omega g^{(1)}_r, \mathbf{E}_{\lambda'/\mu'}(z)\rangle = \frac{1}{1 - v^2}\left(\sum_i (v^{2\mu'_i} - v^{2\lambda'_i})\sigma^{i-1} z\right)\langle \omega g_r, \mathbf{E}_{\lambda'/\mu'}(z)\rangle.$$

Proof. Because $\langle\,,\rangle$ is a Hopf pairing, we have

$$\begin{aligned}\langle \omega g^{(1)}_r, \mathbf{E}_{\lambda'/\mu'}(z)\rangle &= \langle u_{(0,1)} \cdot \omega g_r - \omega g_r \cdot u_{(0,1)}, \mathbf{E}_{\lambda'/\mu'}(z)\rangle \\ &= \langle u_{(0,1)} \otimes \omega g_r, \Delta_{0,r}(\mathbf{E}_{\lambda'/\mu'}(z))\rangle - \langle \omega g_r \otimes u_{(0,1)}, \Delta_{r,0}(\mathbf{E}_{\lambda'/\mu'}(z))\rangle.\end{aligned}$$

Using Proposition 6.2, the coproducts are computed to be

$$\begin{aligned}\Delta_{0,r}(\mathbf{E}_{\lambda'/\mu'}(z)) &= \mathbf{E}_0(v^{2\mu'_1} z) \cdots \mathbf{E}_0(v^{2\mu'_n}\sigma^{n-1} z) \otimes \mathbf{E}_{\lambda'/\mu'}(z) \\ &= \left(1 + \sum_i v^{2\mu'_i - 1}\sigma^{i-1} z \tilde{u}_{(0,1)} + \cdots\right) \otimes \mathbf{E}_{\lambda'/\mu'}(z)\end{aligned}$$

and

$$\begin{aligned}\Delta_{r,0}(\mathbf{E}_{\lambda'/\mu'}(z)) &= \mathbf{E}_{\lambda'/\mu'}(z) \otimes \mathbf{E}_0(v^{2\lambda'_1} z) \cdots \mathbf{E}_0(v^{2\lambda'_n}\sigma^{n-1} z) \\ &= \mathbf{E}_{\lambda'/\mu'}(z) \otimes \left(1 + \sum_i v^{2\lambda'_i - 1}\sigma^{i-1} z \tilde{u}_{(0,1)} + \cdots\right),\end{aligned}$$

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where $\tilde{u}_{(0,1)} = v(1-\sigma)(1-\overline{\sigma})u_{(0,1)}$. The result then follows from the scalar product $\langle u_{(0,1)}, \tilde{u}_{(0,1)} \rangle = v/(1-v^2)$. □

7.4 We now proceed with proving (7.4) and (7.5), arguing by induction on $|\lambda/\mu|$. Assume first that $|\lambda/\mu| = 1$. This means that $\lambda'_i = \mu'_i$ for all but one value of i, say j, for which $\lambda'_j = \mu'_j + 1$. Then, on the one hand,

$$P_{\lambda/\mu}(\sigma^{-1}, v^2) = \psi_{\lambda/\mu} m_1 = \prod_{i<j} \frac{(1 - v^{2(\mu'_i - \mu'_j)}\sigma^{i-j+1})(1 - v^{2(\mu'_i - \mu'_j - 1)}\sigma^{i-j-1})}{(1 - v^{2(\mu'_i - \mu'_j)}\sigma^{i-j})(1 - v^{2(\mu'_i - \mu'_j - 1)}\sigma^{i-j})} \cdot m_1,$$

while on the other hand,

$$\mathbf{E}_{\lambda'/\mu'}(z) = \mathbf{E}_0(v^{2\mu'_1} z) \cdots \mathbf{E}_0(v^{2\mu'_{j-1}}\sigma^{j-2} z)\mathbf{E}_1(v^{2\mu'_j}\sigma^{j-1} z) \cdots \mathbf{E}_0(v^{2\mu'_n}\sigma^{n-1} z).$$

So, by Theorem 6.3, we get

$$\begin{aligned} \mathbf{E}^{(0)}_{\lambda'/\mu'} &= \prod_{i<j} \zeta(v^{2(\mu'_i - \mu'_j)}\sigma^{i-j}) \cdot \mathbf{E}_1(v^{2\mu'_j}\sigma^{j-1} z)^{(0)} \\ &= \prod_{i<j} \zeta(v^{2(\mu'_i - \mu'_j)}\sigma^{i-j}) \cdot m_1. \end{aligned}$$

It remains to notice that

$$\frac{(1 - v^{2(\mu'_i - \mu'_j)}\sigma^{i-j+1})(1 - v^{2(\mu'_i - \mu'_j - 1)}\sigma^{i-j-1})}{(1 - v^{2(\mu'_i - \mu'_j)}\sigma^{i-j})(1 - v^{2(\mu'_i - \mu'_j - 1)}\sigma^{i-j})} = \zeta(v^{2(\mu'_i - \mu'_j)}\sigma^{i-j})$$

so that the ψ-factor and the L-factors do indeed coincide:

$$\psi_{\lambda/\mu}(\sigma^{-1}, v^2) = \prod_{i<j} \zeta(v^{2(\mu'_i - \mu'_j)}\sigma^{i-j}).$$

Next, assume that equations (7.4) and (7.5) hold true for all skew partitions ν/η for which $|\nu/\eta| < r$, and let λ/μ be a skew partition of size r. Combining Lemmas 7.2 and 7.3 gives

$$\begin{aligned} &\frac{1}{1-v^2}\left(\sum_i (v^{2\mu'} - v^{2\lambda'})\sigma^{i-1} z\right)\langle \omega g_r, \mathbf{E}_{\lambda'/\mu'}(z)\rangle \\ &\quad = v\{\langle u_{(1,0)} \otimes \omega g^{(1)}_{r-1}, \Delta_{1,r-1}(\mathbf{E}_{\lambda'/\mu'}(z))\rangle - \langle \omega g^{(1)}_{r-1} \otimes u_{(1,0)}, \Delta_{r-1,1}(\mathbf{E}_{\lambda'/\mu'}(z))\rangle\} \\ &\qquad + (v^{-1} - v)\langle \omega g_{r-1} \otimes u_{(1,1)}, \Delta_{r-1,1}(\mathbf{E}_{\lambda'/\mu'}(z))\rangle. \end{aligned} \tag{7.7}$$

Let us first consider the case of a vertical strip λ'/μ' (so that λ/μ is a horizontal strip). In this case,

$$\mathbf{E}_{\lambda'/\mu'}(z) = \mathbf{E}_{\epsilon_1}(v^{2\mu'_1} z) \cdots \mathbf{E}_{\epsilon_n}(v^{2\mu'_n}\sigma^{n-1} z)$$

for some $\epsilon_i \in \{0, 1\}$. Let I (respectively, J) be the set of $k \in \{1, \ldots, n\}$ for which $\epsilon_k = 0$ (respectively, $\epsilon_k = 1$). Then

$$\mathbf{E}_{\lambda'/\mu'}(z) = \prod_{\substack{i<j \\ i\in I, j\in J}} \zeta(v^{2(\mu'_i - \mu'_j)}\sigma^{i-j}) \cdot \overrightarrow{\prod_{j\in J}} \mathbf{E}_1(v^{2\mu'_j}\sigma^{j-1} z) \cdot \overrightarrow{\prod_{i\in I}} \mathbf{E}_0(v^{2\mu'_i}\sigma^{i-1} z).$$

As above, the ψ-factor and the L-factor coincide,

$$\psi_{\lambda/\mu}(\sigma^{-1}, v^2) = \prod_{\substack{i<j \\ i\in I, j\in J}} \zeta(v^{2(\mu'_i - \mu'_j)}\sigma^{i-j}),$$

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and therefore (7.5) reduces to the simple relation

$$\langle \omega g_r, \mathbf{E}_1(v^{2\mu'_{j_1}}\sigma^{j_1-1}z)\cdots\mathbf{E}_1(v^{2\mu'_{j_r}}\sigma^{j_r-1}z)\rangle = 1.$$

We claim that, in fact, $\langle \omega g_r, \mathbf{E}_1(\alpha_1)\cdots\mathbf{E}_1(\alpha_r)\rangle = 1$ for *any* $\alpha_1,\ldots,\alpha_r$. Upon expanding (7.7) one finds that it is equivalent to the following strange identity, which is proved in Appendix D.

Lemma 7.4. *For any $r \geqslant 1$, the following identity holds over the field of rational functions $\mathcal{K}'(\alpha_1,\ldots,\alpha_r)$:*

$$\sum_{i=1}^{r}\alpha_i = \frac{1}{v^{-2}-1}\sum_{j=1}^{r}\bigg[\bigg(\prod_{l\neq j}\zeta\bigg(\frac{\alpha_l}{\alpha_j}\bigg)-\prod_{l\neq j}\zeta\bigg(\frac{\alpha_j}{\alpha_l}\bigg)\bigg)\cdot\bigg(\sum_{l\neq j}\alpha_l\bigg)\bigg] + \sum_{j=1}^{r}\bigg(\prod_{l\neq j}\zeta\bigg(\frac{\alpha_j}{\alpha_l}\bigg)\bigg)\alpha_j. \tag{7.8}$$

Next, let us assume that λ'/μ' has exactly one part of length two and $r-2$ parts of length one. Arguing as above and cancelling the L-factor, we see that (7.4) is equivalent to

$$\langle \omega g_r, \mathbf{E}_1(v^{2\mu'_{j_1}}\sigma^{j_1-1}z)\cdots\mathbf{E}_2(v^{2\mu'_k}\sigma^{k-1}z)\cdots\mathbf{E}_1(v^{2\mu'_{j_{r-1}}}\sigma^{j_{r-1}-1}z)\rangle = 0.$$

Again, we claim that in fact $\langle \omega g_r, \mathbf{E}_1(\alpha_1)\cdots\mathbf{E}_2(\alpha_k)\cdots\mathbf{E}_1(\alpha_{r-1})\rangle = 0$ for any $\alpha_1,\ldots,\alpha_{r-1}$. This can be checked directly by using (7.7).

In all of the remaining cases, λ'/μ' has at least two parts of length at least two, i.e. $\lambda'_i > \mu'_i + 1$ for more than one value of i. But then no sub-skew-partition of size $r-1$ of λ'/μ' can be vertical. By the induction hypothesis, this implies that all terms on the right-hand side vanish and thus $\langle \omega g_r, \mathbf{E}_{\lambda'/\mu'}(z)\rangle = 0$ as desired. Theorem 7.1 is therefore proved.

Remarks.

(i) A factorization similar to (7.6), involving rank-one difference operators in the context of Pieri rules for skew Macdonald polynomials, appears in [BGHT99]. We thank Mark Haiman for bringing this to our attention.

(ii) In addition to Macdonald's operator Δ_1, one can define an operator ∇, acting on symmetric polynomials in $\Lambda_{(q,t)}$ (see [BGHT99]), which has distinct eigenvalues and whose eigenvectors are the Macdonald polynomials. Specifically, ∇ is defined by

$$\nabla(P_\lambda(q,t^{-1})) = t^{n(\lambda)}q^{n(\lambda')}P_\lambda(q,t^{-1}).$$

Our conventions, taken from [Mac95], differ slightly from those in [BGHT99]. In our framework, this operator ∇ is simply given by the action of the element $A_2 = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by automorphism on the Hall algebra, which is none other than the tensor product with a line bundle $\mathcal{O}(x)$ over X of degree one. Thus we have

$$\rho(A_2)(\mathbf{E}_\lambda(z)) = \mathcal{O}(x)\otimes\mathbf{E}_\lambda(z) = v^{-2n(\lambda')}\sigma^{-n(\lambda)}\mathbf{E}_\lambda(z).$$

(iii) In [Lau90], Laumon defined and studied a 'geometric lift' of Eisenstein series to certain perverse sheaves (or, more precisely, constructible complexes) on the stacks $\underline{\mathrm{Coh}}^{r,d}(X)$, called *Eisenstein sheaves*. The Eisenstein series themselves are recovered from the Eisenstein sheaves via the faisceaux-function correspondence. In the special case of an elliptic curve, *simple* Eisenstein sheaves are determined in [Sch05]. The construction of the (non-simple) Eisenstein sheaves relevant to Macdonald polynomials can easily be translated from Theorem 3.1. Let us denote by $(\overline{\mathbb{Q}}_p)_{r,d}$ the trivial rank-one constructible sheaf on $\underline{\mathrm{Coh}}^{r,d}(X)$, and let us consider the following formal series whose coefficients are semisimple constructible complexes:

$$\mathbb{E}_r(\sigma^l z) = \bigoplus_{d\in\mathbb{Z}} (\overline{\mathbb{Q}}_p)_{r,d}[(r-1)d](ld)z^d,$$

where $[n]$ is the standard shift of complexes and (m) denotes the Tate twist *by the Frobenius eigenvalue* σ in $H^1(X,\overline{\mathbb{Q}}_p)$. Note that a choice of one Frobenius eigenvalue σ is involved here, but choosing the other eigenvalue $\overline{\sigma}$ would, of course, give a similar result. Using the induction functor of [Lau90], we may form the product

$$\mathbb{E}_\lambda(z) = \mathbb{E}_{\lambda_1}(z) \star \mathbb{E}_{\lambda_2}(\sigma z) \star \cdots \star \mathbb{E}_{\lambda_l}(\sigma^{l-1}z).$$

This is still a series with coefficients in semisimple constructible complexes; these will usually be of infinite rank. Upon restricting to the open substack parametrizing semistable sheaves of zero slope, we finally obtain a semisimple constructible complex $\mathbb{E}_\lambda^{(0)}$. Using [Sch05, Proposition 6.1], one can show that the Frobenius eigenvalues of $\mathbb{E}_\lambda(z)$ and $\mathbb{E}_\lambda^{(0)}$ all belong to $v^{\mathbb{Z}}\sigma^{\mathbb{Z}}$. Hence the Frobenius trace $\mathrm{Tr}(\mathbb{E}_\lambda^{(0)})$ is a Laurent series in v and σ. Recall that we have fixed an isomorphism $\mathbb{C}\simeq\overline{\mathbb{Q}}_p$. By Harder's theorem, the series $\mathrm{Tr}(\mathbb{E}_\lambda^{(0)})$ converges (in a suitable domain) to $\mathbf{E}_\lambda^{(0)}$ and hence, by Theorem 3.1, we have $\mathrm{Tr}(\mathbb{E}_\lambda^{(0)}) = \omega P_{\lambda'}(\sigma^{-1}, v^2)$.

(iv) Pick a $\mathbb{F}_l$-rational closed point $x\in X(\mathbb{F}_l)$. Let $i: D_x \to X$ be the embedding of the formal neighborhood of x in X. Given an étale coordinate at x, we get an isomorphism $D_x \simeq \mathrm{Spec}(\mathbb{F}_l((\varpi)))$ where ϖ is a formal variable. Thus the set of isomorphism classes of torsion sheaves on D_x is equal to the set of conjugacy classes of nilpotent matrices. Invariant functions on the nilpotent cone $\mathcal{N}_d$, where $d\geqslant 1$, are canonically identified with elements of the ring $\Lambda^+_{(\sigma,v^2)}$ of symmetric functions. The restriction of coherent sheaves on X to D_x yields a map $\mathcal{I}(X)_{0,d} \to \coprod_{d'\leqslant d}\mathcal{N}_{d'}$. This factors to an algebra isomorphism $\Lambda^+_{(\sigma,v^2)} \simeq \mathbf{U}_X^{(\infty)}$. The Fourier–Mukai transform yields an algebra isomorphism $FM: \mathbf{U}_X^{(0)} \to \mathbf{U}_X^{(\infty)}$. The composed map $\mathbf{U}_X^{(0)} \to \Lambda^+_{(\sigma,v^2)}$ coincides with the isomorphism in Proposition 1.3.
The involution ω in Theorem 7.1 can be removed as follows. We shall give another isomorphism $\mathbf{U}_X^{(\infty)} \simeq \Lambda^+_{(\sigma,v^2)}$ which takes the Laurent series $FM(\mathbf{E}_\lambda^{(0)})$ to $P_{\lambda'}(\sigma^{-1},v^2)$. Let $X^{(d)}$ be the dth symmetric power of E, and let $\widetilde{\underline{\mathrm{Coh}}}^{0,d}(X)$ be the stacks of flags

$$\mathcal{M}_d \to \mathcal{M}_{d-1} \to \cdots \mathcal{M}_1,$$

where each $\mathcal{M}_i$ is a coherent (torsion) sheaf on X of length i. Consider the Cartesian square

$$\begin{array}{ccc} X^d & \xrightarrow{\iota_d} & \widetilde{\underline{\mathrm{Coh}}}^{0,d}(X) \\ \downarrow r_d & & \downarrow \pi_d \\ X^{(d)} & \xrightarrow{\iota_{(d)}} & \underline{\mathrm{Coh}}^{0,d}(X) \end{array}$$

in which π_d is the Springer map, r_d is the ramified finite cover $(x_1,x_2,\ldots x_d)\mapsto x_1+x_2+\cdots x_d$, and $\iota_{(d)}$ takes a divisor D to the sheaf $\mathcal{O}_D$. According to Laumon, the complex $F = R(\pi_d)_*(\overline{\mathbb{Q}}_p)$ is the intermediate extension of its restriction $F|_{U_d}$ to the dense open subset $U_d = \iota_{(d)}(X^{(d)})$. We have $\iota^*_{(d)}F = (r_d)_*(\overline{\mathbb{Q}}_p)$ by base change. Thus the symmetric group $\mathfrak{S}_d$ acts on $F|_{U_d}$. For each irreducible character ϕ of $\mathfrak{S}_d$, let F_ϕ be the intermediate extension of the constructible sheaf $\mathrm{Hom}_{\mathfrak{S}_d}(\phi, F|_{U_d})$. Each F_ϕ is a simple constructible

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complex on $\underline{\mathrm{Coh}}^{0,d}(X)$. The representation ring of $\mathfrak{S}_d$ is canonically identified with a subring of $\Lambda^+_{(\sigma,v^2)}$. We claim that there is an unique isomorphism $\mathbf{U}_X^{(\infty)} \simeq \Lambda^+_{(\sigma,v^2)}$ taking $\mathrm{Tr}(F_\phi)$ to the symmetric function associated to ϕ. This is the map we want.

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Appendix A. Proof of Proposition 3.2

A.1 We begin the proof of Proposition 3.2 with a sequence of lemmas.

Lemma A.1. *We have*

$$S\Big[X_1, \sum_i Y_i\Big]S = (1-q)SX_1Y_1S. \tag{A.1}$$

Proof. Using equations (2.8) and (2.9) in Lemma 2.1, we get

$$\begin{aligned} S\sum_i Y_iX_1S &= SX_1\Big(\sum_{i\geqslant 2} Y_i\Big)S + qt^{1-n}SX_1Y_1S + (t-1)SY_1X_1S \\ &\quad + (t^2-t)SY_1X_1S + \cdots + (t^{n-1}-t^{n-2})SY_1X_1S \\ &= SX_1\Big(\sum_{i\geqslant 2} Y_i\Big)S + qt^{1-n}SX_1Y_1S + qt^{1-n}(t^{n-1}-1)SX_1Y_1S \\ &= SX_1\Big(\sum_i Y_iX_1\Big)S + (q-1)SX_1Y_1S. \end{aligned}$$

□

Lemma A.2. *For any indices $2 \leqslant j_2 < j_3 < \cdots < j_l \leqslant n$ we have*

$$SY_1Y_{j_2}\cdots Y_{j_l}X_1 = qt^{l-n}SX_1Y_1Y_{j_2}\cdots Y_{j_l}. \tag{A.2}$$

Proof. By (2.8) in Lemma 2.1, we have $SY_1X_1 = qt^{1-n}SX_1Y_1$. By (2.9) we have

$$Y_{j_2}X_1 = X_1Y_{j_2} + (t^{1/2}-t^{-1/2})T^{-1}_{j_2-1}\cdots T_1^{-1}\cdots T^{-1}_{j_2-1}Y_1X_1.$$

Multiplying the above equation by $Y_{j_3}\cdots Y_{j_l}$ and using the fact that $[T_k, Y_h] = 0$ if $h > k-1$, we deduce that

$$\begin{aligned} Y_{j_2}Y_{j_3}\cdots Y_{j_l}X_1 &= Y_{j_3}\cdots Y_{j_l}X_1Y_{j_2} \\ &\quad + (t^{1/2}-t^{-1/2})T^{-1}_{j_2-1}\cdots T_1^{-1}\cdots T^{-1}_{j_2-1}Y_1Y_{j_3}\cdots Y_{j_l}X_1. \end{aligned}$$

Now, multiplying by Y_1 and using the relation

$$Y_1T_1^{-1}\cdots Y^{-1}_{j_2-1} = T_1\cdots T_{j_2-1}Y_{j_2}$$

yields

$$\begin{aligned} Y_1Y_{j_2}\cdots Y_{j_l}X_1 &= Y_1Y_{j_3}\cdots Y_{j_l}X_1Y_{j_2} \\ &\quad + (t^{1/2}-t^{-1/2})T^{-1}_{j_2-1}\cdots T_2^{-1}T_1\cdots T_{j_2-1}Y_1Y_{j_2}\cdots Y_{j_l}X_1, \end{aligned}$$

from which it follows that

$$SY_1Y_{j_2}\cdots Y_{j_l}X_1 = SY_1Y_{j_3}\cdots Y_{j_l}X_1Y_{j_2} + (1-t^{-1})SY_1Y_{j_2}\cdots Y_{j_l}X_1$$

and thus that

$$tSY_1Y_{j_2}\cdots Y_{j_l}X_1 = SY_1Y_{j_3}\cdots Y_{j_l}X_1Y_{j_2}.$$

By the same argument,

$$tSY_1Y_{j_3}\cdots Y_{j_l}X_1Y_{j_2} = SY_1Y_{j_4}\cdots Y_{j_l}X_1Y_{j_2}Y_{j_3},$$

and continuing in this manner we finally arrive at

$$SY_1Y_{j_2}\cdots Y_{j_l}X_1 = t^{l-1}SY_1X_1Y_{j_2}\cdots Y_{j_l} = qt^{l-n}SX_1Y_1Y_{j_2}\cdots Y_{j_l}$$

as asserted. □

Lemma A.3. *For any indices $1 < j_1 < j_2 \cdots < j_l \leqslant n$ we have*

$$\begin{aligned} SY_{j_1}\cdots Y_{j_l}X_1 = {} & SX_1Y_{j_1}\cdots Y_{j_l} \\ & + \sum_{u=1}^{l} q(1-t^{-1})t^{l-n+j_u-u}SX_1Y_1Y_{j_1}\cdots \widehat{Y_{j_u}}\cdots Y_{j_l}. \end{aligned} \tag{A.3}$$

Here, $\widehat{x}$ means that the term x is omitted from the product.

Proof. First of all, again by (2.9) in Lemma 2.1, we have

$$Y_jX_1 = X_1Y_j + (t^{1/2}-t^{-1/2})\beta_jY_jX_1$$

for all $j > 1$, where we have set

$$\beta_j = T_{j-1}^{-1}\cdots T_1{-}1\cdots T_{j-1}^{-1}.$$

Define elements

$$A(j_1,\ldots,j_l) = SY_{j_1}\cdots Y_{j_l}X_1 \quad \text{and} \quad B(j_2,\ldots,j_l) = SY_1Y_{j_2}\cdots Y_{j_l}X_1.$$

Using the same arguments as in the previous lemma, we obtain that

$$\begin{aligned} A(j_1,\ldots,j_l) &= A(j_2,\ldots,j_l)Y_{j_1} + (t^{1/2}-t^{-1/2})S\beta_{j_1}Y_1Y_{j_2}\cdots Y_{j_l}X_1 \\ &= A(j_2,\ldots,j_l)Y_{j_1} + (t^{1/2}-t^{-1/2})t^{(j_1-1)-1/2}B(j_2,\ldots,j_l). \end{aligned}$$

By Lemma A.2,

$$B(j_2,\ldots,j_l) = qt^{l-n}SX_1Y_1Y_{j_2}\cdots Y_{j_l},$$

and therefore

$$\begin{aligned} A(j_1,\ldots,j_l) &= A(j_2,\ldots,j_l)Y_{j_1} + q(1-t^{-1})t^{l-n+j_1-1}SX_1Y_1Y_{j_2}\cdots Y_{j_l} \\ &= (A(j_3,\ldots,j_l)Y_{j_2} + q(1-t^{-1})t^{l-n+j_2-2}SX_1Y_1Y_{j_3}\cdots Y_{j_l})Y_{j_1} \\ &\quad + q(1-t^{-1})t^{l-n+j_1-1}SX_1Y_1Y_{j_2}\cdots Y_{j_l} \\ &= \cdots \\ &= SX_1Y_{j_1}\cdots Y_{j_l} + \sum_{u=1}^{l} q(1-t^{-1})t^{l-n+j_u-u}SX_1Y_{j_1}\cdots \widehat{Y_{j_u}}\cdots Y_{j_l}, \end{aligned}$$

which is what we wanted to prove. □

LEMMA A.4. *The following holds:*

$$S\Big[X_1, \sum_{j_1<\cdots<j_l} Y_{j_1}\cdots Y_{j_l}\Big] = (1-q) \sum_{1<j_2<\cdots<j_l} SX_1Y_1Y_{j_2}\cdots Y_{j_l}. \tag{A.4}$$

Proof. Using the previous two lemmas, we compute

$$\begin{aligned}\sum_{j_1<\cdots<j_l} SY_{j_1}\cdots Y_{j_l}X_1 &= \sum_{1<j_2<\cdots<j_l} SY_1Y_{j_2}\cdots Y_{j_l}X_1 + \sum_{1<j_1<\cdots<j_l} SY_{j_1}\cdots Y_{j_l}X_1\\
&= \sum_{1<j_2<\cdots<j_l} qt^{l-n}SX_1Y_1Y_{j_2}\cdots Y_{j_l} + \sum_{1<j_1<\cdots<j_l} SX_1Y_{j_1}\cdots Y_{j_l}\\
&\quad + \sum_{1<j_1<\cdots<j_l}\Big\{\sum_{u=1}^{l} q(1-t^{-1})t^{l-n+j_u-u}SX_1Y_1Y_{j_1}\cdots\widehat{Y_{j_u}}\cdots Y_{j_l}\Big\}\\
&= \sum_{1<j_2<\cdots<j_l} qt^{n-l}SX_1Y_1Y_{j_2}\cdots Y_{j_l} + \sum_{1<j_1<\cdots<j_l} SX_1Y_{j_1}\cdots Y_{j_l}\\
&\quad + \sum_{1<k_1<\cdots<k_{l-1}} qt^{l-n}(1-t^{-1})\sigma_{k_1,\ldots,k_{l-1}}SX_1Y_1Y_{k_1}\cdots Y_{k_{l-1}},\end{aligned}$$

where

$$\begin{aligned}\sigma_{k_1,\ldots,k_{l-1}} &= \{(t+\cdots+t^{(k_1-1)-1}) + (t^{(k_1+1)-2}+\cdots+t^{(k_2-1)-2})\\
&\quad +\cdots+(t^{(k_{l-1}+1)-l}+\cdots+t^{n-l})\}\\
&= \frac{t^{n-l}-1}{1-t^{-1}}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j_1<\cdots<j_l} SY_{j_1}\cdots Y_{j_l}X_1 &= \sum_{1<j_2<\cdots<j_l} qt^{l-n}SX_1Y_1Y_{j_2}\cdots Y_{j_l} + \sum_{1<j_1<\cdots<j_l} SX_1Y_{j_1}\cdots Y_{j_l}\\
&\quad + \sum_{1<k_1<\cdots<k_{l-1}} qt^{l-n}(t^{n-l}-1)SX_1Y_1Y_{k_1}\cdots Y_{k_{l-1}}\\
&= q\sum_{1<j_2<\cdots<j_l} qt^{l-n}SX_1Y_1Y_{j_2}\cdots Y_{j_l} + \sum_{1<j_1<\cdots<j_l} SX_1Y_{j_1}\cdots Y_{j_l},\end{aligned}$$

from which the assertion follows. □

We are finally ready to give the proof of Proposition 3.2. We will argue by induction, with Lemma A.1 being the $l=1$ base case. So fix $l\in\mathbb{N}$ and assume that Proposition 3.2 has been proved for all $l'<l$. It is necessary to distinguish two cases.

Case 1. Let us assume that $l \leqslant n$. We will use the formula

$$\begin{aligned}\sum_i Y_i^l &= \Big(\sum_i Y_i^{l-1}\cdot\sum_i Y_i\Big) - \Big(\sum_i Y_i^{l-2}\cdot\sum_{i<j} Y_iY_j\Big)\\
&\quad +\cdots+(-1)^l\Big(\sum_i Y_i\cdot\sum_{j_1<\cdots<j_{l-1}} Y_{j_1}\cdots Y_{j_{l-1}}\Big)\\
&\quad + (-1)^{l+1}l\sum_{j_1<\cdots<j_l} Y_{j_1}\cdots Y_{j_l}.\end{aligned}$$

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According to the above, we have

$$\begin{aligned}
S\Big[X_1, \sum_i Y_i^l\Big]S = {} & S\Big[X_1, \sum_i Y_i\Big]S \cdot \sum_i Y_i^{l-1} + \sum_i Y_i \cdot S\Big[X_1, \sum_i Y_i^{l-1}\Big]S \\
& - S\Big[X_1, \sum_{j_1<j_2} Y_{j_1}Y_{j_2}\Big]S \cdot \sum_i Y_i^{l-2} - \sum_{j_1<j_2} Y_{j_1}Y_{j_2} \cdot S\Big[X_1, \sum_i Y_i^{l-2}\Big]S \\
& + \cdots + (-1)^l S\Big[X_1, \sum_{j_1<\cdots<j_{l-1}} Y_{j_1}\cdots Y_{j_{l-1}}\Big]S \cdot \sum_i Y_i \\
& + (-1)^l \sum_{j_1<\cdots<j_{l-1}} Y_{j_1}\cdots Y_{j_{l-1}} \cdot S\Big[X_1, \sum_i Y_i\Big]S \\
& + (-1)^{l+1} l S\Big[X_1, \sum_{j_1<\cdots<j_l} Y_{j_1}\cdots Y_{j_l}\Big]S.
\end{aligned} \tag{A.5}$$

By the induction hypothesis and Lemma A.4,

$$S\Big[X_1, \sum_i Y_i^m\Big]S = (1-q^m)SY_1^t X_1 S,$$

$$S\Big[X_1, \sum_{j_1<\cdots<j_s} Y_{j_1}\cdots Y_{j_s}\Big]S = (1-q) \sum_{1<j_2<\cdots<j_s} SX_1Y_1Y_{j_2}\cdots Y_{j_s}S$$

for all $m<l$ and all s. Substituting these into (A.5), we deduce that

$$\begin{aligned}
S\Big[X_1, \sum_i Y_i^l\Big]S = {} & (1-q)SX_1Y_1 \sum_i Y_i^{l-1}S + (1-q^{l-1})S \sum_i Y_iX_1Y_1^{l-1}S \\
& - (1-q)SX_1Y_1 \sum_{1<j_2} Y_{j_2} \sum_i Y_i^{l-2}S - (1-q^{l-2})S \sum_{j_1<j_2} Y_{j_1}Y_{j_2}X_1Y_1S \\
& + \cdots + (-1)^l(1-q)SX_1Y_1 \sum_{1<j_2<\cdots<j_{l-1}} Y_{j_2}\cdots Y_{j_{l-1}} \sum_i Y_iS \\
& + (-1)^l(1-q)S \sum_{j_1<\cdots<j_{l-1}} Y_{j_1}\cdots Y_{j_{l-1}}X_1Y_1S \\
& + (-1)^{l+1} l(1-q)SX_1Y_1 \sum_{1<j_2<\cdots<j_l} Y_{j_2}\cdots Y_{j_l}S.
\end{aligned} \tag{A.6}$$

This is where we use Lemma A.4 again, in the form

$$S \sum_{j_1<\cdots<j_t} Y_{j_1}\cdots Y_{j_t}X_1 = SX_1 \sum_{j_1<\cdots<j_t} Y_{j_1}\cdots Y_{j_t} + (q-1)SX_1Y_1 \sum_{1<j_2<\cdots<j_t} Y_{j_2}\cdots Y_{j_t},$$

to obtain the following expression for the bracket $S[X_1, \sum_i Y_i^l]S$:

$$\begin{aligned}
& S\Big[X_1, \sum_i Y_i^l\Big]S \\
& = (1-q)SX_1\Big\{Y_1 \sum_i Y_i^{l-1} - Y_1 \sum_{1<j_2} Y_{j_2} \sum_i Y_i^{l-2} + Y_1 \sum_{1<j_2<j_3} Y_{j_2}Y_{j_3} \sum_i Y_i^{l-3} - \cdots
\end{aligned}$$

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$$+\cdots+(-1)^l Y_1 \sum_{1<j_2<\cdots<j_{l-1}} Y_{j_2}\cdots Y_{j_{l-1}} \sum_i Y_i + (-1)^{l+1} l Y_1 \sum_{1<j_2<\cdots<j_l} Y_{j_2}\cdots Y_{j_l}\Big\}$$
$$+ (1-q^{l-1}) S X_1 Y_1^{l-1} \sum_i Y_i S + (1-q^{l-1})(q-1) S X_1 Y_1^l S$$
$$- (1-q^{l-2}) S X_1 Y_1^{l-2} \sum_{j_1<j_2} Y_{j_1} Y_{j_2} - (1-q^{l-2})(q-1) S X_1 Y_1^{l-1} \sum_{1<j_2} Y_{j_2} S$$
$$+\cdots+(-1)^l (1-q) S X_1 Y_1 \sum_{j_1<\cdots<j_{l-1}} Y_{j_1}\cdots Y_{j_{l-1}} S$$
$$+ (-1)^l (1-q)(q-1) S X_1 Y_1^2 \sum_{j_2<\cdots<j_{l-1}} Y_{j_2}\cdots Y_{j_{l-1}} S.$$

After collecting terms, we get

$$\begin{aligned}
&S\Big[X_1, \sum_i Y_i^l\Big] S\\
&= S X_1 Y_1^l S\{(1-q) + (1-q^{l-1}) + (1-q^{l-1})(q-1)\}\\
&\quad + S X_1 Y_1^{l-1} \sum_{1<j_2} Y_{j_2} S\{(q-1) + (1-q^{l-1}) - (1-q^{l-2})(q-1) - (1-q^{l-2})\}\\
&\quad + S X_1 Y_1^{l-2} \sum_{1<j_2<j_3} Y_{j_2} Y_{j_3} S\{(1-q) + (q^{l-2}-1) - (q^{l-3}-1)(q-1) - (q^{l-3}-1)\}\\
&\quad +\cdots+ S X_1 Y_1^2 \sum_{1<j_2<\cdots<j_{l-1}} Y_{j_2}\cdots Y_{j_{l-1}} S\{(-1)^l((1-q) + (q^2-1) - (q-1)^2 - (q-1))\}\\
&\quad + S X_1 Y_1 \sum_{1<j_2<\cdots<j_l} Y_{j_2}\cdots Y_{j_l} S\{(1-q)((-1)^l(l-1) + (-1)^{l+1} l + (-1)^l)\}\\
&\quad + \sum_{j>1} \sum_{m=1}^{l-2} S X_1 Y_1 Y_j^{l-m} \sum_{\substack{1<j_2<\cdots<j_m\\ j_u\neq j}} Y_{j_2}\cdots Y_{j_m}\{(1-q)((-1)^{m+1} + (-1)^{m+2})\}\\
&= (1-q^l) S X_1 Y_1^l S.
\end{aligned}$$

This concludes the proof of Proposition 3.2 in the first case.

Case 2. Let us deal with the situation where $l>n$. The method is very similar to that used in the proof of Case 1 above. This time we use the following identity:

$$\sum_i Y_i^l = \sum_i Y_i^{l-1} \cdot \sum_i Y_i - \sum_i Y_i^{l-2} \cdot \sum_{j_1<j_2} Y_{j_1} Y_{j_2} + \cdots + (-1)^{n-1} \sum_i Y_i^{l-n} Y_1 \cdots Y_n.$$

Based on the above decomposition, we write

$$\begin{aligned}
S\Big[X_1, \sum_i Y_i^l\Big] S &= S\Big[X_1, \sum_i Y_i\Big] S \cdot \sum_i Y_i^{l-1} + \sum_i Y_i \cdot S\Big[X_1, \sum_i Y_i^{l-1}\Big] S\\
&\quad - S\Big[X_1, \sum_{j_1<j_2} Y_{j_1} Y_{j_2}\Big] S \cdot \sum_i Y_i^{l-2} - \sum_{j_1<j_2} Y_{j_1} Y_{j_2} \cdot S\Big[X_1, \sum_i Y_i^{l-2}\Big] S
\end{aligned}$$

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$$+\cdots+(-1)^{n-1}S[X_1,Y_1\cdots Y_n]S\cdot\sum_i Y_i^{l-n}$$
$$+(-1)^{n-1}Y_1\cdots Y_nS\Big[X_1,\sum_i Y_i^{l-n}\Big]S. \tag{A.7}$$

By the induction hypothesis and Lemma A.4, this simplifies to

$$\begin{aligned}
&S\Big[X_1,\sum_i Y_i^l\Big]S\\
&=(1-q)SX_1Y_1\sum_i Y_i^{l-1}S+(1-q^{l-1})S\sum_i Y_iX_1Y_1^{l-1}S\\
&\quad-(1-q)SX_1\sum_{1<j_2}Y_{j_2}Y_1\sum_i Y_i^{l-2}S-(1-q^{l-2})S\sum_{j_1<j_2}Y_{j_1}Y_{j_2}X_1Y_1^{l-2}S\\
&\quad+\cdots+(-1)^{n-1}\Big((1-q)SX_1Y_1\cdots Y_n\sum_i Y_i^{l-n}S+(1-q^{l-n})SY_1\cdots Y_nX_1Y_1^{l-n}S\Big)\\
&=\Big\{(1-q)SX_1Y_1\sum_i Y_i^{l-1}S+(1-q^{l-1})SX_1\sum_i Y_iY_1^{l-1}S\\
&\qquad+(1-q^{l-1})(q-1)SX_1Y_1^lS\Big\}\\
&\quad-\Big\{(1-q)SX_1\sum_{1<j_2}Y_{j_2}Y_1\sum_i Y_i^{l-2}S+(1-q^{l-2})SX_1\sum_{j_1<j_2}Y_{j_1}Y_{j_2}Y_1^{l-2}S\\
&\qquad+(1-q^{l-2})(q-1)SX_1\sum_{1<j_2}Y_{j_2}Y_1^{l-1}S\Big\}\\
&\quad+\cdots+(-1)^{n-1}\Big\{(1-q)SX_1Y_1\cdots Y_n\sum_i Y_i^{l-n}S+(1-q^{l-n})SX_1Y_1\cdots Y_nY_1^{l-n}S\\
&\qquad\qquad+(1-q^{l-n})(q-1)SX_1Y_2\cdots Y_nY_1^{l-n+1}S\Big\}.
\end{aligned}$$

Upon gathering terms, we obtain

$$\begin{aligned}
&S\Big[X_1,\sum_i Y_i^l\Big]S\\
&=SX_1Y_1^lS\{(1-q)+(1-q^{l-1})+(q-1)(1-q^{l-1})\}\\
&\quad+SX_1\sum_{1<j_2}Y_{j_2}Y_1^{l-1}S\{-(1-q)+(1-q^{l-1})-(1-q^{l-2})-(q-1)(1-q^{l-2})\}\\
&\quad+\cdots+SX_1Y_2\cdots Y_nY_1^{l-n+1}S\{(-1)^{n-1}((1-q)-(1-q^{l-n+1})+(1-q^{l-n})\\
&\qquad\qquad+(q-1)(1-q^{l-n}))\}\\
&\quad+\sum_{i>1}\sum_{m=1}^n SX_1\sum_{\substack{1<j_2<\cdots<j_{m-1}\\ j_u\neq i}}Y_{j_2}\cdots Y_{j_{m-1}}Y_i^{l-m}S\{(-1)^{m+1}(q-1)+(-1)^m(q-1)\}\\
&=(1-q^l)SX_1Y_1^lS
\end{aligned}$$

as desired. This concludes the proof of Case 2 as well as that of Proposition 3.2.

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Appendix B. Proof of Proposition 3.3

We start by giving a closed expression for the commutator $S[\sum_i Y_i, X_1^l Y_1^{-1}]S$.

Lemma B.1. *For any $l \geqslant 1$ we have*

$$S\left[\sum_i Y_i, X_1^l Y_1^{-1}\right]S = (t^{n-1}-1)S\{qX_1^{l-1}X_n + q^2 X_1^{l-2}X_n^2 + \cdots + q^l X_n^l\}S$$
$$+ q^l S X_n^l S - S X_1^l S. \tag{B.1}$$

Proof. First of all, by (2.10) we have

$$\begin{aligned} Y_1 X_1 Y_1^{-1} &= qT_1 \cdots T_{n-2}T_{n-1}^2 T_{n-2} \cdots T_1 X_1 \\ &= qT_1 \cdots T_{n-2}T_{n-1}X_n T_{n-1}^{-1} \cdots T_1^{-1}, \end{aligned}$$

from which it follows that

$$Y_1 X_1^l Y_1^{-1} = q^l T_1 \cdots T_{n-2}T_{n-1}X_n^l T_{n-1}^{-1} \cdots T_1^{-1} \tag{B.2}$$

and hence that $SY_1X_1^lY_1^{-1}S = q^l S X_n^l S$. Now we compute

$$\begin{aligned} S\left[\sum_i Y_i, X_1^l Y_1^{-1}\right]S &= SY_1X_1^lY_1^{-1}S - SX_1^lS + \sum_{m=2}^{n} S[Y_m, X_1^lY_1^{-1}]S \\ &= q^l S X_n^l S - S X_1^l S + \sum_{m=2}^{n} S[Y_m, X_1^lY_1^{-1}]S. \end{aligned}$$

The lemma will therefore be proved once we have shown that

$$S[Y_m, X_1^lY_1^{-1}]S = (1-t^{-1})t^{m-1}S\{qX_1^{l-1}X_n + q^2X_1^{l-2}X_n^2 + \cdots + q^lX_n^l\}S. \tag{B.3}$$

For this, we need a preparatory result.

Sublemma B.2. *For any m, the following identity holds:*

$$\begin{aligned} &T_{m-1}^{-1} \cdots T_2^{-1}T_1^{-1}T_2^{-1} \cdots T_{m-1}^{-1}T_1 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_1 \\ &\quad = T_1^{-1} \cdots T_{m-2}^{-1}T_mT_{m+1} \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdot T_1. \end{aligned} \tag{B.4}$$

Proof of sublemma. We argue by induction. The relation can easily be checked directly for $m = 2$. Fix m and assume that (B.4) holds for $m-1$. We have

$$\begin{aligned} &T_{m-1}^{-1} \cdots T_2^{-1}T_1^{-1}T_2^{-1} \cdots T_{m-1}^{-1}T_1 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_1 \\ &\quad = T_{m-1}^{-1} \cdots T_1^{-1}T_2^{-1}T_1^{-1} \cdots T_{m-1}^{-1}T_1 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_1 \\ &\quad = T_1^{-1}T_{m-1}^{-1} \cdots T_3^{-1}T_2^{-1}T_3^{-1} \cdots T_{m-1}^{-1}T_1^{-1}T_1 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_1 \\ &\quad = T_1^{-1}(T_{m-1}^{-1} \cdots T_3^{-1}T_2^{-1}T_3^{-1} \cdots T_{m-1}^{-1}T_2 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_2)T_1. \end{aligned}$$

Using the induction hypothesis applied to the set of indices $2, 3, \ldots, n$ rather than $1, 2, \ldots, n$, we may simplify the expression enclosed in parentheses to get

$$\begin{aligned} &T_{m-1}^{-1} \cdots T_2^{-1}T_1^{-1}T_2^{-1} \cdots T_{m-1}^{-1}T_1 \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_1 \\ &\quad = T_1^{-1}(T_2^{-1} \cdots T_{m-2}^{-1}T_mT_{m+1} \cdots T_{n-2}T_{n-1}^2T_{n-2} \cdots T_2)T_1, \end{aligned}$$

which proves (B.4) for the integer m. This finishes the induction step and hence the proof of the sublemma. □

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We may now prove Lemma B.1. Once again, we argue by induction. Fix m and set $u_l = Y_m X_1^l Y_1^{-1}$. We compute u_1 directly, using (2.9) and (B.4), as follows:

$$\begin{aligned} u_1 &= X_1 Y_m Y_1^{-1} + (t^{1/2} - t^{-1/2}) T_{m-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{m-1}^{-1} Y_1 X_1 Y_1^{-1} \\ &= X_1 Y_m Y_1^{-1} + q(t^{1/2} - t^{-1/2}) T_{m-1}^{-1} \cdots T_1^{-1} \cdots T_{m-1}^{-1} T_1 \cdots T_{n-1}^2 \cdots T_1 X_1 Y_1 Y_1^{-1} \\ &= X_1 Y_m Y_1^{-1} + q(t^{1/2} - t^{-1/2}) T_1^{-1} \cdots T_{m-2}^{-1} T_m \cdots T_{n-1}^2 \cdots T_1 X_1 \\ &= X_1 Y_m Y_1^{-1} + q(t^{1/2} - t^{-1/2}) T_1^{-1} \cdots T_{m-2}^{-1} T_m \cdots T_{n-1} X_n T_{n-1}^{-1} \cdots T_1^{-1}. \end{aligned}$$

We will now prove the following formula by induction on l:

$$u_l = X_1^l Y_1^{-1} Y_m + (t^{1/2} - t^{-1/2}) T_1^{-1} \cdots T_{m-2}^{-1} T_m \cdots T_{n-1} e_l T_{n-1}^{-1} \cdots T_1^{-1} \tag{B.5}$$

where

$$e_l = q X_n X_1^{l-1} + q^2 X_n^2 X_1^{l-2} + \cdots + q^l X_n^l.$$

The $l = 1$ case was proved above. Let us assume that formula (B.5) holds for a certain integer l. We have

$$\begin{aligned} u_{l+1} &= X_1 u_l + q(t^{1/2} - t^{-1/2}) T_1^{-1} \cdots T_{m-2}^{-1} T_m \cdots T_{n-1} X_n T_{n-1}^{-1} \cdots T_1^{-1} Y_1 X_1^l Y_1^{-1} \\ &= X_1 u_l + q^{l+1}(t^{1/2} - t^{-1/2}) T_1^{-1} \cdots T_{m-2}^{-1} T_m \cdots T_{n-1} X_n T_{n-1}^{-1} \cdots T_1^{-1} \\ &\quad \times T_1 \cdots T_{n-1} X_n^l T_{n-1}^{-1} \cdots T_1^{-1} \\ &= X_1 u_l + q^{l+1}(t^{1/2} - t^{-1/2}) T_1^{-1} \cdots T_{m-2}^{-1} T_m \cdots T_{n-1} X_n^{l+1} T_{n-1}^{-1} \cdots T_1^{-1}, \end{aligned} \tag{B.6}$$

and (B.5) for the integer $l + 1$ follows from this by the induction hypothesis.

Equation (B.3) is obtained from multiplying (B.5) by S on both sides. Lemma B.1 is now proved. □

We can now proceed with the proof of Proposition 3.3. Let us form the generating series for $S[\sum_i Y_i, X_1^l Y_1^{-1}]S$. By Lemma B.1, we find

$$\begin{aligned} \sum_{r \geqslant 1} S\Big[\sum_i Y_i, X_1^r Y_1^{-1}\Big] S u^r &= S\bigg\{-\frac{X_1 u}{1 - X_1 u} + (t^{n-1} - 1)\frac{\sum_{i \geqslant 1} q^i X_n^i u^i \cdot X_1 u}{1 - X_1 u} + t^{n-1}\frac{q X_n u}{1 - X_n u}\bigg\} \\ &= S\frac{-X_1 u + t^{n-1} q X_n u}{(1 - X_1 u)(1 - q X_n u)} S. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\exp\bigg(\sum_{r \geqslant 1} \frac{(t^{r/2} - t^{-r/2})(t^{-r/2} - q^r t^{r/2})}{r} \sum_i X_i^r u^r\bigg) \\ &\quad = \frac{\exp(\sum_{r \geqslant 1}(1/r)\sum_i X_i^r u^r)\exp(\sum_{r \geqslant 1}(q^r/r)\sum_i X_i^r u^r)}{\exp(\sum_{r \geqslant 1}(t^{-r}/r)\sum_i X_i^r u^r)\exp(\sum_{r \geqslant 1}(q^r t^r/r)\sum_i X_i^r u^r)} \\ &\quad = \prod_{i=1}^n \frac{\exp(\sum_{r \geqslant 1}(1/r) X_i^r u^r)\exp(\sum_{r \geqslant 1}(q^r/r) X_i^r u^r)}{\exp(\sum_{r \geqslant 1}(t^{-r}/r) X_i^r u^r)\exp(\sum_{r \geqslant 1}(q^r t^r/r) X_i^r u^r)} \\ &\quad = \prod_{i=1}^n \frac{(1 - t^{-1} X_i u)(1 - q t X_i u)}{(1 - q X_i u)(1 - X_i u)}. \end{aligned}$$

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Hence we are reduced to proving the following relation:

$$S\prod_{i=1}^{n}(1-t^{-1}X_iu)(1-tqX_iu)S = S\prod_{i=1}^{n}(1-X_iu)(1-qX_iu)S + \frac{(1-t^{-n})(1-tq)}{1-q}S \times \left\{(X_1u-t^{n-1}qX_nu)\prod_{2}^{n}(1-X_iu)\prod_{i=1}^{n-1}(1-qX_iu)\right\}S. \quad (B.7)$$

Of course, owing to homogeneity, we may drop the dummy variable u from this formula. A brute force approach based on the equalities

$$SX_1S = \frac{1}{1+t^{-1}}S(X_1+X_2)S = tSX_2S,$$
$$SX_1^2S = \frac{1}{1+t^{-1}}S(X_1^2+X_2^2+(1-t^{-1})X_1X_2)S,$$
$$SX_2^2S = \frac{1}{1+t}S(X_1^2+X_2^2+(1-t)X_1X_2)S$$

allows one to check (B.7) directly for $n=2$. We will now prove (B.7) by induction on n. So fix n and assume that (B.7) holds for the integer $n-1$, with $n-1\geqslant 2$. For any subset $\{i_1,\ldots,i_r\}$ of $\{1,\ldots,n\}$, we denote by $S_{i_1,\ldots,i_r}$ the partial symmetrizer with respect to the indices $\{i_1,\ldots,i_r\}$.

Using the relation

$$S_{12}(X_1-t^{n-1}qX_n)(1-X_2)S_{12} = tS_{12}(X_2-t^{n-2}qX_n)(1-t^{-1}X_1)S_{12},$$

we get

$$\frac{(1-t^{-n})(1-tq)}{1-q}S\left\{(X_1-t^{n-1}qX_n)\prod_{2}^{n}(1-X_i)\prod_{i=1}^{n-1}(1-qX_i)\right\}S$$
$$= \frac{(1-t^{-n})(1-tq)}{1-q}tS\left\{(1-qX_1)(1-t^{-1}X_1)(X_2-t^{n-2}qX_n)\prod_{3}^{n}(1-X_i)\prod_{i=2}^{n-1}(1-qX_i)\right\}S$$
$$= \frac{1-t^{-n}}{1-t^{1-n}}tS\left\{(1-qX_1)(1-t^{-1}X_1)\left(\prod_{i=2}^{n}(1-t^{-1}X_i)(1-tqX_i)-\prod_{i=2}^{n}(1-X_i)(1-qX_i)\right)\right\}S.$$

Next, we use the formulas

$$S(1-qX_1)\prod_{i=2}^{n}(1-tqX_i)S = t^{-1}S\prod_{i=1}^{n}(1-tqX_i)S + (1-t^{-1})S\prod_{i=2}^{n}(1-tqX_i)S,$$
$$S(1-t^{-1}X_1)\prod_{i=2}^{n}(1-X_i)S = t^{-1}S\prod_{i=1}^{n}(1-X_i)S + (1-t^{-1})S\prod_{i=2}^{n}(1-X_i)S$$

to simplify (B.7) to the following relation:

$$(t^{-n}-t^{1-n})S\left\{\prod_{i=1}^{n}(1-t^{-1}X_i)(1-tqX_i)-\prod_{i=1}^{n}(1-X_i)(1-qX_i)\right\}S$$
$$= (1-t^{-n})(t-1)\left(\prod_{i=1}^{n}(1-t^{-1}X_i)S\prod_{i=2}^{n}(1-tqX_i)S-\prod_{i=1}^{n}(1-qX_i)S\prod_{i=2}^{n}(1-X_i)S\right). \quad (B.8)$$

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As usual, let

$$m_\lambda(z_1,\ldots,z_k)=\sum_{\sigma\in\mathfrak{S}_k} z_{\sigma(1)}^{\lambda_1}\cdots z_{\sigma(k)}^{\lambda_k}$$

stand for the monomial symmetric function. The computation of the $(t^{1/2})$-symmetrization of a monomial symmetric function $m_{(1^r)}$ is an easy exercise which we leave to the reader.

Sublemma B.3. *For any $1\leqslant r\leqslant n$ we have*

$$Sm_{(1^r)}(X_2,\ldots X_n)S=t^{-r}\frac{\begin{bmatrix} n-1\\ r\end{bmatrix}_t^-}{\begin{bmatrix} n\\ r\end{bmatrix}_t^-}Sm_{(1^r)}(X_1,\ldots,X_n)S.$$

We take the convention that $\begin{bmatrix} n-1\\ r\end{bmatrix}_t^+=0$ if $r=n$ and drop the index t for simplicity. Using Sublemma B.3, we can now write down closed and symmetric expressions for all terms involved in (B.8):

$$\prod_{i=1}^n(1-t^{-1}X_i)=\sum_{r=0}^n(-1)^r t^{-r}m_{(1^r)}(X_2,\ldots,X_n),$$

$$\prod_{i=1}^n(1-qX_i)=\sum_{r=0}^n(-1)^r q^{r}m_{(1^r)}(X_2,\ldots,X_n),$$

$$S\prod_{i=2}^n(1-X_i)S=\sum_{r=0}^n(-1)^r t^{-r}\frac{\begin{bmatrix} n-1\\ r\end{bmatrix}^-}{\begin{bmatrix} n\\ r\end{bmatrix}^-}Sm_{(1^r)}(X_1,\ldots,X_n)S,$$

$$S\prod_{i=2}^n(1-tqX_i)S=\sum_{r=0}^n(-1)^r q^{r}\frac{\begin{bmatrix} n-1\\ r\end{bmatrix}^-}{\begin{bmatrix} n\\ r\end{bmatrix}^-}Sm_{(1^r)}(X_1,\ldots,X_n)S.$$

This allows us to write

$$\prod_{i=1}^n(1-t^{-1}X_i)S\prod_{i=2}^n(1-tqX_i)S=\sum_{r,k}(-1)^r\left\{\sum_{u=0}^r\binom{r}{u}q^{u+k}\frac{\begin{bmatrix} n-1\\ u+k\end{bmatrix}^-}{\begin{bmatrix} n\\ u+k\end{bmatrix}^-}t^{u-r-k}\right\}$$
$$\times Sm_{(1^r2^k)}(X_1,\ldots,X_n)S$$

and

$$\prod_{i=1}^n(1-qX_i)S\prod_{i=2}^n(1-X_i)S=\sum_{r,k}(-1)^r\left(\sum_{u=0}^r\binom{r}{u}t^{-u-k}\frac{\begin{bmatrix} n-1\\ u+k\end{bmatrix}^-}{\begin{bmatrix} n\\ u+k\end{bmatrix}^-}q^{r-u+k}\right)$$
$$\times Sm_{(1^r2^k)}(X_1,\ldots,X_n)S.$$

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Hence

$$\prod_{i=1}^{n}(1-t^{-1}X_i)S\prod_{i=2}^{n}(1-tqX_i)S-\prod_{i=1}^{n}(1-qX_i)S\prod_{i=2}^{n}(1-X_i)S$$
$$=\sum_{r,k}(-1)^r\sum_{u=0}^{r}\binom{r}{u}\frac{\begin{bmatrix}n-1\\u+k\end{bmatrix}^-}{\begin{bmatrix}n\\u+k\end{bmatrix}^-}\{q^{u+k}t^{u-r-k}-t^{-u-k}q^{r-u+k}\}Sm_{(1^r2^k)}(X_1,\ldots,X_n)S \tag{B.9}$$

while, of course,

$$S\left\{\prod_{i=1}^{n}(1-t^{-1}X_i)(1-tqX_i)-\prod_{i=1}^{n}(1-X_i)(1-qX_i)\right\}S$$
$$=\sum_{r,k}(-1)^r\sum_{u=0}^{r}\binom{r}{u}(t^{2u-r}-1)q^{u+k}m_{(1^r2^k)}(X_1,\ldots,X_n). \tag{B.10}$$

Let A denote the right-hand side of (B.9) multiplied by $(1-t^{-n})(t-1)$, and let B stand for the right-hand side of (B.10) multiplied by $(t^{-n}-t^{1-n})$. Equation (B.8) says simply that $A=B$. To show this, we check that the term $q^{u+k}m_{(1^r2^k)}(X_1,\ldots,X_n)$ appears in A and in B with the same coefficient. In B this coefficient is clearly equal to

$$(-1)^r\binom{r}{u}(t^{2u-r}-1)(t^{-n}-t^{1-n})$$

whereas, as far as A is concerned, it is equal to

$$(1-t^{-n})(t-1)(-1)^r\binom{r}{u}\left\{t^{u-r-k}\frac{\begin{bmatrix}n-1\\u+k\end{bmatrix}^-}{\begin{bmatrix}n\\u+k\end{bmatrix}^-}-t^{u-r-k}\frac{\begin{bmatrix}n-1\\r-u+k\end{bmatrix}^-}{\begin{bmatrix}n\\r-u+k\end{bmatrix}^-}\right\}$$
$$=(1-t^{-n})(t-1)(-1)^r\binom{r}{u}t^{u-r-k}\left(\frac{t^{r+k-n-u}-t^{u-n+k}}{1-t^{-n}}\right)$$
$$=(t-1)t^{-n}(1-t^{2u-r})(-1)^r\binom{r}{u}$$
$$=(t^{-n}-t^{1-n})(t^{2u-r}-1)(-1)^r\binom{r}{u},$$

as desired. Thus equation (B.8) and Proposition 3.3 are (finally!) proved.

Appendix C. Proof of Theorem 6.3

C.1 We begin with (6.3). Since

$$\mathbf{E}_r(z_2)=\mathbf{E}_r^{\mathrm{vec}}(z_2)\mathbf{E}_0(v^{2r}z_2)\quad\text{and}\quad[\mathbf{E}_0(z_1),\mathbf{E}_0(v^{2r}z_2)]=0,$$

the relation (6.3) is equivalent to

$$[T_{(0,1)},\mathbf{E}_r^{\mathrm{vec}}(z)]=v\#X(\mathbb{F}_l)\frac{v^{-2r}-1}{v^{-2}-1}z^{-1}\mathbf{E}_r^{\mathrm{vec}}(z). \tag{C.1}$$

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We prove (C.1) by showing that for any $x \in X(\mathbb{F}_l)$,

$$[\mathbf{1}_{\mathcal{O}_x}, \mathbf{1}^{\mathrm{vec}}_{r,d}] = \frac{v^{-r} - v^{r}}{v^{-2} - 1} \mathbf{1}^{\mathrm{vec}}_{r,d+1},$$

where $\mathcal{O}_x$ is the structure sheaf at x. Indeed, we have

$$\mathbf{1}^{\mathrm{vec}}_{r,d} \cdot \mathbf{1}_{\mathcal{O}_x} = v^{-r} \sum_{\substack{\mathcal{F} \text{ vector bundle} \\ \overline{\mathcal{F}}=(r,d)}} \mathbf{1}_{\mathcal{F} \oplus \mathcal{O}_x},$$

whereas, since every non-zero map to $\mathcal{O}_x$ is onto,

$$\mathbf{1}_{\mathcal{O}_x} \cdot \mathbf{1}^{\mathrm{vec}}_{r,d} = v^{r} \bigg\{ \sum_{\substack{\mathcal{G} \text{ vector bundle} \\ \overline{\mathcal{G}}=(r,d+1)}} \frac{\#\,\mathrm{Hom}(\mathcal{G}, \mathcal{O}_x) - 1}{v^{-2} - 1} \mathbf{1}_{\mathcal{G}} + \sum_{\substack{\mathcal{F} \text{ vector bundle} \\ \overline{\mathcal{F}}=(r,d)}} \#\,\mathrm{Hom}(\mathcal{F}, \mathcal{O}_x) \mathbf{1}_{\mathcal{F} \oplus \mathcal{O}_x} \bigg\}.$$

We conclude by using $\dim \mathrm{Hom}(\mathcal{G}, \mathcal{O}_x) = \dim \mathrm{Hom}(\mathcal{F}, \mathcal{O}_x) = r$.

C.2 We now turn to the proof of (6.4). We begin with a lemma.

Lemma C.1. *For any $d \geqslant 1$ and any r, the series $\mathbf{E}_r(z)$ is an eigenvector for the adjoint action of $\mathcal{T}_{(0,d)}$.*

Proof. We will first show that for any vector bundle $\mathcal{V}$ of rank r, the commutator $[T_{(0,d)}, \mathbf{1}_{\mathcal{V}}]$ is supported on the set of vector bundles. To see this, let $\mathcal{F}$ be a coherent sheaf of rank r, and write $\mathcal{F} = \nu_{\mathcal{F}} \oplus \tau_{\mathcal{F}}$ where $\nu_{\mathcal{F}}$ is a vector bundle and $\tau_{\mathcal{F}}$ is a torsion sheaf. Let us assume that $\tau_{\mathcal{F}} \neq 0$ and compute

$$\begin{aligned} \langle T_{(0,d)} 1_{\mathcal{V}}, \mathbf{1}_{\mathcal{F}} \rangle_G &= \langle T_{(0,d)} \mathbf{1}_{\mathcal{V}}, v^{\langle \nu_{\mathcal{F}}, \tau_{\mathcal{F}} \rangle} \mathbf{1}_{\nu_{\mathcal{F}}} \mathbf{1}_{\tau_{\mathcal{F}}} \rangle_G \\ &= v^{\langle \nu_{\mathcal{F}}, \tau_{\mathcal{F}} \rangle} \langle \Delta(T_{(0,d)}) \cdot \Delta(\mathbf{1}_{\mathcal{V}}), \mathbf{1}_{\nu_{\mathcal{F}}} \otimes \mathbf{1}_{\tau_{\mathcal{F}}} \rangle_G. \end{aligned} \tag{C.2}$$

Since $\Delta(T_{(0,d)}) = T_{(0,d)} \otimes 1 + 1 \otimes T_{(0,d)}$, $\mathcal{V}$ and $\mathcal{F}$ are both of rank r and no subsheaf of $\mathcal{V}$ is torsion, we may simplify (C.2) to

$$\langle T_{(0,d)} \mathbf{1}_{\mathcal{V}}, \mathbf{1}_{\mathcal{F}}) = v^{\langle \nu_{\mathcal{F}}, \tau_{\mathcal{F}} \rangle} (T_{(0,d)} \mathbf{1}_{\mathcal{V}} \otimes 1 + \mathbf{1}_{\mathcal{V}} \otimes T_{(0,d)}, \mathbf{1}_{\nu_{\mathcal{F}}} \otimes \mathbf{1}_{\tau_{\mathcal{F}}} \rangle_G. \tag{C.3}$$

This is non-zero only if $\deg(\tau_{\mathcal{F}}) \in \{0, d\}$. A very similar computation shows that

$$\langle \mathbf{1}_{\mathcal{V}} T_{(0,d)}, \mathbf{1}_{\mathcal{F}}) = v^{\langle \nu_{\mathcal{F}}, \tau_{\mathcal{F}} \rangle} (\mathbf{1}_{\mathcal{V}} T_{(0,d)} \otimes 1 + \mathbf{1}_{\mathcal{V}} \otimes T_{(0,d)}, \mathbf{1}_{\nu_{\mathcal{F}}} \otimes \mathbf{1}_{\tau_{\mathcal{F}}} \rangle_G, \tag{C.4}$$

which is also non-zero only if $\deg(\tau_{\mathcal{F}}) \in \{0, d\}$. Furthermore, if $\deg(\tau_{\mathcal{F}}) = d$ (i.e. if $\mathcal{F}$ is not a vector bundle), then (C.3) and (C.4) actually coincide, so that $\langle [T_{(0,d)}, \mathbf{1}_{\mathcal{V}}], \mathbf{1}_{\mathcal{F}} \rangle_G = 0$. This proves that $[T_{(0,d)}, \mathbf{1}_{\mathcal{V}}]$ is indeed supported on the set of vector bundles.

To finish the proof of Lemma C.1, we need to show that the scalar product

$$\langle [T_{(0,d)}, \mathbf{1}^{\mathrm{vec}}_{(r,l)}], \mathbf{1}_{\mathcal{F}} \rangle_G = \langle T_{(0,d)} \mathbf{1}^{\mathrm{vec}}_{(r,l)}, \mathbf{1}_{\mathcal{F}} \rangle_G$$

is independent of the particular choice of a vector bundle $\mathcal{F}$ of rank r and degree $d + l$ and that, furthermore, this value is itself independent of l. The proof of this is essentially the same as for the $d = 1$ case above (see § C.1). It suffices to notice that the number $\mathrm{Surj}(\mathcal{G}, \mathcal{T})$ of surjective maps from a vector bundle $\mathcal{F}$ of rank r to a torsion sheaf $\mathcal{T}$ is independent of the choice (and degree) of $\mathcal{F}$. This last statement is clear when $\mathcal{T}$ is stable and, in general form, can be proved by induction using the formula

$$\#\,\mathrm{Hom}(\mathcal{F}, \mathcal{T}) = \sum_{\mathcal{T}' \subseteq \mathcal{T}} \#\mathrm{Surj}(\mathcal{F}, \mathcal{T}'). \qquad \square$$

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From Lemma C.1 and the formula

$$\mathbf{E}_0(z) = \exp\bigg(\sum_r \frac{T_{(0,r)}}{[r]} z^r\bigg),$$

we deduce that there exists a series $\mathcal{S}_r(z_1/z_2) \in \mathbb{C}[[z_1/z_2]]$ such that

$$\mathbf{E}_0(z_1)\mathbf{E}_r(z_2) = \mathcal{S}_r\bigg(\frac{z_1}{z_2}\bigg)\mathbf{E}_r(z_2)\mathbf{E}_0(z_1). \tag{C.5}$$

Let us first determine $\mathcal{S}_1(z_1/z_2)$. Relation (C.5) for $r=1$ is equivalent to

$$\mathbf{E}_0(z_1)\mathbf{E}_1^{\mathrm{vec}}(z_2) = \mathcal{S}_1\bigg(\frac{z_1}{z_2}\bigg)\mathbf{E}_1^{\mathrm{vec}}(z_2)\mathbf{E}_0(z_1). \tag{C.6}$$

Thus, in order to compute $\mathcal{S}_1(z_1/z_2)$, it is enough to consider the restriction of $\mathbf{E}_0(z_1)\mathbf{E}_1(z_2)$ to line bundles of degree zero, say. If $\mathcal{L}$ is such a line bundle, then for any $d>0$ we have

$$\begin{aligned}\mathbf{1}_{(0,d)} \cdot \mathbf{1}_{(1,-d)}(\mathcal{L}) &= v^d \sum_{\mathcal{L}_{-d} \in \mathrm{Pic}^{-d}(X)} \frac{\#\,\mathrm{Hom}(\mathcal{L}_{-d},\mathcal{L}) - 1}{v^{-2}-1}\\ &= v^d \# X(\mathbb{F}_l)\frac{v^{-2d}-1}{v^{-2}-1},\end{aligned}$$

from which we get

$$\begin{aligned}\mathbf{E}_0(z_1)\mathbf{E}_1(z_2)(\mathcal{L}) &= 1 + \sum_{d>0} v^{-d}\bigg(\frac{z_1}{z_2}\bigg)^d \mathbf{1}_{(0,d)}\mathbf{1}_{(1,-d)}(\mathcal{L})\\ &= 1 + \frac{\#X(\mathbb{F}_l)}{v^{-2}-1}\sum_{d>0}\bigg(\frac{z_1}{z_2}\bigg)^d (v^{-2d}-1)\\ &= 1 + \frac{z_1}{z_2}\frac{\#X(\mathbb{F}_l)}{(1-(z_1/z_2))(1-v^{-2}(z_1/z_2))}\\ &= \zeta\bigg(\frac{z_1}{z_2}\bigg).\end{aligned}$$

This shows that $\mathcal{S}_1(z_1/z_2) = \zeta(z_1/z_2)$. Finally, to determine $\mathcal{S}_r(z_1/z_2)$, observe that by the coproduct formulas in Proposition 6.2,

$$\begin{aligned}\mathcal{S}_r\bigg(\frac{z_1}{z_2}\bigg)\Delta_{1,\ldots,1}(\mathbf{E}_r(z_2)\mathbf{E}_0(z_1)) &= \Delta_{1,\ldots,1}(\mathbf{E}_0(z_1)\mathbf{E}_r(z_2))\\ &= \mathbf{E}_0(z_1)\mathbf{E}_1(z_2) \otimes \mathbf{E}_0(z_1)\mathbf{E}_1(v^2 z_2) \otimes \cdots \otimes \mathbf{E}_0(z_1)\mathbf{E}_1(v^{2(r-1)}z_2)\\ &= \prod_{i=0}^{r-1}\zeta\bigg(v^{-2i}\frac{z_1}{z_2}\bigg)\mathbf{E}_1(z_2)\mathbf{E}_0(z_1) \otimes \cdots \otimes \mathbf{E}_1(v^{2(r-1)}z_2)\mathbf{E}_0(z_1)\\ &= \prod_{i=0}^{r-1}\zeta\bigg(v^{-2i}\frac{z_1}{z_2}\bigg)\Delta_{1,\ldots,1}(\mathbf{E}_r(z_2)\mathbf{E}_0(z_1)).\end{aligned}$$

It follows that

$$\mathcal{S}_r(z_1/z_2) = \prod_{i=0}^{r-1}\zeta(v^{-2i}z_1/z_2),$$

as desired. So Theorem 6.3 is proved.

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Appendix D. Proof of Lemma 7.4

Let us denote by $F(\alpha_1, \dots, \alpha_r)$ the right-hand side of (7.8). Each $\zeta(\alpha_i/\alpha_j)$ is a rational function of degree zero and leading coefficient one in both α_i and α_j. From this and the expression (7.8) we see that $F(\alpha_1, \dots, \alpha_r)$ is a rational function of degree one in each of the variables $\alpha_1, \dots, \alpha_r$ whose leading coefficient in any of these variables is also equal to one. Next, since $\zeta(z)$ has a simple pole at $z=1$ and at $z=v^2$, the function $F(\alpha_1, \dots, \alpha_r)$ has at most simple poles, and these are located along the hyperplanes $\alpha_i=\alpha_j$ and $\alpha_i=v^2\alpha_j$. We claim that the residues on each of these hyperplanes vanish, so that $F(\alpha_1, \dots, \alpha_r)$ is a polynomial in $\alpha_1, \dots, \alpha_r$.

In fact, the residues along hyperplanes $\alpha_i=\alpha_j$ vanish because $F(\alpha_1, \dots, \alpha_r)$ is symmetric in $\alpha_1, \dots, \alpha_r$; as for the hyperplanes $a_i=v^2\alpha_j$, we compute

$$\begin{aligned}
&\mathrm{Res}_{v^2\alpha_j-\alpha_i} F(\alpha_1, \dots, \alpha_r)\\
&\quad= \frac{1}{v^{-2}-1}\Bigg[\prod_{\substack{l\neq i\\ l\neq j}} \zeta\Big(\frac{\alpha_l}{\alpha_j}\Big) \cdot \mathrm{Res}_{v^2\alpha_j-\alpha_i}\zeta\Big(\frac{\alpha_i}{\alpha_j}\Big)\cdot\Big(\sum_{\substack{l\neq i\\ l\neq j}}\alpha_l + v^2\alpha_j\Big)\\
&\qquad\qquad - \prod_{\substack{l\neq i\\ l\neq j}} \zeta\Big(\frac{v^2\alpha_j}{\alpha_l}\Big) \cdot \mathrm{Res}_{v^2\alpha_j-\alpha_i}\zeta\Big(\frac{\alpha_i}{\alpha_j}\Big)\cdot\Big(\sum_{\substack{l\neq i\\ l\neq j}}\alpha_l + \alpha_j\Big)\Bigg]\\
&\qquad + \prod_{\substack{l\neq i\\ l\neq j}} \zeta\Big(\frac{v^2\alpha_j}{\alpha_l}\Big) \cdot \mathrm{Res}_{v^2\alpha_j-\alpha_i}\zeta\Big(\frac{\alpha_i}{\alpha_j}\Big) v^2\alpha_j.
\end{aligned}$$

Using the relation

$$\zeta\Big(\frac{v^2\alpha_j}{\alpha_l}\Big) = \zeta\Big(\frac{\alpha_l}{\alpha_j}\Big),$$

we can simplify this to

$$\begin{aligned}
\mathrm{Res}_{v^2\alpha_j-\alpha_i} F(\alpha_1, \dots, \alpha_r) &= \prod_{\substack{l\neq i\\ l\neq j}} \zeta\Big(\frac{\alpha_l}{\alpha_j}\Big) \cdot \mathrm{Res}_{v^2\alpha_j-\alpha_i}\zeta\Big(\frac{\alpha_i}{\alpha_j}\Big)\cdot\Big\{\frac{v^2\alpha_j-\alpha_j}{v^{-2}-1} - v^2\alpha_j\Big\}\\
&= 0,
\end{aligned}$$

as desired. Upon combining all the information we have on the function $F(\alpha_1, \dots, \alpha_r)$, we see that necessarily $F(\alpha_1, \dots, \alpha_r) = \alpha_1 + \dots + \alpha + r + u$ for some $u \in \mathcal{K}'$. It remains to observe that (for instance) we have $F(1, \dots, 1) = r$. So we are done.

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O. Schiffmann olive@math.jussieu.fr

Institut Mathématique de Jussieu, Université de Paris 6, 175 rue du Chevaleret, 75013 Paris, France

E. Vasserot vasserot@math.jussieu.fr

Institut Mathématique de Jussieu, Université de Paris 7, 175 rue du Chevaleret, 75013 Paris, France