Nagoya Math. J., **256** (2024), 761–784
DOI 10.1017/nmj.2023.38

A HOMOMORPHISM BETWEEN BOTT–SAMELSON BIMODULES

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Abstract. In the previous paper, we defined a new category which categorifies the Hecke algebra. This is a generalization of the theory of Soergel bimodules. To prove theorems, the existences of certain homomorphisms between Bott–Samelson bimodules are assumed. In this paper, we prove this assumption. We only assume the vanishing of certain two-colored quantum binomial coefficients.

§1. Introduction

In recent development of representation theory of algebraic reductive groups, the Hecke category plays central role. Here, the Hecke category means a categorification of the Hecke algebra of Coxeter groups. One can find the importance of the Hecke category in representation theory in Williamson's survey [13].

There are several incarnations of the Hecke category. They can be roughly divided into two types: geometric ones and combinatorial ones. The geometric Hecke category which appeared in representation theory first is the category of semisimple complexes on the flag variety. This category is the Hecke category with a field of characteristic zero. Juteau–Mauter–Williamson [10] introduced the notion of parity sheaves. The category of parity complexes on the flag variety is a geometric incarnation of the Hecke category with any field. When the characteristic of the ground field is zero, parity complexes are the same as semisimple complexes.

Soergel [12] introduced a category which is now called the category of Soergel bimodules. Under some conditions, he proved that Soergel's category is a Hecke category and it is equivalent to the category of semisimple complexes on the flag variety. This fact is used to prove the Koszul duality of the category $\mathcal{O}$ [3].

Soergel's definition starts with a certain representation of the Coxeter group and Soergel assumed that this representation is reflection faithful. However, because of this assumption, the theory of Soergel cannot apply to representation theory of algebraic reductive group. In such applications, we take an affine Weyl group as the Coxeter group and also take a natural representation coming from the root datum. However, it is not reflection faithful. A combinatorial incarnation which works in this situation was introduced by Elias–Williamson [7] and the category introduced is called the diagrammatic category. This category is defined by generators and relations. A priori, the definition seems very different from Soergel's category. An approach closer to Soergel's category was introduced in [1]. This incarnation is used by Bezrukavnikov–Riche [4] to prove a conjecture of Riche–Williamson [11] which implies the tilting character formula, and hence an irreducible character formula of algebraic representations of reductive groups when the characteristic is not too small.

Received January 26, 2023. Revised August 13, 2023. Accepted November 6, 2023.
2020 Mathematics subject classification: Primary 20F55.
Keywords: Soergel bimodule, Bott-Samelson bimodule, Coxeter group.

We remark that these categories are equivalent to each others when they behave well [1], [2], [11].

It is proved that these theories work well very general, including most cases over a field of positive characteristic. However, we still need some assumptions. The situation is subtle. In [1], we need one non-trivial assumption which we recall later. One problem is that this assumption is not easy to check. In [1], a sufficient condition for this assumption which we can check easier is given. However, the author thought that the assumption holds more generally. The aim of this paper is to prove this assumption under a mild condition. In particular, we prove that the assumption is always satisfied if the representation is obtained from a root system. (The situation is also subtle for the diagrammatic Hecke category. See [8, 5.1] and also Hazi's result [9]. We do not discuss about it in this paper.)

1.1 Soergel bimodules

We recall the category introduced in [1] and the assumption. Let (W,S) be a Coxeter system such that $\#S<\infty$ and $\mathbb{K}$ a commutative integral domain. We fix a realization [7, Def. 3.1] $(V,\{\alpha_s\}_{s\in S},\{\alpha_s^\vee\}_{s\in S})$ of (W,S) over $\mathbb{K}$ (here, V corresponds to $\mathfrak{h}^*$ in [7]). Namely, V is a free $\mathbb{K}$-module of finite rank with an action of W, $\alpha_s\in V$, $\alpha_s^\vee\in\operatorname{Hom}_{\mathbb{K}}(V,\mathbb{K})$ such that:

(1) $s(v)=v-\langle\alpha_s^\vee,v\rangle\alpha_s$ for any $s\in S$ and $v\in V$.
(2) $\langle\alpha_s^\vee,\alpha_s\rangle=2$.
(3) Let $s,t\in S$ $(s\neq t)$ and $m_{s,t}$ the order of st. If $m_{s,t}<\infty$, then the two-colored quantum numbers $[m_{s,t}]_X,[m_{s,t}]_Y$ attached to $\{s,t\}$ are both zero. (See §3.1 for the definition of these numbers.)

We also assume the Demazure surjectivity, namely we assume that $\alpha_s\colon\operatorname{Hom}_{\mathbb{K}}(V,\mathbb{K})\to\mathbb{K}$ and $\alpha_s^\vee\colon V\to\mathbb{K}$ are both surjective for any $s\in S$.

We define the category $\mathcal{C}$ as follows: Let $R=S(V)$ be the symmetric algebra and $Q=R[w(\alpha_s)^{-1}\mid w\in W,s\in S]$. An object of $\mathcal{C}$ is $(M,(M_Q^x)_{x\in W})$ such that M is a graded R-bimodule, M_Q^x is a Q-bimodule such that $mp=x(p)m$ for any $m\in M_Q^x$, $p\in Q$ and $M\otimes_R Q=\bigoplus_{x\in W}M_Q^x$. We also assume that M is flat as a right R-module. A morphism $\varphi\colon(M,(M_Q^x)_{x\in W})\to(N,(N_Q^x)_{x\in W})$ is an R-bimodule homomorphism $\varphi\colon M\to N$ of degree zero such that $(\varphi\otimes\mathrm{id}_Q)(M_Q^x)\subset N_Q^x$. We often write M for $(M,(M_Q^x))$. For $M,N\in\mathcal{C}$, we define $M\otimes N\in\mathcal{C}$ as follows. As an R-bimodule, we have $M\otimes N=M\otimes_R N$ and $(M\otimes N)_Q^x=\bigoplus_{yz=x}M_Q^y\otimes_Q N_Q^z$.

For each $s\in S$, we have an object denoted by B_s. As a graded R-bimodule, $B_s=R\otimes_{R^s}R(1)$, where (1) is the grading shift and $R^s=\{f\in R\mid s(f)=f\}$. Then B_s has a unique lift in $\mathcal{C}$ such that $(B_s)_Q^x=0$ unless $x=e,s$. An object of a form

$$B_{s_1}\otimes B_{s_2}\otimes\cdots\otimes B_{s_l}(n)$$

for $s_1,\dots,s_l\in S$ and $n\in\mathbb{Z}$ is called a *Bott–Samelson bimodule*. Let $\mathcal{BS}$ denote the category of Bott–Samelson bimodules.

In [1], we proved that $\mathcal{BS}$ gives a categorification of the Hecke algebra assuming the following. We refer it as [1, Assumption 3.2].

Let $s,t \in S$, $s \neq t$ such that $m_{s,t}$ is finite. Then, there exists a morphism

$$\overbrace{B_s \otimes B_t \otimes \cdots}^{m_{s,t}} \to \overbrace{B_t \otimes B_s \otimes \cdots}^{m_{s,t}}$$

which sends $(1\otimes 1)\otimes(1\otimes 1)\otimes\cdots\otimes(1\otimes 1)$ to $(1\otimes 1)\otimes(1\otimes 1)\otimes\cdots\otimes(1\otimes 1)$.

We introduce the following assumption.

ASSUMPTION 1.1. For any $s,t \in S$ such that $m_{s,t} < \infty$, the two-colored quantum binomial coefficients $\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_X$ and $\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_Y$ are both zero for any $k = 1,\ldots,m_{s,t}-1$.

For the definition of two-colored quantum binomial coefficients, see §§2.1 and 3.3. This assumption is related to the existence of Jones–Wenzl projectors (see Proposition 3.4 and [8, Conj. 6.27], which is now a theorem [9]). After this paper was written, Hazi proved that this condition is equivalent to the existence of rotatable Jones–Wenzl operators [9]. The main theorem of this paper is the following.

THEOREM 1.2 (Theorem 3.10). *Under Assumption 1.1, [1, Assumption 3.2] holds.*

Note that to check Assumption 1.1 is easy and it is a very mild condition. For example, one can easily check that if a realization comes from a root system then Assumption 1.1 is always satisfied (Proposition 3.7).

1.2 Diagrammatic category

Let $\mathcal{D}$ be the diagrammatic Hecke category defined in [7]. We assume that the category $\mathcal{D}$ is "well-defined" [8, 5.1]. (After this paper was written, Hazi [9] proved that if Assumption 1.1 is satisfied then $\mathcal{D}$ is well-defined.) In [7], under some assumptions [8, 5.3], a functor $\mathcal{F}$ from $\mathcal{D}$ to $\mathcal{BS}$ is constructed. The construction of $\mathcal{F}$ is deeply related to [1, Assumption 3.2] as we explain here.

The morphisms in the category $\mathcal{D}$ are defined by generators and relations. So to define $\mathcal{F}$, we have to define the images of generators. Except the generators called $2m_{s,t}$-valent vertices ($s,t \in S$), the images of generators are given easily. For $2m_{s,t}$-valent vertices, the images should be morphisms in [1, Assumption 3.2]. Hence, to prove [1, Assumption 3.2] is almost equivalent to the construction of $\mathcal{F}$. Therefore as a consequence of our main theorem, we can prove the following.

THEOREM 1.3. *Assume that $\mathbb{K}$ is Noetherian. Under Assumption 1.1, the category $\mathcal{D}$ is equivalent to $\mathcal{BS}$.*

In particular, there is a natural functor from $\mathcal{D}$ to the category of graded R-bimodules.

1.3 Localized calculus

In the proof, we use localized calculus. Ideas of localized calculus are found in [1], [7] and more systematic treatment recently appeared in [8].

Let $\mathcal{C}_Q$ be the category of $(P^x)_{x\in W}$, where P^x is a graded Q-bimodule such that $mp = x(p)m$ for $p \in Q$ and $m \in P^x$. A morphism $(P_1^x)_{x\in W} \to (P_2^x)_{x\in W}$ is $(\varphi_x)_{x\in W}$, where $\varphi_x \colon P_1^x \to P_2^x$ is a Q-bimodule homomorphism of degree zero for any $x \in W$. Then for $M \in \mathcal{C}$, $(M_Q^x)_{x\in W} \in \mathcal{C}_Q$. We denote this object by M_Q. For $M,N \in \mathcal{C}$ and a morphism $\varphi \colon M \to N$, we have a morphism $\varphi_Q \colon M_Q \to N_Q$. Conversely, assume that $\varphi_Q \colon M_Q \to N_Q$ is given and if φ_Q sends $M \subset M \otimes_R Q = \bigoplus_{x\in W} M_Q^x$ to N, then the restriction of φ_Q to M gives a morphism $M \to N$ in $\mathcal{C}$.

Let M, N be two Bott–Samelson bimodules in [1, Assumption 3.2]. A candidate of $\varphi_Q\colon M_Q \to N_Q$ is given in [8]. Hence, we have to prove that φ_Q sends M to N. This is the aim of this paper.

We check that φ_Q gives a desired homomorphism by calculations. One of the things which we need to prove is the following. Let $s,t \in S$ such that $m_{s,t} < \infty$. For simplicity, assume that V is balanced, namely $[m_{s,t}-1]_X = [m_{s,t}-1]_Y = 1$. Let $s_1 \dots s_{m_{s,t}}$ be a reduced expression of the longest element in the group $\langle s,t\rangle$ generated by $\{s,t\}$. Then, for any $g \in \langle s,t\rangle$, we have

$$\sum_{e=(e_i)\in\{0,1\}^{m_{s,t}},\, s_1^{e_1}\dots s_{m_{s,t}}^{e_{m_{s,t}}}=g} \prod_{i=1}^{m_{s,t}} s_1^{e_1}\dots s_{i-1}^{e_{i-1}}\left(\frac{1}{\alpha_{s_i}}\right) = \frac{1}{\prod_{i=1}^{m_{s,t}} s_1\dots s_{i-1}(\alpha_{s_i})}. \tag{1.1}$$

(If V comes from a root system, then this formula can be proved by applying the localization formula to the Bott–Samelson resolution of the flag variety. The author learned this from Syu Kato.)

In §2, we calculate the left-hand side of (1.1). Moreover, we give an explicit formula of the left-hand side for any sequence $(s_1, s_2, \dots)$ of $\{s,t\}$.

For a general element $m \in M$, we first give a formula to express $\varphi_Q(m)$ using the left-hand side of (1.1) (with any $s_1, s_2, \dots$). We also have an algorithm to check $\varphi_Q(m) \in N$ (Lemma 3.9). In §3, using this algorithm and an explicit formula obtained in §2, we prove the main theorem. We also give a remark for an error in the previous paper [1] pointed out by Simon Riche, see Remark 3.8.

1.4 On Assumption 1.1

In [1], a sufficient condition for [1, Assumption 3.2] was given. In [7], a sufficient condition for the existence of $\mathcal{F}$ was given. Both conditions are stronger than Assumption 1.1. It was expected that these theorems are proved under the weaker condition but concrete conditions were not known.

In this paper, we prove these theorems under Assumption 1.1. Moreover, we prove that the theorems are almost equivalent to Assumption 1.1. More precisely, we prove the following. Let $\varphi_Q\colon M_Q \to N_Q$ be the morphism in $\mathcal{C}_Q$ introduced in [8] and $\psi_Q\colon N_Q \to M_Q$ the morphism obtaining by the same way as φ_Q. Then φ_Q and ψ_Q give desired morphisms if and only if Assumption 1.1 holds (Proposition 3.11). Therefore, the author thinks that Assumption 1.1 is the final form in this direction

§2. A calculation in the universal Coxeter system of rank two

Since our main theorem is concerned with a rank two Coxeter system, in almost all part of this paper, we only consider a Coxeter system of rank two. In this section, we give an explicit formula of the left-hand side of (1.1). Such formula can be proved in a universal form. Hence, we work with the universal Coxeter system of rank two in this section.

2.1 Two-colored quantum numbers

In this subsection, we introduce two-colored quantum numbers [6], [7]. Let $\mathbb{Z}[X,Y]$ be the polynomial ring with two variables over $\mathbb{Z}$.

DEFINITION 2.1 (Two-colored quantum numbers, [7, Def. 3.6]). For $n \in \mathbb{Z}_{\geq 0}$, we define $[n]_X, [n]_Y \in \mathbb{Z}[X,Y]$ by

$$[0]_X = [0]_Y = 0, \quad [1]_X = [1]_Y = 1,$$
$$[n+1]_X = X[n]_Y - [n-1]_X,$$
$$[n+1]_Y = Y[n]_Y - [n-1]_Y.$$

Note that $[2]_X = X$ and $[2]_Y = Y$. Define $\sigma\colon \{X,Y\} \to \{X,Y\}$ by $\sigma(X) = Y$ and $\sigma(Y) = X$. Then for $Z \in \{X,Y\}$, we have

$$[n+1]_Z = [2]_Z[n]_{\sigma(Z)} - [n-1]_Z.$$

We prove some properties of these polynomials which we will use later. Some of them are known well or immediately follow from known results. We give proofs for the sake of completeness.

LEMMA 2.2. *Let $n \in \mathbb{Z}_{\geq 0}$.*

(1) *If n is odd, then $[n]_X = [n]_Y$.*
(2) *If n is even, $[n]_X/X, [n]_Y/Y \in \mathbb{Z}[X,Y]$ and $[n]_X/X = [n]_Y/Y$.*
(3) *We have $[n]_Z = [n]_{\sigma^n(Z)}$ for $Z \in \{X,Y\}$.*
(4) *We have $[n]_X, [n]_Y \neq 0$ if $n > 0$.*

Proof. The first two statements follow from the definition using induction. For the third, if n is odd then it follows from (1). If n is even then it is obvious. We also have $[n]_X(2,2) = [n]_Y(2,2) = n$ which follows easily by induction. Hence $[n]_X, [n]_Y \neq 0$. □

An obvious consequence of (1) (2) which will be used several times in this paper is the following. For $k_1, \ldots, k_r, l_1, \ldots, l_s \in \mathbb{Z}_{>0}$ such that $\#(2\mathbb{Z} \cap \{k_1, \ldots, k_r\}) = \#(2\mathbb{Z} \cap \{l_1, \ldots, l_s\})$, then $([k_1]_X \ldots [k_r]_X)/([l_1]_X \ldots [l_s]_X) = ([k_1]_Y \ldots [k_r]_Y)/([l_1]_Y \ldots [l_s]_Y)$.

LEMMA 2.3. *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $Z \in \{X,Y\}$. Then we have*

$$[m+n+1]_{\sigma^n(Z)} = [m+1]_Z[n+1]_{\sigma(Z)} - [m]_{\sigma(Z)}[n]_Z.$$

Proof. We prove by induction on n. The cases of $n = 0$ and $n = 1$ follow from the definitions. Assume that the lemma holds for $n-1, n-2$. Then

$$\begin{aligned}
[m+n+1]_{\sigma^n(Z)} &= [2]_{\sigma^n(Z)}[m+n]_{\sigma^{n+1}(Z)} - [m+n-1]_{\sigma^n(Z)} \\
&= [2]_{\sigma^n(Z)}([m+1]_Z[n]_{\sigma(Z)} - [m]_{\sigma(Z)}[n-1]_Z) \\
&\quad - ([m+1]_Z[n-1]_{\sigma(Z)} - [m]_{\sigma(Z)}[n-2]_Z) \\
&= [m+1]_Z([2]_{\sigma^n(Z)}[n]_{\sigma(Z)} - [n-1]_{\sigma(Z)}) \\
&\quad - [m]_{\sigma(Z)}([2]_{\sigma^n(Z)}[n-1]_Z - [n-2]_Z).
\end{aligned}$$

By Lemma 2.2(3), we have $[n]_{\sigma(Z)} = [n]_{\sigma^{n+1}(Z)}$ and $[n-1]_{\sigma(Z)} = [n-1]_{\sigma^n(Z)}$. Hence $[2]_{\sigma^n(Z)}[n]_{\sigma(Z)} - [n-1]_{\sigma(Z)} = [2]_{\sigma^n(Z)}[n]_{\sigma^{n+1}(Z)} - [n-1]_{\sigma^n(Z)} = [n+1]_{\sigma^n(Z)}$and we have $[n+1]_{\sigma^n(Z)} = [n+1]_{\sigma(Z)}$ by Lemma 2.2(3). Similarly, we have $[2]_{\sigma^n(Z)}[n-1]_Z - [n-2]_Z = [2]_{\sigma^n(Z)}[n-1]_{\sigma^{n+1}(Z)} - [n-2]_{\sigma^n(Z)} = [n]_{\sigma^n(Z)} = [n]_Z$. □

LEMMA 2.4. *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $Z \in \{X,Y\}$. Then we have*

$$[m]_{\sigma^n(Z)} = [m+n]_Z[n+1]_{\sigma(Z)} - [m+n+1]_{\sigma(Z)}[n]_Z.$$

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Proof. If $n=0$, then the lemma is obvious. We assume $n>0$. Apply the previous lemma as $(n,m,Z)=(m+n-1,n,\sigma(Z))$, we have $[m+n]_Z[n+1]_{\sigma(Z)}=[m+2n]_{\sigma^{m+n}(Z)}+[m+n-1]_{\sigma(Z)}[n]_Z$. By Lemma 2.2(3), we have $[m+2n]_{\sigma^{m+n}(Z)}=[m+2n]_{\sigma^n(Z)}$. Hence $[m+n]_Z[n+1]_{\sigma(Z)}-[m+n+1]_{\sigma(Z)}[n]_Z=[m+n-1]_{\sigma(Z)}[n]_Z-[m+n]_Z[n-1]_{\sigma(Z)}$. Therefore, by induction on n, $[m+n]_Z[n+1]_{\sigma(Z)}-[m+n+1]_{\sigma(Z)}[n]_Z=[m]_{\sigma^n(Z)}[1]_{\sigma^{n+1}(Z)}-[m+1]_{\sigma^{n+1}(Z)}[0]_{\sigma^n(Z)}=[m]_{\sigma^n(Z)}$. ∎

LEMMA 2.5. *For $m,n\in\mathbb{Z}_{\geq 0}$ and $Z\in\{X,Y\}$, we have*

$$\begin{aligned}[m+n+1]_{\sigma^n(Z)}[m+n]_Z-[m+1]_Z[m]_{\sigma^n(Z)}&=[n]_Z[2m+n+1]_{\sigma^{m+1}(Z)}\\&=[n]_{\sigma^n(Z)}[2m+n+1]_{\sigma^{n+m}(Z)},\\ [m+n+1]_{\sigma^{n+1}(Z)}[m+n+1]_Z-[m]_Z[m]_{\sigma^{n+1}(Z)}&=[n+1]_Z[2m+n+1]_{\sigma^m(Z)}.\end{aligned}$$

Proof. Applying Lemma 2.3 to $[m+n+1]_{\sigma^n(Z)}$ (resp., Lemma 2.4 to $[m]_{\sigma^n(Z)}$), we have

$$\begin{aligned}&[m+n+1]_{\sigma^n(Z)}[m+n]_Z-[m+1]_Z[m]_{\sigma^n(Z)}\\&=([m+1]_Z[n+1]_{\sigma(Z)}-[m]_{\sigma(Z)}[n]_Z)[m+n]_Z\\&\quad-[m+1]_Z([m+n]_Z[n+1]_{\sigma(Z)}-[m+n+1]_{\sigma(Z)}[n]_Z)\\&=-[m]_{\sigma(Z)}[n]_Z[m+n]_Z+[m+1]_Z[m+n+1]_{\sigma(Z)}[n]_Z\\&=[n]_Z([m+n+1]_{\sigma(Z)}[m+1]_Z-[m+n]_Z[m]_{\sigma(Z)}).\end{aligned}$$

The first formula of the lemma follows from Lemma 2.3. The second follows from the first and Lemma 2.2(3). The third formula follows from a similar calculation. ∎

For $m,n\in\mathbb{Z}_{\geq 0}$ such that $n\leq m$ and $Z\in\{X,Y\}$, define the two-colored quantum binomial coefficient $\begin{bmatrix}m\\n\end{bmatrix}_Z$ [8, Def. 6.1] by

$$\begin{bmatrix}m\\n\end{bmatrix}_Z=\frac{[m]_Z[m-1]_Z\dots[m-n+1]_Z}{[n]_Z[n-1]_Z\dots[1]_Z}.$$

By Lemma 2.6(2) and induction, we have $\begin{bmatrix}m\\n\end{bmatrix}_Z\in\mathbb{Z}[X,Y]$.

LEMMA 2.6. *Let $m,n\in\mathbb{Z}$ such that $1\leq n\leq m$ and $Z\in\{X,Y\}$.*

(1)

$$\begin{bmatrix}m\\n\end{bmatrix}_Z=\begin{bmatrix}m+1\\n\end{bmatrix}_{\sigma^n(Z)}[n+1]_Z-\begin{bmatrix}m\\n-1\end{bmatrix}_Z[m+2]_{\sigma^{n+1}(Z)}.$$

(2)

$$\begin{bmatrix}m+1\\n\end{bmatrix}_Z=\begin{bmatrix}m\\n\end{bmatrix}_{\sigma^n(Z)}[n+1]_Z-\begin{bmatrix}m\\n-1\end{bmatrix}_Z[m-n]_{\sigma^{n+1}(Z)}.$$

Proof. (1) By Lemma 2.4, we have $[m-n+1]_Z=[m+1]_{\sigma^n(Z)}[n+1]_{\sigma^{n+1}(Z)}-[m+2]_{\sigma^{n+1}(Z)}[n]_{\sigma^n(Z)}$. By Lemma 2.2(3), we have $[n]_{\sigma^n(Z)}=[n]_Z$ and $[n+1]_{\sigma^{n+1}(Z)}=[n+1]_Z$. Therefore

$$\begin{aligned}\begin{bmatrix}m\\n\end{bmatrix}_Z&=\frac{[m]_Z\dots[m-n+2]_Z}{[n]_Z\cdots[1]_Z}([m+1]_{\sigma^n(Z)}[n+1]_Z-[m+2]_{\sigma^{n+1}(Z)}[n]_Z)\\&=\frac{[m+1]_{\sigma^n(Z)}[m]_Z\dots[m-n+2]_Z}{[n]_Z\dots[1]_Z}[n+1]_Z-\begin{bmatrix}m\\n-1\end{bmatrix}_Z[m+2]_{\sigma^{n+1}(Z)}.\end{aligned}$$

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Therefore, it is sufficient to prove

$$\frac{[m+1]_{\sigma^n(Z)}[m]_Z\ldots[m-n+2]_Z}{[n]_Z\ldots[1]_Z}=\frac{[m+1]_{\sigma^n(Z)}[m]_{\sigma^n(Z)}\ldots[m-n+2]_{\sigma^n(Z)}}{[n]_{\sigma^n(Z)}\ldots[1]_{\sigma^n(Z)}}.$$

If n is even, then we have nothing to prove. If n is odd, then $\#(2\mathbb{Z}\cap\{m,\ldots,m-n+2\})=\#(2\mathbb{Z}\cap\{n,\ldots,1\})$. Hence, it follows from Lemma 2.2.

(2) By Lemma 2.3, we have $[m+1]_Z=[m-n+1]_{\sigma^n(Z)}[n+1]_{\sigma^{n+1}(Z)}-[m-n]_{\sigma^{n+1}(Z)}[n]_{\sigma^n(Z)}$. By Lemma 2.2(3), we have $[n+1]_{\sigma^{n+1}(Z)}=[n+1]_Z$ and $[n]_{\sigma^n(Z)}=[n]_Z$. Hence

$$\begin{aligned}\begin{bmatrix}m+1\\n\end{bmatrix}_Z&=\frac{[m]_Z\ldots[m-n+2]_Z}{[n]_Z[n-1]_Z\ldots[1]_Z}[m+1]_Z\\&=\frac{[m]_Z\ldots[m-n+2]_Z[m-n+1]_{\sigma^n(Z)}}{[n]_Z\ldots[1]_Z}[n+1]_Z-\begin{bmatrix}m\\n-1\end{bmatrix}_Z[m-n]_{\sigma^{n+1}(Z)}.\end{aligned}$$

It is sufficient to prove

$$\frac{[m]_Z\ldots[m-n+2]_Z[m-n+1]_{\sigma^n(Z)}}{[n]_Z\ldots[1]_Z}=\begin{bmatrix}m\\n\end{bmatrix}_{\sigma^n(Z)}.$$

If n is even, we have nothing to prove. If n is odd, then we have $\#(2\mathbb{Z}\cap\{m,m-1,\ldots,m-n+2\})=\#(2\mathbb{Z}\cap\{n,\ldots,1\})$. Hence, we get (2) by Lemma 2.2. ∎

LEMMA 2.7. *We have*

$$\frac{[2m+n+1]_{\sigma^{m+1}(Z)}}{[m]_{\sigma^n(Z)}}\begin{bmatrix}2m+n\\m-1\end{bmatrix}_{\sigma^n(Z)}=\begin{bmatrix}2m+n+1\\m\end{bmatrix}_{\sigma^{n+1}(Z)}.$$

Proof. Replacing Z with $\sigma^n(Z)$, the lemma is equivalent to

$$\frac{[2m+n+1]_{\sigma^{m+n+1}(Z)}[2m+n]_Z\ldots[m+n+2]_Z}{[m]_Z[m-1]_Z\ldots[1]_Z}=\frac{[2m+n+1]_{\sigma(Z)}\ldots[m+n+2]_{\sigma(Z)}}{[m]_{\sigma(Z)}[m-1]_{\sigma(Z)}\ldots[1]_{\sigma(Z)}}.$$

If $m+n+1$ is even, then we have $\#(2\mathbb{Z}\cap\{2m+n+1,\ldots,m+n+2\})=\#(2\mathbb{Z}\cap\{m,\ldots,1\})$. Hence, the lemma follows from Lemma 2.2. If $m+n+1$ is odd, then $\sigma^{m+n+1}(Z)=\sigma(Z)$ and $\#(2\mathbb{Z}\cap\{2m+n,\ldots,m+n+2\})=\#(2\mathbb{Z}\cap\{m,\ldots,1\})$. Hence again, the lemma follows from Lemma 2.2. ∎

2.2 A formula

Let (W,S) be the universal Coxeter system of rank two, namely the group W is generated by the set of two elements $S=\{s,t\}$ and defined by relations $s^2=t^2=1$. The length function is denoted by ℓ and the Bruhat order is denoted by $\leq$. Let $V=\mathbb{Z}[X,Y]\alpha_s\oplus\mathbb{Z}[X,Y]\alpha_t$ be the free $\mathbb{Z}[X,Y]$-module of rank two with a basis $\{\alpha_s,\alpha_t\}$. We define an action of W on V by

$$s(\alpha_s)=-\alpha_s,\quad s(\alpha_t)=\alpha_t+X\alpha_s,\quad t(\alpha_s)=\alpha_s+Y\alpha_t,\quad t(\alpha_t)=-\alpha_t.$$

Let $\Phi=\{w(\alpha_s),w(\alpha_t)\mid w\in W\}$ be the set of roots and the set of positive roots Φ^+ is defined by $\Phi^+=\{w(\alpha_s)\mid ws>w\}\cup\{w(\alpha_t)\mid wt>w\}$. For each $\alpha\in\Phi$, we have the reflection $s_\alpha\in W$. By Lemma 2.8, the stabilizer of $\alpha\in\{\alpha_s,\alpha_t\}$ is trivial. Therefore, for each $\beta\in\Phi$, a pair

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$(w,\alpha) \in W \times \{\alpha_s, \alpha_t\}$ is unique. Hence, we can define $s_{\alpha_s} = s$, $s_{\alpha_t} = t$, $s_{w(\alpha)} = ws_\alpha w^{-1}$ for $\alpha \in \{\alpha_s, \alpha_t\}$ and $w \in W$.

The following formula can be proved by induction.

Lemma 2.8. *We have*

$$(st)^k\alpha_s = [2k+1]_X\alpha_s + [2k]_Y\alpha_t, \qquad (st)^k\alpha_t = -[2k]_X\alpha_s - [2k-1]_Y\alpha_t,$$
$$(ts)^k\alpha_t = [2k]_X\alpha_s + [2k+1]_Y\alpha_t, \qquad (ts)^k\alpha_s = -[2k-1]_X\alpha_s - [2k]_Y\alpha_t.$$

Lemma 2.9. *Let $\gamma \in \Phi^+$ and $g = s_\gamma$.*

(1) *If $sg > g$, then*

$$\gamma = \left[\frac{\ell(g)-1}{2}\right]_X \alpha_s + \left[\frac{\ell(g)+1}{2}\right]_Y \alpha_t.$$

(2) *If $tg > g$, then*

$$\gamma = \left[\frac{\ell(g)+1}{2}\right]_X \alpha_s + \left[\frac{\ell(g)-1}{2}\right]_Y \alpha_t.$$

Proof. We have $\gamma = (ts)^k(\alpha_t)$ or $t(st)^k(\alpha_s)$ or $(st)^k(\alpha_s)$ or $s(ts)^k(\alpha_t)$. If $\gamma = (ts)^k(\alpha_t)$, then $sg > g$ and $\ell(g) = 4k+1$. The lemma follows from the previous lemma. If $\gamma = t(st)^k(\alpha_s)$, then $sg > g$ and $\ell(g) = 4k+3$. We have

$$\begin{aligned}\gamma &= t([2k+1]_X\alpha_s + [2k]_Y\alpha_t) = [2k+1]_X(\alpha_s + Y\alpha_t) - [2k]_Y\alpha_t \\ &= [2k+1]_X\alpha_s + ([2k+1]_XY - [2k]_Y)\alpha_t = [2k+1]_X\alpha_s + [2k+2]_Y\alpha_t\end{aligned}$$

and the lemma follows. The proof of the other cases are similar. ∎

We define some elements which will be needed for our main formula. We use the following notation for sequences in S. A sequence in S will be written with the underline like $\underline{w} = (s_1,\dots,s_l)$. We write $w = s_1\dots s_l$. For $u \in S$, put $(\underline{w},u) = (s_1,\dots,s_l,u)$. For $e = (e_1,\dots,e_l) \in \{0,1\}^l$, we put $\underline{w}^e = s_1^{e_1}\dots s_l^{e_l}$. We set $\ell(\underline{w}) = l$.

For $g, w \in W$, we put

$$X_g^w = \{\alpha \in \Phi^+ \mid s_\alpha g \le w\}.$$

Let $\underline{w} = (s_1,\dots,s_l) \in S^l$ be a sequence of elements in S and $g \in W$. For a real number r, let $\lfloor r \rfloor$ be the integral part of r. We define $k_g^{\underline{w}} \in \mathbb{Z}[X,Y]$ as follows. If $s_i = s_{i+1}$ for some i or $g \not\le w$, then $k_g^{\underline{w}} = 0$. If $\ell(\underline{w}) = 0$, then $k_1^{\underline{w}} = 1$ and $k_g^{\underline{w}} = 0$ if $g \ne 1$. Otherwise, we put

$$k_g^{\underline{w}} = \begin{cases} \left[\begin{matrix}\ell(\underline{w})-1 \\ \lfloor \frac{\ell(\underline{w})-\ell(g)-1}{2} \rfloor\end{matrix}\right]_{\sigma^{\ell(\underline{w})-1}(Z)} & (s_1g > g), \\ \left[\begin{matrix}\ell(\underline{w})-1 \\ \lfloor \frac{\ell(\underline{w})-\ell(g)}{2} \rfloor\end{matrix}\right]_{\sigma^{\ell(\underline{w})-1}(Z)} & (s_1g < g), \end{cases}$$

where $Z = X$ if $s_1 = s$ and $Z = Y$ if $s_1 = t$.

Let R be the symmetric algebra of V and $R^\emptyset = \Phi^{-1}R$ the ring of fractions. We define an element $a^{\underline{w}}(g)$ of $R^\emptyset$ by

$$a^{\underline{w}}(g) = \sum_{\underline{w}^e = g} \frac{1}{\alpha_{s_1}} s_1^{e_1}\left(\frac{1}{\alpha_{s_2}} s_2^{e_2}\left(\dots \frac{1}{\alpha_{s_{l-1}}} s_{l-1}^{e_{l-1}}\left(\frac{1}{\alpha_{s_l}}\right)\dots\right)\right) = \sum_{\underline{w}^e = g}\prod_{i=1}^{l}(s_1^{e_1}\dots s_{i-1}^{e_{i-1}})\left(\frac{1}{\alpha_{s_i}}\right).$$

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LEMMA 2.10. *If $s_i = s_{i-1}$ for some i, namely if $\underline{w}$ is not a reduced expression, then $a^{\underline{w}}(g) = 0$.*

Proof. Set $A = \{e \in \{0,1\}^l \mid \underline{w}^e = g\}$. Define $f\colon A \to A$ by $f(e) = (e'_1,\dots,e'_l)$, where $e'_i = 1-e_i$, $e'_{i-1} = 1-e_{i-1}$ and $e'_j = e_j$ for $j \neq i, i-1$. Set $b_{e,j} = (s_1^{e_1}\cdots s_{j-1}^{e_{j-1}})\left(\frac{1}{\alpha_j}\right)$. If $j<i$, then obviously we have $b_{e,j} = b_{f(e),j}$. If $j>i$, then since $s_{i-1}^{1-e_{i-1}}s_i^{1-e_i} = s_{i-1}^{e_{i-1}}s_i^{e_i}$, we have $b_{e,j} = b_{f(e),j}$. If $j=i$, then

$$b_{f(e),i} = (s_1^{e_1}\cdots s_{i-1}^{e_{i-1}})s_{i-1}\left(\frac{1}{\alpha_{s_i}}\right) = -b_{f(e),i}$$

since $s_{i-1}^{1-e_{i-1}} = s_{i-1}^{e_{i-1}}s_{i-1}$ and $s_{i-1} = s_i$. Therefore, $b_e = \prod_{i=1}^l b_{e,i}$ satisfies $b_{f(e)} = -b_e$. Let B be a set of complete representatives of $A/\langle f\rangle$. Then $a^{\underline{w}}(g) = \sum_{e\in A} b_e = \sum_{e\in B}(b_e + b_{f(e)}) = 0$. ∎

The aim of this section is to prove the following theorem.

THEOREM 2.11. *For $\underline{w} \in S^l$, we have*

$$a^{\underline{w}}(g) = \frac{k_g^{\underline{w}}}{\prod_{\alpha\in X_g^w}\alpha}.$$

From the above lemma, we may assume $s_{i-1} \neq s_i$ for any i. By definitions, we also may assume $g \leq w$, otherwise both sides are zero.

2.3 Proof of Theorem 2.11

In this subsection, we prove Theorem 2.11.

We split the sum in the definition of $a^{\underline{w}}(g)$ to $e_l = 0$ part and $e_l = 1$ part. If $e_l = 0$, then $s_1^{e_1}\dots s_{l-1}^{e_{l-1}} = g$. Hence $(s_1^{e_1}\dots s_{l-1}^{e_{l-1}})\left(\frac{1}{\alpha_{s_l}}\right) = g\left(\frac{1}{\alpha_{s_l}}\right)$. Therefore

$$\prod_{i=1}^{l}(s_1^{e_1}\dots s_{i-1}^{e_{i-1}})\left(\frac{1}{\alpha_{s_i}}\right) = g\left(\frac{1}{\alpha_{s_l}}\right)\prod_{i=1}^{l-1}(s_1^{e_1}\dots s_{i-1}^{e_{i-1}})\left(\frac{1}{\alpha_{s_i}}\right).$$

Similarly, if $e_l = 1$, then $(s_1^{e_1}\dots s_{l-1}^{e_{l-1}})\left(\frac{1}{\alpha_{s_l}}\right) = gs_l\left(\frac{1}{\alpha_{s_l}}\right) = -g\left(\frac{1}{\alpha_{s_l}}\right)$. Therefore, we have

$$\begin{aligned}a^{\underline{w}}(g) &= \frac{1}{g(\alpha_{s_l})}\left(\sum_{s_1^{e_1}\dots s_{l-1}^{e_{l-1}}=g}\prod_{i=1}^{l-1}(s_1^{e_1}\dots s_{i-1}^{e_{i-1}})\left(\frac{1}{\alpha_{s_i}}\right) - \sum_{s_1^{e_1}\dots s_{l-1}^{e_{l-1}}=gs_l}\prod_{i=1}^{l-1}(s_1^{e_1}\dots s_{i-1}^{e_{i-1}})\left(\frac{1}{\alpha_{s_i}}\right)\right)\\ &= \frac{1}{g(\alpha_{s_l})}(a^{(s_1,\dots,s_{l-1})}(g) - a^{(s_1,\dots,s_{l-1})}(gs_l)).\end{aligned}$$

We change the notation slightly and we get the following lemma.

LEMMA 2.12. *Let $\underline{w} \in S^l$ and $u \in S$. Then we have*

$$a^{(\underline{w},u)}(g) = \frac{1}{g(\alpha_u)}(a^{\underline{w}}(g) - a^{\underline{w}}(gu)).$$

To prove the theorem, we need the following lemmas. In the rest of the proof, we sometimes use the following Deodhar's "Property Z" [5]. Let $v,w \in W$, $s \in S$ and assume that $ws < w$. Then we have $v \leq w$ if and only if $vs \leq w$, and, $ws \leq v$ if and only if $ws \leq vs$.

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LEMMA 2.13. *Let $w, g \in W$ and $u \in S$ such that $wu > w$, $sw < w$, $g \le w$ and $gu \le w$.*

(1) *There exists a unique element $\beta \in X_g^w$ such that $s_\beta \in \{wg^{-1}, swg^{-1}\}$.*
(2) *There exists a unique element $\gamma \in X_{gu}^w$ such that $s_\gamma \in \{wug^{-1}, swug^{-1}\}$.*
(3) *We have $X_g^w \setminus \{\beta\} = X_{gu}^w \setminus \{\gamma\}$ and $X_{gu}^{wu} = X_g^w \cup \{\gamma\}$.*

Proof. Since our Coxeter system has rank two, for $x \in W$, there exists $\alpha \in \Phi^+$ such that $s_\alpha = x$ if and only if $\ell(x)$ is odd. One of elements in wg^{-1}, swg^{-1} has the odd length. Hence, there exists $\beta \in \Phi^+$ such that $s_\beta \in \{wg^{-1}, swg^{-1}\}$. If $s_\beta = wg^{-1}$, then $s_\beta g = w \le w$. If $s_\beta = swg^{-1}$, then $s_\beta g = sw \le w$. Hence $\beta \in X_g^w$ and we get (1). The proof of (2) is similar.

We prove (3). Let $\delta \in X_g^w$. Then $s_\delta g \le w$. Since our Coxeter system is of rank two, if $\ell(s_\delta gu) \le \ell(w) - 1$, we have $s_\delta gu \le w$. Hence $\delta \in X_{gu}^w$. Therefore, if $\ell(s_\delta g) \le \ell(w) - 2$, then since $\ell(s_\delta gu) \le \ell(s_\delta g) + 1$, we have $\delta \in X_{gu}^w$.

Let u' be the element in S which is not u. Then we have $sw < w$, $wu' < w$.

- If $\ell(s_\delta g) = \ell(w) - 1$, then $s_\delta g = sw$ or wu'. If $s_\delta g = sw$, then $s_\delta = swg^{-1}$, hence $\delta = \beta$. If $s_\delta g = wu'$ and $w \ne u'$, the reduced expression of wu' ends with u. Hence $s_\delta gu = wu'u \le w$. Therefore $\delta \in X_{gu}^w$. If $w = u'$, then $u' = s$ since $sw < w$. We have $\ell(s_\delta g) = \ell(w) - 1 = 0$, hence $s_\delta g = 1$. Since $g \le w$, we have $g = u'$ or $g = 1$. Since $\ell(s_\delta)$ is odd, by $s_\delta g = 1$, we have $g = u'$ and $s_\delta = u' = swg^{-1}$. Hence $\delta = \beta$.
- If $\ell(s_\delta g) = \ell(w)$, then $s_\delta g = w$. Hence $s_\delta = wg^{-1}$. Therefore $\delta = \beta$.

In any case, if $\delta \in X_g^w$, then $\delta = \beta$ or $\delta \in X_{gu}^w$. Hence $X_g^w \setminus \{\beta\} \subset X_{gu}^w$. If $\delta = \gamma$, the element $s_\delta g$ is wu or swu. Since $wu > w$, we have $s_\delta g \le w$ only when $s_\delta g = swu = w$. Therefore $\delta = \beta$. Hence $X_g^w \setminus \{\beta\} \subset X_{gu}^w \setminus \{\gamma\}$. By replacing g with gu, we get the reverse inclusion.

Since $wu > w$, for any $v \in W$, $vu \le wu$ if and only if $v \le w$ or $vu \le w$ by Property Z. Hence $X_{gu}^{wu} = X_g^w \cup X_{gu}^w$. Therefore, we get the last part of (3). ∎

LEMMA 2.14. *Let $w, g \in W$, $u \in S$ such that $wu > w$, $sw < w$, $g \le w$ and $gu \not\le w$. Then $X_{gu}^{wu} = X_g^w \cup \{g(\alpha_u)\}$.*

Proof. By Property Z, for any $x \in W$, $x \le w$ implies $xu \le wu$. Applying this to $x = s_\gamma g$ for $\gamma \in X_g^w$, we have $X_g^w \subset X_{gu}^{wu}$. Since $g \le w$, we have $s_{g(\alpha_u)} g = gu \le wu$. Therefore $g(\alpha_u) \in X_{gu}^{wu}$. Hence $X_g^w \cup \{g(\alpha_u)\} \subset X_{gu}^{wu}$.

If $\ell(g) \le \ell(w) - 2$, then $\ell(gu) \le \ell(w) - 1$, hence $gu \le w$ since $\#S = 2$. Therefore $\ell(g) = \ell(w) - 1$ or $\ell(w)$. If $\ell(g) = \ell(w) - 1$, then $g = sw$ since $gu \not\le w$. If $\ell(g) = \ell(w)$, then $g = w$. Hence $g = w$ or sw.

Let $\delta \in X_{gu}^{wu} \setminus X_g^w$. Then $s_\delta gu \le wu$ and $s_\delta g \not\le w$. By Property Z, $s_\delta gu < s_\delta g$ and $s_\delta gu \le w$. Therefore, from the discussion in the previous paragraph, $s_\delta gu = w$ or $s_\delta gu = sw$. Combining $g \in \{w, sw\}$, we have $(g, s_\delta) = (w, wuw^{-1})$ or $(sw, swu(sw)^{-1})$. In any case, we have $s_\delta = gug^{-1}$ and $\delta = g(\alpha_u)$. ∎

LEMMA 2.15. *Let $\underline{w} = (s_1, \ldots, s_l) \in S^l$ such that $s_{i-1} \ne s_i$ for any i and $g \in W$. Set $u = s_l$.*

(1) $a^{\underline{w}}(g) = a^{\underline{w}}(gu)$.
(2) $k_g^{\underline{w}} = k_{gu}^{\underline{w}}$.
(3) $X_g^w = X_{gu}^w$.

Proof. We may assume $g < gu$ by replacing g with gu if necessary. We also may assume that $s_1 = s$ by swapping s with t if necessary. By Lemma 2.10, we have $a^{(\underline{w},u)}(g) = 0$. Hence, (1) follows from Lemma 2.12.

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For (2), first we assume $sg > g$ and $g \neq 1$. Then the reduced expression of g has a form $g = t\dots u'$, where $u' \in S$ is the element which is not u, namely the reduced expression starts with t and ends with u'. Since $\underline{w} = (s,\dots,u)$ and $s_{i-1} \neq s_i$ for any i, we have $\ell(g) \equiv \ell(\underline{w}) \pmod 2$. Hence. the lemma follows from the definition of $k_g^{\underline{w}}$. The proof in the case of $sg < g$, $g \neq 1$ is similar.

Assume $g = 1$. If $u = s$, then $s_1 = s_l = s$, hence $\ell(\underline{w})$ is odd. If $u = t$, then $s_1 = s$ and $s_l = t$. Hence $\ell(\underline{w})$ is even. In both cases, we can confirm $k_g^{\underline{w}} = k_{gu}^{\underline{w}}$ by the definition.

Since $wu < w$, by Property Z, we have $s_\gamma g \leq w$ if and only if $s_\gamma gu \leq w$. (3) follows. ∎

Proof of Theorem 2.11. We prove the theorem by induction on $\ell(\underline{w})$. If $\ell(\underline{w}) = 0$, then this is trivial. Let $u \in S$ and we prove that the theorem is true for $(\underline{w},u)$ assuming that the theorem is true for $\underline{w}$. If $(\underline{w},u)$ is not a reduced expression, then both sides of the theorem are zero. Hence we may assume $(\underline{w},u)$ is a reduced expression. By the previous lemma, we also may assume $gu > g$. If $g \not\leq w$, then by Property Z, $g \not\leq wu$. Hence, both sides are zero.

Take $s_1,\dots,s_l \in S$ such that $\underline{w} = (s_1,\dots,s_l)$. If $g \leq w$ and $gu \not\leq w$, then $a^{\underline{w}}(gu) = 0$. By Lemma 2.12, inductive hypothesis and Lemma 2.14,

$$a^{(\underline{w},u)}(g) = \frac{a^{\underline{w}}(g)}{g(\alpha_u)} = \frac{k_g^{\underline{w}}}{\prod_{\gamma\in X_g^w}\gamma}\frac{1}{g(\alpha_u)} = \frac{k_g^{\underline{w}}}{\prod_{\gamma\in X_{gu}^{(\underline{w},u)}}\gamma}.$$

As in the proof of Lemma 2.14, we have $g = w$ or $g = s_1 w$ (the latter does not happen when $l = 0$). Hence $k_g^{\underline{w}} = k_g^{(\underline{w},u)} = 1$ from the definitions. Therefore, the theorem holds in this case.

We assume $g, gu \leq w$. Then $\ell(\underline{w}) > 0$. We may assume $s_1 = s$ by swapping (s,X) with (t,Y) if necessary. By Lemma 2.12 and inductive hypothesis, we have

$$a^{(\underline{w},u)}(g) = \frac{1}{g(\alpha_u)}(a^{\underline{w}}(g) - a^{\underline{w}}(gu)) = \frac{1}{g(\alpha_u)}\left(\frac{k_g^{\underline{w}}}{\prod_{\delta\in X_g^w}\delta} - \frac{k_{gu}^{\underline{w}}}{\prod_{\delta\in X_{gu}^w}\delta}\right).$$

Take $\beta,\gamma \in \Phi^+$ as in Lemma 2.13. Then by Lemma 2.13, the right-hand side is

$$\frac{1}{\prod_{\delta\in X_g^w\setminus\{\beta\}}\delta}\frac{1}{\beta\gamma}\frac{1}{g(\alpha_u)}(k_g^{\underline{w}}\gamma - k_{gu}^{\underline{w}}\delta) = \frac{1}{\prod_{\delta\in X_g^{wu}}\delta}\frac{1}{g(\alpha_u)}(k_g^{\underline{w}}\gamma - k_{gu}^{\underline{w}}\delta).$$

Hence, it is sufficient to prove that $k_g^{\underline{w}}\gamma - k_{gu}^{\underline{w}}\delta = k_g^{(\underline{w},u)}g(\alpha_u)$. Since $gu > g$, the reduced expression of g ends with the simple reflection which is not u. Hence, the reduced expression of gug^{-1} can be obtained by concatenating the reduced expressions of g, u and g^{-1}. Therefore, we have $\ell(gug^{-1}) = \ell(g) + \ell(u) + \ell(g^{-1}) = 2\ell(g) + 1$. Moreover, if $sg > g$, then we have $sgug^{-1} > gug^{-1}$.

First, we assume $sg > g$ and $g \neq 1$. Then $ss_{g(u)} = sgug^{-1} > gug^{-1}$. Hence

$$g(\alpha_u) = [\ell(g)]_X\alpha_s + [\ell(g)+1]_Y\alpha_t$$

by Lemma 2.9. Since $gu > g$ and $wu > w$, the reduced expressions of g and w end with the same simple reflection. Namely, if $u' \in S$ is the element which is not u, then the reduced expression of w is $w = s\dots u'$ and the reduced expression of g is $g = t\dots u'$ since we assumed $sg > g$. Since $g \leq w$, the last $\ell(g)$-letters of the reduced expression of w is the reduced expression of g. Hence, $\ell(wg^{-1}) + \ell(g) = \ell(w)$ and the reduced expression of wg^{-1} starts

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with s and ends with s. Therefore, $twg^{-1} > wg^{-1}$ and $s_\beta = wg^{-1}$. Hence, by Lemma 2.9, we have

$$\beta = \left[\frac{\ell(wg^{-1})+1}{2}\right]_X \alpha_s + \left[\frac{\ell(wg^{-1})-1}{2}\right]_Y \alpha_t$$
$$= \left[\frac{\ell(w)-\ell(g)+1}{2}\right]_X \alpha_s + \left[\frac{\ell(w)-\ell(g)-1}{2}\right]_Y \alpha_t.$$

A calculation of γ is similar. We have $\ell(wug^{-1}) = \ell(g)+\ell(u)+\ell(w^{-1})$ and the reduced expression of wug^{-1} starts with s and ends with t. Therefore, $\ell(swug^{-1}) = \ell(wug^{-1})-1$, $s_\gamma = swug^{-1}$ and $s(swug^{-1}) > swug^{-1}$. Hence, by Lemma 2.9, we have

$$\gamma = \left[\frac{\ell(w)+\ell(g)-1}{2}\right]_X \alpha_s + \left[\frac{\ell(w)+\ell(g)+1}{2}\right]_Y \alpha_t.$$

Put $m = (\ell(w)-\ell(g)-1)/2$ and $n = \ell(g)$. Then we have

$$g(\alpha_u) = [n]_X\alpha_s + [n+1]_Y\alpha_t,$$
$$\beta = [m+1]_X\alpha_s + [m]_Y\alpha_t,$$
$$\gamma = [m+n]_X\alpha_s + [m+n+1]_Y\alpha_t.$$

Therefore, we have

$$k_g^{\underline{w}}\gamma - k_{gu}^{\underline{w}}\beta = (k_g^{\underline{w}}[m+n]_X - k_{gu}^{\underline{w}}[m+1]_X)\alpha_s + (k_g^{\underline{w}}[m+n+1]_Y - k_{gu}^{\underline{w}}[m]_Y)\alpha_t.$$

By the definition, we have

$$k_g^{\underline{w}} = \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{\sigma^{2m+n}(X)} = \frac{[m+n+1]_{\sigma^{2m+n}(X)}}{[m]_{\sigma^{2m+n}(X)}} \begin{bmatrix} 2m+n \\ m-1 \end{bmatrix}_{\sigma^{2m+n}(X)}$$
$$= \frac{[m+n+1]_{\sigma^n(X)}}{[m]_{\sigma^n(X)}} k_{gu}^{\underline{w}}.$$

Hence,

$$k_g^{\underline{w}}\gamma - k_{gu}^{\underline{w}}\beta = \frac{k_{gu}^{\underline{w}}}{[m]_\sigma^{n(X)}} \Big(([m+n+1]_{\sigma^n(X)}[m+n]_X - [m+1]_X[m]_{\sigma^n(X)})\alpha_s$$
$$+ ([m+n+1]_\sigma^{n(X)}[m+n+1]_Y - [m]_Y[m]_\sigma^{n(X)})\alpha_t \Big).$$

By Lemma 2.5, this is equal to

$$\frac{k_{gu}^{\underline{w}}}{[m]_\sigma^{n(X)}}([n]_X[2m+n+1]_{\sigma^{m+1}(X)}\alpha_s + [n+1]_Y[2m+n+1]_{\sigma^{m+1}(X)}\alpha_t)$$
$$= \frac{k_{gu}^{\underline{w}}}{[m]_\sigma^{n(X)}}[2m+n+1]_{\sigma^{m+1}(X)} g(\alpha_u).$$

Hence, it is sufficient to prove

$$k_{gu}^{\underline{w}} \frac{[2m+n+1]_{\sigma^{m+1}(X)}}{[m]_\sigma^{n(X)}} = k_{gu}^{(\underline{w},u)}.$$

This follows immediately from Lemma 2.7.

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The case of $tg > g$ and $g \neq 1$ is similar. By Lemma 2.9, we have

$$g(\alpha_u) = [\ell(g)+1]_X \alpha_s + [\ell(g)]_Y \alpha_t.$$

The reduced expressions of w and g end with the same reflection, hence $\ell(wg^{-1}) = \ell(w) - \ell(g)$. The reduced expression of g starts with s. Hence, the reduced expression of wg^{-1} starts with s, ends with t. Hence, $s_\beta = swg^{-1}$, $s(swg^{-1}) > swg^{-1}$ and $\ell(s_\beta) = \ell(w) - \ell(g) - 1$. Hence, by Lemma 2.9, we have

$$\beta = \left[\frac{\ell(w)-\ell(g)}{2} - 1\right]_X \alpha_s + \left[\frac{\ell(w)-\ell(g)}{2}\right]_Y \alpha_t.$$

We have $\ell(wug^{-1}) = \ell(w) + \ell(g) + 1$ and the reduced expression starts with s and ends with s. Hence $s_\gamma = wug^{-1}$, $ts_\gamma > s_\gamma$, and $\ell(s_\gamma) = \ell(g) + \ell(w) + 1$. Therefore, by Lemma 2.9, we have

$$\gamma = \left[\frac{\ell(w)+\ell(g)}{2} + 1\right]_X \alpha_s + \left[\frac{\ell(w)+\ell(g)}{2}\right]_Y \alpha_t.$$

Put $m = (\ell(w) - \ell(g))/2 - 1$ and $n = \ell(g) + 1$. Then

$$\begin{aligned} g(\alpha_u) &= [n]_X \alpha_s + [n-1]_Y \alpha_t, \\ \beta &= [m]_X \alpha_s + [m+1]_Y \alpha_t, \\ \gamma &= [m+n+1]_X \alpha_s + [m+n]_Y \alpha_t. \end{aligned}$$

We have

$$k_g^{\underline{w}} = \frac{[m+n]_{\sigma^n(X)}}{[m+1]_{\sigma^n(X)}} k_{gu}^{\underline{w}}.$$

Therefore, by Lemma 2.5, we have

$$\begin{aligned} &k_g^{\underline{w}}\gamma - k_{gu}^{\underline{w}}\beta \\ &= \frac{k_{gu}^{\underline{w}}}{[m+1]_\sigma^{n(X)}} \big(([m+n]_{\sigma^n(X)}[m+n+1]_X - [m]_X[m+1]_{\sigma^n(X)})\alpha_s \\ &\qquad + ([m+n]_Y[m+n]_{\sigma^n(X)} - [m+1]_Y[m+1]_{\sigma^n(X)})\alpha_t\big) \\ &= \frac{k_{gu}^{\underline{w}}}{[m+1]_\sigma^{n(X)}}([n]_X[2m+n+1]_{\sigma^m(X)} + [n-1]_Y[2m+n+1]_{\sigma^m(X)}) \\ &= \frac{k_{gu}^{\underline{w}}}{[m+1]_\sigma^{n(X)}}[2m+n+1]_{\sigma^m(X)} g(\alpha_u). \end{aligned}$$

Therefore, it is sufficient to prove

$$\frac{[2m+n+1]_{\sigma^m(X)}}{[m+1]_\sigma^{n(X)}} k_{gu}^{\underline{w}} = k_{gu}^{(\underline{w},u)},$$

which is again an immediate consequence of Lemma 2.7.

We assume $g = 1$ and $u = t$. Then one can check that the formulas for $g(\alpha_u), \beta, \gamma$ in the case of $sg > g, g \neq 1$ hold. Hence, the theorem follows from the calculations in this case. If $g = 1$ and $u = s$, then one can use the calculations in the case of $tg > g, g \neq 1$. ∎

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§3. A homomorphism between Bott–Samelson bimodules

3.1 Finite Coxeter group of rank two and a realization

We add the tilde to the notation in the previous section, namely $(\widetilde{W},\widetilde{S})$ is the universal Coxeter system of rank 2, $\widetilde{V}$ is the free $\mathbb{Z}[\widetilde{X},\widetilde{Y}]$-module with the action of $\widetilde{W}$, $[n]_{\widetilde{X}},[n]_{\widetilde{Y}} \in \mathbb{Z}[\widetilde{X},\widetilde{Y}]$ is the two-colored quantum numbers, etc.

The notation without tilde will be used for non-universal version. Let (W,S) be a Coxeter system such that $S=\{s,t\}$, $s\neq t$. We assume that the order $m_{s,t}$ of st is finite. Let $\mathbb{K}$ be a commutative integral domain and $(V,\{\alpha_s,\alpha_t\},\{\alpha_s^\vee,\alpha_t^\vee\})$ a realization [7, Def. 3.1], namely V is a free $\mathbb{K}$-module of finite rank with an action of W, $\alpha_s,\alpha_t\in V$ and $\alpha_s^\vee,\alpha_t^\vee\in \operatorname{Hom}_{\mathbb{K}}(V,\mathbb{K})$ such that:

- $\langle\alpha_s^\vee,\alpha_s\rangle=\langle\alpha_t^\vee,\alpha_t\rangle=2$.
- $s(v)=v-\langle\alpha_s^\vee,v\rangle\alpha_s$, $t(v)=v-\langle\alpha_t^\vee,v\rangle\alpha_t$ for any $v\in V$.
- $[m_{s,t}]_{\widetilde{X}}(-\langle\alpha_s^\vee,\alpha_t\rangle,-\langle\alpha_t^\vee,\alpha_s\rangle)=[m_{s,t}]_{\widetilde{Y}}(-\langle\alpha_s^\vee,\alpha_t\rangle,-\langle\alpha_t^\vee,\alpha_s\rangle)=0$.

We also assume the following (part of) Demazure surjectivity:

- $\alpha_s,\alpha_t\neq 0$ and $\alpha_s^\vee,\alpha_t^\vee\colon V\to\mathbb{K}$ are surjective.

The map $\tilde{s}\mapsto s$, $\tilde{t}\mapsto t$ gives a surjective homomorphism $\widetilde{W}\to W$. Set $X=-\langle\alpha_s^\vee,\alpha_t\rangle$, $Y=-\langle\alpha_t^\vee,\alpha_s\rangle$. Then $\tilde{\alpha}_s\mapsto\alpha_s$, $\tilde{\alpha}_t\mapsto\alpha_t$ gives a $\mathbb{Z}[\widetilde{X},\widetilde{Y}]$-module homomorphism $\widetilde{V}\to V$ which commutes with the actions of $\widetilde{W}$, where we regard V as a $\mathbb{Z}[\widetilde{X},\widetilde{Y}]$-module via $\mathbb{Z}[\widetilde{X},\widetilde{Y}]\to\mathbb{K}$ defined by $\widetilde{X}\mapsto X$ and $\widetilde{Y}\mapsto Y$. The image of $[n]_{\widetilde{X}}$ (resp., $[n]_{\widetilde{Y}}$) is denoted by $[n]_X$ (resp., $[n]_Y$). We also have $\begin{bmatrix}n\\m\end{bmatrix}_X,\begin{bmatrix}n\\m\end{bmatrix}_Y\in\mathbb{K}$. Note that by $[m_{s,t}]_X=[m_{s,t}]_Y=0$, we have $[m_{s,t}-1]_X[m_{s,t}-1]_Y=1$ [8, (6.11), (6.12)].

Let R (resp., $\widetilde{R}$) be the symmetric algebra of V (resp., $\widetilde{V}$). We regard R as a graded $\mathbb{K}$-algebra via $\deg(V)=2$. We put $\partial_u(p)=(p-u(p))/\alpha_u$ for $p\in R$. The maps $\widetilde{V}\to V$ and $\mathbb{Z}[\widetilde{X},\widetilde{Y}]\to\mathbb{K}$ induce $\widetilde{R}\to R$. We defined an element $\tilde{a}^{\underline{\tilde{w}}}(\tilde{g})\in\widetilde{R}[\tilde{w}(\tilde{\alpha}_{\tilde{u}})^{-1}\mid\tilde{w}\in\widetilde{W},\tilde{u}\in\widetilde{S}]$. Set $Q=R[w(\alpha_u)^{-1}\mid w\in W,u\in S]$. The image of $\tilde{a}^{\underline{\tilde{w}}}(\tilde{g})$ in Q is denoted by $a^{\underline{\tilde{w}}}(\tilde{g})\in Q$.

As some of them appeared already, objects related to the universal Coxeter system is denoted with the tilde and the corresponding letter without the tilde means the image in the finite Coxeter system. For example, if $\underline{\tilde{w}}=(\tilde{s}_1,\tilde{s}_2,\dots)$ is a sequence of elements in $\widetilde{S}$, then $\underline{w}=(s_1,s_2,\dots)$ is the corresponding sequence in S. As we have already explained, a sequence is denoted with the underline and removing the underline means the product of elements in the sequence. Hence $\tilde{w}=\tilde{s}_1\tilde{s}_2\dots\in\widetilde{W}$ and $w=s_1s_2\dots\in W$. For each root $\tilde{\alpha}\in\widetilde{\Phi}$, we have $\tilde{s}_{\tilde{\alpha}}\in\widetilde{W}$ and $s_{\tilde{\alpha}}\in W$.

Set $\underline{\tilde{x}}=(\tilde{s},\tilde{t},\dots)\in S^{m_{s,t}}$ and $\underline{\tilde{y}}=(\tilde{t},\tilde{s},\dots)\in S^{m_{s,t}}$. The sequences $\underline{x}$ and $\underline{y}$ are the two reduced expressions of the longest element. In general, for a sequence $\underline{w}=(s_1,s_2,\dots,s_l)\in S^l$, we put

$$\pi_{\underline{w}}=\prod_{i=1}^{l}s_1\dots s_{i-1}(\alpha_{s_i})\in R.$$

The two elements $\pi_{\underline{x}}$ and $\pi_{\underline{y}}$ are not the same in general. By [8, (7.9), (7.11)], $\pi_{\underline{y}}=\pi_{\underline{x}}$ if $m_{s,t}$ is even and $\pi_{\underline{y}}=[m_{s,t}-1]_X\pi_{\underline{x}}$ if $m_{s,t}$ is odd. Put $\xi=[m_{s,t}-1]_X=[m_{s,t}-1]_Y$ if $m_{s,t}$ is even and $\xi=1$ if $m_{s,t}$ is odd. Then we have $\pi_{\underline{y}}=\xi[m_{s,t}-1]_X\pi_{\underline{x}}$ since $[m_{s,t}-1]_X[m_{s,t}-1]_Y=1$. In particular, $\pi_{\underline{y}}\in\mathbb{K}^\times\pi_{\underline{x}}$ [8, (6.11), (6.12)]. The realization is even-balanced if and only if $\xi=1$.

LEMMA 3.1. *We have* $[k]_Z[m_{s,t}-1]_{\sigma^{k-1}(Z)} = [m_{s,t}-k]_Z$.

Proof. This follows from [8, (6.10)]. ∎

LEMMA 3.2. *Let* $\tilde{g} \in \widetilde{W}$ *such that* $\tilde{g} \leq \tilde{x}$. *Then we have*

$$\frac{\prod_{\tilde{\delta}\in \tilde{X}^{\tilde{x}}_{\tilde{g}}} \delta}{\pi_{\underline{x}}} = \begin{cases} \xi \displaystyle\prod_{i=1}^{\left\lfloor \frac{m_{s,t}-\ell(g)-1}{2} \right\rfloor} [m_{s,t}-1]_{\sigma^{i-1}(X)} & (\tilde{s}\tilde{g} > \tilde{g}), \\ \displaystyle\prod_{i=1}^{\left\lfloor \frac{m_{s,t}-\ell(g)}{2} \right\rfloor} [m_{s,t}-1]_{\sigma^{i-1}(X)} & (\tilde{s}\tilde{g} < \tilde{g}). \end{cases}$$

Proof. We prove the lemma by backward induction on $\ell(\tilde{g})$. If $\tilde{g} = \tilde{x}$, then by Theorem 2.11, we have $a^{\underline{\tilde{x}}}(\tilde{x}) = (\prod_{\tilde{\delta}\in \tilde{X}^{\tilde{x}}_{\tilde{x}}} \delta)^{-1}$. On the other hand, for $e \in \{0,1\}^{m_{s,t}}$, we have $\underline{\tilde{x}}^e = \tilde{x}$ if and only if $e = (1,\dots,1)$. Hence, by the definition of $a^{\underline{\tilde{x}}}(\tilde{x})$, we have $a^{\underline{\tilde{x}}}(\tilde{x}) = 1/\pi_{\underline{x}}$.

Next, assume that $\tilde{g} = \tilde{s}\tilde{x}$. Define $\tilde{s}_i = \tilde{s}$ if i is odd and $\tilde{s}_i = \tilde{t}$ if i is even. Then $\underline{x} = (\tilde{s}_1,\dots,\tilde{s}_{m_{s,t}})$. For $e \in \{0,1\}^{m_{s,t}}$, $\underline{\tilde{x}}^e = \tilde{g}$ if and only if $e = (0,1,\dots,1)$. Hence, by the definition, $a^{\underline{\tilde{x}}}(\tilde{g}) = 1/\alpha_{s_1}\prod_{i=2}^{m_{s,t}} s_2\dots s_{i-1}(\alpha_{s_i})$. Since $\underline{\tilde{y}} = (\tilde{s}_2,\tilde{s}_3,\dots,\tilde{s}_{m_{s,t}+1})$, we have $\pi_{\underline{y}} = \prod_{i=2}^{m_{s,t}+1} s_2\dots s_{i-1}(\alpha_{s_i}) = (1/a^{\underline{\tilde{x}}}(\tilde{g}))(s_2s_3\dots s_{m_{s,t}}(\alpha_{s_{m_{s,t}+1}})/\alpha_{s_1})$. Since $\tilde{s}_1 = \tilde{s}$, $\tilde{s}\tilde{s}_{s_2\tilde{s}_3\dots\tilde{s}_{m_{s,t}}(\tilde{\alpha}_{\tilde{s}_{m_{s,t}+1}})} > \tilde{s}_{s_2\tilde{s}_3\dots\tilde{s}_{m_{s,t}}(\tilde{\alpha}_{\tilde{s}_{m_{s,t}+1}})}$. Hence, we have $s_2s_3\dots s_{m_{s,t}}(\alpha_{s_{m_{s,t}+1}}) = [m_{s,t}-1]_X\alpha_s + [m_{s,t}]_Y\alpha_t = [m_{s,t}-1]_X\alpha_s$ by Lemma 2.9. Since $s_1 = s$, we get $\pi_{\underline{y}} = [m_{s,t}-1]_X/a^{\underline{\tilde{x}}}(\tilde{g})$. By $\pi_{\underline{y}} = \xi[m_{s,t}-1]_X\pi_{\underline{x}}$, we have $\xi\pi_{\underline{x}}a^{\underline{\tilde{x}}}(\tilde{g}) = 1$. By Theorem 2.11, the left-hand side of the lemma is $(a^{\underline{\tilde{x}}}(\tilde{g})\pi_{\underline{x}})^{-1}$. Hence, we get the lemma in this case.

Assume that $\tilde{g} \neq \tilde{x}, \tilde{s}\tilde{x}$. Then there exists $\tilde{u} \in \widetilde{S}$ such that $\tilde{x} \geq \tilde{g}\tilde{u} > \tilde{g}$. When $\tilde{g} = 1$, we take $\tilde{u} = \tilde{t}$.

- First assume that $\tilde{x}\tilde{u} < \tilde{x}$. By Lemma 2.15, the left-hand side is not changed if we replace $\tilde{g}$ with $\tilde{g}\tilde{u}$. We prove that the right-hand side is also not changed. Then this gives the lemma by inductive hypothesis.
  - Assume $\tilde{g} \neq 1$. The reduced expression of $\tilde{x}$ is given as $\tilde{x} = \tilde{s}\dots\tilde{u}$. Let $\tilde{u}' \in \widetilde{S}$ be the element which is not $\tilde{u}$. If $\tilde{s}\tilde{g} > \tilde{g}$, then the reduced expression of $\tilde{g}$ is $\tilde{g} = \tilde{t}\dots\tilde{u}'$. Hence $\ell(\tilde{g}) \equiv \ell(\tilde{x}) (\mathrm{mod}\ 2)$. If $\tilde{s}\tilde{g} < \tilde{g}$, then the reduced expression of $\tilde{g}$ is $\tilde{g} = \tilde{s}\dots\tilde{u}'$. Hence $\ell(\tilde{g}) \equiv \ell(\tilde{x}) + 1 (\mathrm{mod}\ 2)$. Therefore, the right-hand side is not changed.
  - If $\tilde{g} = 1$, then by $\tilde{x}\tilde{t} < \tilde{x}$ (recall that we took $\tilde{u} = \tilde{t}$), the reduced expression of $\tilde{x}$ is $\tilde{x} = \tilde{s}\dots\tilde{t}$. Hence, $\ell(\tilde{x})$ is even and the right-hand side is not changed.
- Assume that $\tilde{x}\tilde{u} > \tilde{x}$. Take $\tilde{\beta}$ and $\tilde{\gamma}$ such that $\tilde{s}_{\tilde{\beta}} \in \{\tilde{x}\tilde{g}^{-1}, \tilde{s}\tilde{x}\tilde{g}^{-1}\}$ and $\tilde{s}_{\tilde{\gamma}} \in \{\tilde{x}\tilde{u}\tilde{g}^{-1}, \tilde{s}\tilde{x}\tilde{u}\tilde{g}^{-1}\}$. By Lemma 2.13, we have $(\prod_{\tilde{\delta}\in \tilde{X}^{\tilde{x}}_{\tilde{g}}} \delta)/(\prod_{\tilde{\delta}\in \tilde{X}^{\tilde{x}}_{\tilde{g}\tilde{u}}} \delta) = \beta/\gamma$. We calculate β/γ. We use calculations in the proof of Theorem 2.11.
  - If $\tilde{s}\tilde{g} > \tilde{g}$, $\tilde{g} \neq 1$ or $\tilde{g} = 1$, then by the proof of Theorem 2.11, we have $\beta = [(m_{s,t}-\ell(\tilde{g})+1)/2]_X\alpha_s + [(m_{s,t}-\ell(\tilde{g})-1)/2]_Y\alpha_t$ and $\gamma = [(m_{s,t}+\ell(\tilde{g})-1)/2]_X\alpha_s + [(m_{s,t}+\ell(\tilde{g})+1)/2]_Y\alpha_t$. Therefore, by the previous lemma, we have $\gamma = [m_{s,t}-1]_{\sigma^{(m_{s,t}-\ell(\tilde{g})-1)/2}(X)}\beta$. We have $[m_{s,t}-1]_X[m_{s,t}-1]_Y = 1$ by [8, (6.11), (6.12)]. Hence, we have $\beta/\gamma = [m_{s,t}-1]_{\sigma^{(m_{s,t}-\ell(\tilde{g})-1)/2-1}(X)}$. By inductive hypothesis, we get the lemma in this case.
  - Finally, assume that $\tilde{s}\tilde{g} < \tilde{g}$. By the proof of Theorem 2.11, we have $\beta = [(m_{s,t}-\ell(\tilde{g}))/2-1]_X\alpha_s + [(m_{s,t}-\ell(\tilde{g}))/2]_Y\alpha_t$, $\gamma = [(m_{s,t}+\ell(\tilde{g}))/2+1]_X\alpha_s + [(m_{s,t}+\ell(\tilde{g}))/2]_Y\alpha_t$. Hence $\gamma = [m_{s,t}-1]_{\sigma^{(m_{s,t}-\ell(g))/2-2}(X)}\beta$. Therefore $\beta = [m_{s,t}-1]_{\sigma^{(m_{s,t}-\ell(g))/2-1}(X)}\gamma$ and we get the lemma. ∎

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LEMMA 3.3. *Let $\underline{\tilde{w}} \in \widetilde{S}^l$. If $0 \le l \le m_{s,t}$, then $\pi_{\underline{x}} a^{\underline{\tilde{w}}}(1) \in R$.*

Proof. By Theorem 2.11, the lemma follows from $\pi_{\underline{x}}/\prod_{\tilde{\gamma}\in \tilde{X}_1^{\tilde{w}}}\gamma \in R$. If $\tilde{w} = \tilde{x}$, then it follows from Lemma 3.2. By swapping s with t, $\pi_{\underline{y}} a^{\underline{\tilde{y}}}(1) \in R$. Since $\pi_{\underline{y}} \in \mathbb{K}^\times \pi_{\underline{x}}$, we get the lemma for $\tilde{w} = \tilde{y}$. In general, we have $\tilde{w} \le \tilde{x}$ or $\tilde{w} \le \tilde{y}$. If $\tilde{w} \le \tilde{x}$ then $X_1^{\tilde{w}} \subset X_1^{\tilde{x}}$. Hence $\pi_{\underline{x}}/\prod_{\tilde{\gamma}\in \tilde{X}_1^{\tilde{w}}}\gamma = (\pi_{\underline{x}}/\prod_{\tilde{\gamma}\in \tilde{X}_1^{\tilde{x}}}\gamma)(\prod_{\tilde{\gamma}\in \tilde{X}_1^{\tilde{x}}\setminus \tilde{X}_1^{\tilde{w}}}\gamma) \in R$. The same discussion implies the lemma when $\tilde{w} \le \tilde{y}$. ∎

3.2 An assumption

To prove the main theorem, we need one more assumption. In this subsection, we discuss on the assumption. We start with the following proposition.

PROPOSITION 3.4. *The following are equivalent.*

(1) $\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_Z = 0$ *for any* $1 \le k \le m_{s,t} - 1$ *and* $Z \in \{X, Y\}$.
(2) *We have* $\begin{bmatrix} m_{s,t}-1 \\ k \end{bmatrix}_Z = \prod_{i=1}^k [m_{s,t}-1]_{\sigma^{i-1}(Z)}$ *for* $0 \le k \le m_{s,t}-1$ *and* $Z \in \{X, Y\}$.
(3) *The realization is even-balanced and* $\begin{bmatrix} m_{s,t}-1 \\ k \end{bmatrix}_Z = \prod_{i=1}^k [m_{s,t}-1]_{\sigma^{i-1}(Z)}$ *for* $0 \le k \le (m_{s,t}-1)/2$ *and* $Z \in \{X, Y\}$.

Proof. Assume (1). By Lemma 2.6 and (1), we have $\begin{bmatrix} m_{s,t}-1 \\ k \end{bmatrix}_Z = -\begin{bmatrix} m_{s,t}-1 \\ k-1 \end{bmatrix}_Z [m_{s,t}+1]_{\sigma^{k-1}(Z)}$. We have $[m_{s,t}+1]_{\sigma^{k-1}(Z)} = -[m_{s,t}-1]_{\sigma^{k-1}(Z)}$ [8, (6.9)]. Hence, (2) follows from induction on k.

Conversely assume (2) and we prove (1). By Lemma 2.6, we have

$$\begin{aligned}\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_Z &= \begin{bmatrix} m_{s,t}-1 \\ k \end{bmatrix}_{\sigma^k(Z)} [k+1]_Z - \begin{bmatrix} m_{s,t}-1 \\ k-1 \end{bmatrix}_Z [m_{s,t}-k-1]_{\sigma^{k+1}(Z)} \\ &= \prod_{i=1}^k [m_{s,t}-1]_{\sigma^{k+i-1}(Z)}[k+1]_Z - \prod_{i=1}^{k-1}[m_{s,t}-1]_{\sigma^{i-1}(Z)}[m-k-1]_{\sigma^{k+1}(Z)}.\end{aligned}$$

By replacing i with $k-i$, we have $\prod_{i=1}^k [m_{s,t}-1]_{\sigma^{k+i-1}(Z)} = \prod_{i=0}^{k-1}[m_{s,t}-1]_{\sigma^{i-1}(Z)} = [m_{s,t}-1]_{\sigma(Z)}\prod_{i=1}^{k-1}[m_{s,t}-1]_{\sigma^{i-1}(Z)}$. Therefore, it is sufficient to prove $[m_{s,t}-1]_{\sigma(Z)}[k+1]_Z - [m-k-1]_{\sigma^{k-1}(Z)} = 0$. By Lemma 2.3, we have $[m_{s,t}-1]_{\sigma(Z)}[k+1]_Z = [m_{s,t}]_Z[k+2]_{\sigma(Z)} - [m_{s,t}+k+1]_{\sigma^{k+1}(Z)}$. Since $[m_{s,t}]_Z = 0$ and $[m_{s,t}+k+1]_{\sigma^{k+1}(Z)} = -[m_{s,t}-k-1]_{\sigma^{k+1}(Z)}$ [8, (6.9)], we get (1).

We assume (2) and we prove (3). By putting $k = m_{s,t}-1$, we have $\prod_{i=1}^{m_{s,t}-1}[m_{s,t}-1]_{\sigma^{i-1}(Z)} = 1$. If $m_{s,t}$ is even, by Lemma 2.2 (1) and $[m_{s,t}-1]_Z^2 = 1$ [8, (6.12)], we get $[m_{s,t}-1]_Z = 1$. Hence, V is even-balanced and we get (3).

Assume (3) and we prove (2). It is sufficient to prove that $\prod_{i=1}^k [m_{s,t}-1]_{\sigma^{i-1}(Z)} = \prod_{i=1}^{m_{s,t}-1-k}[m_{s,t}-1]_{\sigma^{i-1}(Z)}$. By [8, (6.11)], since the realization is even-balanced, we have $\prod_{i=1}^{m_{s,t}-1}[m_{s,t}-1]_{\sigma^{i-1}(Z)} = 1$. Hence, the right-hand side is $\prod_{i=m_{s,t}-k}^{m_{s,t}-1}[m_{s,t}-1]_{\sigma^{i-1}(Z)}^{-1} = \prod_{i=1}^k [m_{s,t}-1]_{\sigma^{m_{s,t}-i}(Z)}$. Here in the last part, we replaced i with $m_{s,t}-i$ and used $[m_{s,t}-1]_X[m_{s,t}-1]_Y = 1$ [8, (6.11), (6.12)]. By Lemma 2.2(3), we have $[m_{s,t}-1]_{\sigma^{m_{s,t}-i}(Z)} = [m_{s,t}-1]_{\sigma^{i-1}(Z)}$ and we get (2). ∎

We need the following assumption to prove the main theorem.

ASSUMPTION 3.5. The equivalent conditions in Proposition 3.4 hold.

We have a sufficient condition of Assumption 3.5.

PROPOSITION 3.6. *If the action of W on $\mathbb{K}\alpha_s+\mathbb{K}\alpha_t$ is faithful, then Assumption 3.5 holds.*

Proof. If $[k]_X=[k]_Y=0$ for some k such that $1\le k\le m_{s,t}-1$, then by [6, before Claims 3.2 and 3.5], $(st)^k$ is the identity on $\mathbb{K}\alpha_s+\mathbb{K}\alpha_t$. This is a contradiction. Hence $[k]_X\neq 0$ or $[k]_Y\neq 0$ for any $1\le k\le m_{s,t}-1$. For $1\le k\le m_{s,t}-1$, we have $[k]_X\begin{bmatrix}m_{s,t}\\k\end{bmatrix}_X=[m_{s,t}]_X\begin{bmatrix}m_{s,t}-1\\k-1\end{bmatrix}_X=0$. Hence, if $[k]_X\neq 0$, then $\begin{bmatrix}m_{s,t}\\k\end{bmatrix}_X=0$. Therefore, if $[k]_X,[k]_Y\neq 0$ for any $1\le k\le m_{s,t}-1$, we get the proposition. Assume that there exists $k=1,\dots,m_{s,t}-1$ such that $[k]_X=0$. Then $[k]_Y\neq 0$. By Lemma 2.2(1), k is even and by Lemma 2.2(2), we have $[k]_YX=[k]_XY=0$. Hence $X=0$. Therefore by induction, we have $[2n]_X=0$ and $[2n+1]_X=(-1)^n$ for any $n\in\mathbb{Z}_{\ge 0}$. Hence, $[2n+1]_Y=[2n+1]_X=(-1)^n\neq 0$ for any $n\in\mathbb{Z}_{\ge 0}$ by Lemma 2.2 (1). We also have $[2n]_Y\neq 0$ if $1\le 2n\le m_{s,t}-1$ since $[2n]_X=0$. Therefore, for any $1\le l\le m_{s,t}-1$, $[l]_Y\neq 0$. Therefore $\begin{bmatrix}m_{s,t}\\l\end{bmatrix}_Y=0$.

Since $[2n+1]_X\neq 0$ for any $n\in\mathbb{Z}_{\ge 0}$, $m_{s,t}$ is even. Therefore, if l is even, $\#(2\mathbb{Z}\cap\{m_{s,t},\dots,m_{s,t}-l+1\})=\#(2\mathbb{Z}\cap\{1,\dots,l\})$. Hence, by Lemma 2.2, we have $\begin{bmatrix}m_{s,t}\\l\end{bmatrix}_X=\begin{bmatrix}m_{s,t}\\l\end{bmatrix}_Y$ which is zero as we have proved. On the other hand, if l is odd, then $[l]_X\neq 0$. Hence $\begin{bmatrix}m_{s,t}\\l\end{bmatrix}_X=0$. ∎

Maybe more useful criterion is the following.

PROPOSITION 3.7. *If the realization comes from a root datum and W is the Weyl group, then Assumption 3.5 holds.*

Proof. We are in one of the following situation:

- $m_{s,t}=2$, $\langle\alpha_s,\alpha_t^\vee\rangle=\langle\alpha_t,\alpha_s^\vee\rangle=0$.
- $m_{s,t}=3$, $\langle\alpha_s,\alpha_t^\vee\rangle=\langle\alpha_t,\alpha_s^\vee\rangle=-1$.
- $m_{s,t}=4$, $\langle\alpha_s,\alpha_t^\vee\rangle=-1$, $\langle\alpha_t,\alpha_s^\vee\rangle=-2$.
- $m_{s,t}=6$, $\langle\alpha_s,\alpha_t^\vee\rangle=-1$, $\langle\alpha_t,\alpha_s^\vee\rangle=-3$.

We can check the assumption by direct calculations. ∎

The assumption is related to the existence of Jones–Wenzl projectors. If Assumtion 3.5 holds, then $\begin{bmatrix}m_{s,t}-1\\k\end{bmatrix}_Z$ is invertible by [8, (6.11), (6.12)] and Proposition 3.4 (2). By [8, Conj. 6.27] (this is now a theorem of Hazi [9]), the assumption implies the existence of the Jones–Wenzl projector $JW_{m_{s,t}-1}$. Moreover, Hazi proved that this condition is equivalent to the existence of rotatable Jones–Wenzl projector [9].

3.3 Soergel bimodules

For a graded R-bimodule $M=\bigoplus_{i\in\mathbb{Z}}M^i$ and $k\in\mathbb{Z}$, we define the grading shift $M(k)$ by $M(k)^i=M^{i+k}$.

We define a category $\mathcal{C}$ as follows. An object of $\mathcal{C}$ is $(M,(M_Q^x)_{x\in W})$, where:

- M is a graded R-bimodule.
- M_Q^x is a graded Q-bimodule such that $mp=x(p)m$ for $m\in M_Q^x$ and $p\in Q$.
- $M\otimes_R Q=\bigoplus_{x\in W}M_Q^x$ as graded (R,Q)-bimodules.
- There exist only finite $x\in W$ such that $M_Q^x\neq 0$.
- The R-bimodule M is flat as a right R-module.

By the third condition, $M\otimes_R Q$ is also a left Q-module [1, Rem. 2.2]. A morphism $(M,(M_Q^x))\to(N,(N_Q^x))$ is an R-bimodule homomorphism φ of degree zero such that

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$(\varphi\otimes \mathrm{id}_Q)(M_Q^x)\subset N_Q^x$ for any $x\in W$. Usually, we denote just M for $(M,(M_Q^x))$. For $M,N\in\mathcal{C}$, we define the tensor product $M\otimes N=(M\otimes_R N,((M\otimes N)_Q^x))$ by $(M\otimes N)_Q^x=\bigoplus_{yz=x}M_Q^y\otimes_Q M_Q^z$.

REMARK 3.8. The category introduced here is slightly different from the one in [1]. In [1], Q is the field of fractions of R and M_Q^x is a Q-bimodule which is not graded. However, Simon Riche pointed out that it is not clear (probably not true) that the bimodules M^x, M_x introduced in [1] are graded. The problem is solved with this modification and arguments in [1] work with this modification. In particular, one can define the category of Soergel bimodules inside $\mathcal{C}$ as in [1] and prove that this gives a categorification of the Hecke algebra. We also note that there is a natural fully faithful functor from $\mathcal{C}$ here to the category introduced in [1] (namely the category $\mathcal{C}$ defined with the field of fractions). Therefore, the category of Soergel bimodules defined as a full subcategory of $\mathcal{C}$ is equivalent to the category of Soergel bimodules defined in [1].

Let $\mathcal{C}_Q$ be the category consisting of objects $(P^x)_{x\in W}$, where P^x is a Q-bimodule such that $mp=x(p)m$ for $m\in P^x$, $p\in Q$ and there exists only finite $x\in W$ such that $P^x\neq 0$. A morphism $(P_1^x)\to(P_2^x)$ in $\mathcal{C}_Q$ is $(\phi_x)_{x\in W}$, where $\phi_x\colon P_1^x\to P_2^x$ is a Q-bimodule homomorphism. Obviously, $M\mapsto (M_Q^x)_{x\in W}$ is a functor $\mathcal{C}\to\mathcal{C}_Q$. We denote this functor by $M\mapsto M_Q$. Since $M\to M\otimes_R Q$ is injective, this functor is faithful. For $P_1=(P_1^x),P_2=(P_2^x)\in\mathcal{C}_Q$, we define $P_1\otimes P_2=((P_1\otimes P_2)^x)$ by $(P_1\otimes P_2)^x=\bigoplus_{yz=x}P_1^y\otimes_Q P_2^z$. We have $(M\otimes N)_Q=M_Q\otimes N_Q$.

For $x\in W$, we define $Q_x\in\mathcal{C}_Q$ by:

- $(Q_x)^x=Q$ as a left Q-module and the right action of $q\in Q$ is given by $m\cdot q=x(q)m$.
- $(Q_x)^y=0$ if $y\neq x$.

If M is in the category $\mathcal{S}$ defined in [1], then M_Q is isomorphic to a direct sum of Q_x's. We have $Q_x\otimes Q_y\simeq Q_{xy}$ via $f\otimes g\mapsto fx(g)$.

Let $u\in S$ and we put $R^u=\{f\in R\mid u(f)=f\}$, $B_u=R\otimes_{R^u}R(1)$. Then there exists a unique decomposition $B_u\otimes_R Q=(B_u)_Q^e\oplus(B_u)_Q^u$ as in the definition of the category $\mathcal{C}$. Explicitly, it is given by the following. Take $\delta_u\in V$ such that $\langle\alpha_u^\vee,\delta_u\rangle=1$. Then

$$(B_u)_Q^e=(\delta_u\otimes 1-1\otimes u(\delta_u))Q,$$
$$(B_u)_Q^u=(\delta_u\otimes 1-1\otimes\delta_u)Q.$$

Therefore $B_u\in\mathcal{C}$. We have $(B_u)_Q\simeq Q_e\oplus Q_s$ and an isomorphism is given by

$$f\otimes g\mapsto\left(\frac{fg}{\alpha_u},\frac{fu(g)}{\alpha_u}\right).$$

We always use this isomorphism to identify $(B_u)_Q$ with $Q_e\oplus Q_u$.

Let $M\in\mathcal{C}$ and consider $M\otimes B_u$. Then $(M\otimes B_u)_Q\simeq M_Q\otimes_Q Q_e\oplus M_Q\otimes_Q Q_u$. As a left Q-module, this is isomorphic to $M_Q\oplus M_Q$. The right action is given by $(m_1,m_2)p=(m_1p,m_2u(p))$ for $p\in Q$.

LEMMA 3.9. *Let $(m_1,m_2)\in M_Q\oplus M_Q$. Then $(m_1,m_2)\in M\otimes B_u$ if and only $m_1\alpha_u\in M$ and $m_1-m_2\in M$.*

Proof. Let $m\in M$, $p_1,p_2\in R$. Then the image of $m\otimes(p_1\otimes p_2)\in M\otimes B_u$ in $(M\otimes B_u)_Q\simeq M_Q\oplus M_Q$ is $(mp_1p_2\alpha_u^{-1},mp_1u(p_2)\alpha_u^{-1})$. Hence $(mp_1p_2\alpha_u^{-1})\alpha_u=mp_1p_2\in M$ and $(mp_1p_2\alpha_u^{-1})-(mp_1u(p_2)\alpha_u^{-1})=mp_1\partial_u(p_2)\in M$.

On the other hand, assume that $m_1\alpha_u \in M$ and $m_1 - m_2 \in M$. Take $\delta_u \in V$ such that $\langle \alpha_u^\vee, \delta_u\rangle = 1$. Then we have $u(\delta_u) = \delta_u - \alpha_u$. Hence the image of $(m_1\alpha_u)\otimes(1\otimes 1) + (m_2 - m_1)\otimes(\delta_u\otimes 1 - 1\otimes\delta_u) \in M\otimes B_u$ is $(m_1, m_1) + ((m_2-m_1)(\delta_u/\alpha_u), (m_2-m_1)(\delta_u/\alpha_u)) - ((m_2-m_1)(\delta_u/\alpha_u), (m_2-m_1)(u(\delta_u)/\alpha_u)) = (m_1, m_2)$. ∎

In general, for a sequence $\underline{w} = (s_1, s_2, \dots, s_l) \in S^l$ of elements in S, we put $B_{\underline{w}} = B_{s_1}\otimes\cdots\otimes B_{s_l}$. Set $b_{\underline{w}} = (1\otimes 1)\otimes\cdots\otimes(1\otimes 1)\in B_{\underline{w}}$. The main theorem of this paper is the following.

THEOREM 3.10. *Assume Assumption 3.5. There exists a morphism $\varphi\colon B_{\underline{x}} \to B_{\underline{y}}$ such that $\varphi(b_{\underline{x}}) = b_{\underline{y}}$.*

3.4 Localized calculus

Since $(B_u)_Q = (B_u)_Q^e \oplus (B_u)_Q^u \simeq Q_e \oplus Q_u$, for $\underline{w} = (s_1, \dots, s_l) \in S$, we have

$$(B_{\underline{w}})_Q \simeq \bigoplus_{e=(e_i)\in\{0,1\}^l} Q_{s_1^{e_1}}\otimes\cdots\otimes Q_{s_l^{e_l}} \simeq \bigoplus_{e\in\{0,1\}^l} Q_{\underline{w}^e}.$$

We call the component corresponding to e the e-component of $(B_{\underline{w}})_Q$. As an R-bimodule,

$$B_{\underline{w}} = (R\otimes_{R^{s_1}} R)\otimes_R (R\otimes_{R^{s_2}} R)\otimes_R\cdots\otimes_R (R\otimes_{R^{s_l}} R)(l) \simeq R\otimes_{R^{s_1}} R\otimes_{R^{s_2}}\cdots\otimes_{R^{s_l}} R(l).$$

The e-component of $p_0\otimes p_1\otimes\cdots\otimes p_l \in R\otimes_{R^{s_1}} R\otimes_{R^{s_2}}\cdots\otimes_{R^{s_l}} R(l)$ is

$$\left(\prod_{i=1}^{l} s_1^{e_1}\dots s_{i-1}^{e_{i-1}}\left(\frac{p_{i-1}}{\alpha_{s_i}}\right)\right) s_1^{e_1}\dots s_l^{e_l}(p_l).$$

We construct $\varphi\colon B_{\underline{x}} \to B_{\underline{y}}$ as follows. First, we define $\varphi_Q\colon (B_{\underline{x}})_Q \simeq \bigoplus_{e\in\{0,1\}^{m_{s,t}}} Q_{\underline{x}^e} \to \bigoplus_{f\in\{0,1\}^{m_{s,t}}} Q_{\underline{y}^f} \simeq (B_{\underline{y}})_Q$ explicitly and we will prove that φ_Q satisfies $\varphi_Q(B_{\underline{x}}) \subset B_{\underline{y}}$. The definition of φ_Q is given in [8, 2.6]. For $\underline{w} = (s_1, \dots, s_l) \in S^l$ and $e = (e_1, \dots, e_l) \in \{0,1\}^l$, we put $\zeta_{\underline{w}}(e) = \prod_{i=1}^{l} s_1^{e_1}\dots s_{i-1}^{e_{i-1}}(\alpha_{s_i})$. Then set

$$G_e^f = \begin{cases} \dfrac{\pi_{\underline{x}}}{\zeta_{\underline{y}}(f)} & (\underline{x}^e = \underline{y}^f),\\ 0 & (\underline{x}^e \neq \underline{y}^f).\end{cases}$$

Now, we define $\varphi_Q\colon \bigoplus_{e\in\{0,1\}^{m_{s,t}}} Q_{\underline{x}^e} \to \bigoplus_{f\in\{0,1\}^{m_{s,t}}} Q_{\underline{y}^f}$ by

$$\varphi_Q((q_e)) = \left(\sum_{e\in\{0,1\}^{m_{s,t}}} G_e^f q_e\right)_f = \left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{y}}(f)}\sum_{\underline{x}^e=\underline{y}^f} q_e\right)_f.$$

By the same way, we also define $\psi_Q\colon (B_{\underline{y}})_Q \to (B_{\underline{x}})_Q$. From the definition, we have

$$\varphi_Q(b_{\underline{x}}) = \left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{y}}(f)}\sum_{\underline{x}^e=\underline{y}^f}\prod_{i=1}^{m_{s,t}} s_1^{e_1}\dots s_{i-1}^{e_{i-1}}\left(\frac{1}{\alpha_{s_i}}\right)\right)_f.$$

Define $r\colon W \to \widetilde{W}$ as follows. If $w\in W$ is not the longest element, $r(w) = \tilde{s}_1\dots\tilde{s}_l$, where $w = s_1\dots s_l$ is the reduced expression of w. If w is the longest element, then $r(w) = \tilde{x}$. Then,

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for $e\in\{0,1\}^{m_{s,t}}$, $\underline{x}^e=g$ if and only if $\underline{\tilde{x}}^e=r(g)$. Therefore, we have

$$\varphi_Q(b_{\underline{x}})=\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{y}}(f)}a^{\underline{\tilde{x}}}(r(\underline{y}^f))\right).$$

PROPOSITION 3.11. *We have $\varphi_Q(b_{\underline{x}})=b_{\underline{y}}$ and $\psi_Q(b_{\underline{y}})=b_{\underline{x}}$ if and only if Assumption 3.5 holds.*

Proof. Set $\varepsilon(f)=1$ if $\tilde{s}r(\underline{y}^f)>r(\underline{y}^f)$ and $\varepsilon(f)=0$ otherwise. By Theorem 2.11 and Lemma 3.2, the f-component of $\varphi_Q(b_{\underline{x}})$ is

$$\frac{1}{\zeta_{\underline{y}}(f)}\begin{bmatrix}m_{s,t}-1\\ \left\lfloor\frac{m_{s,t}-\ell(\underline{y}^f)-\varepsilon(f)}{2}\right\rfloor\end{bmatrix}_{\sigma^{m_{s,t}-1}(X)}\left(\xi^{\varepsilon(f)}\prod_{i=1}^{\left\lfloor\frac{m_{s,t}-\ell(\underline{y}^f)-\varepsilon(f)}{2}\right\rfloor}[m_{s,t}-1]_{\sigma^{i-1}(X)}\right)^{-1}.$$

On the other hand, the f-component of $b_{\underline{y}}$ is $1/\zeta_{\underline{y}}(f)$. Therefore, $\varphi_Q(b_{\underline{x}})=b_{\underline{y}}$ if and only if

$$\begin{bmatrix}m_{s,t}-1\\ \left\lfloor\frac{m_{s,t}-\ell(\underline{y}^f)-\varepsilon(f)}{2}\right\rfloor\end{bmatrix}_{Z}=\xi^{\varepsilon(f)}\prod_{i=1}^{\left\lfloor\frac{m_{s,t}-\ell(\underline{y}^f)-\varepsilon(f)}{2}\right\rfloor}[m_{s,t}-1]_{\sigma^{i-1}(Z)},\tag{3.1}$$

for any $f\in\{0,1\}^l$ where $Z=\sigma^{m_{s,t}-1}(X)$. Here, we used $[m_{s,t}-1]_{\sigma^{i-m_{s,t}}(Z)}=[m_{s,t}-1]_{\sigma^{i-1}(Z)}$ which follows from Lemma 2.2(3). With $Z=\sigma^{m_{s,t}-1}(Y)$, we have another equation which is equivalent to $\psi(b_{\underline{y}})=b_{\underline{x}}$. Hence, $\varphi(b_{\underline{x}})=b_{\underline{y}}$ and $\psi(b_{\underline{y}})=b_{\underline{x}}$ if and only if (3.1) holds for any $f\in\{0,1\}^{m_{s,t}}$ and $Z\in\{X,Y\}$.

We assume that $\varphi_Q(b_{\underline{x}})=b_{\underline{y}}$ and $\psi_Q(b_{\underline{y}})=b_{\underline{x}}$. Set $f_k=(1^{m_{s,t}-k-1},0^{k+1})\in\{0,1\}^{m_{s,t}}$ for $0\le k\le m_{s,t}-1$. Then $\tilde{s}r(\underline{y}^{f_k})>r(\underline{y}^{f_k})$ and $\ell(\underline{y}^{f_k})=m_{s,t}-k-1$. Take $f=f_k$ in (3.1). Then we have $\begin{bmatrix}m_{s,t}-1\\ \lfloor k/2\rfloor\end{bmatrix}_Z=\xi\prod_{i=1}^{\lfloor k/2\rfloor}[m_{s,t}-1]_{\sigma^{i-1}(Z)}$. Let $k=0$. Then $\xi=1$. Therefore V is even-balanced. Hence, for any $0\le k\le m_{s,t}-1$, we have $\begin{bmatrix}m_{s,t}-1\\ \lfloor k/2\rfloor\end{bmatrix}_Z=\prod_{i=1}^{\lfloor k/2\rfloor}[m_{s,t}-1]_{\sigma^{i-1}(Z)}$. Therefore, we have Assumption 3.5.

On the other hand, assume Assumption 3.5. Then, by Proposition 3.4(3), the realization is even-balanced. Hence $\xi=1$. Therefore, (3.1) follows from Proposition 3.4(3). ∎

For $\underline{\tilde{w}}=(\tilde{s}_1,\dots,\tilde{s}_l)\in S^l$ and $c=(c_1,\dots,c_l)\in\{0,1\}^l$, we define the sequence $\underline{\tilde{w}}^{(c)}$ by removing i-th entry from $\underline{\tilde{w}}$ when $c_i=0$. For $u\in S$, we put $D_u^{(0)}=\partial_u$ and $D_u^{(1)}=u$.

LEMMA 3.12. *Let $\underline{\tilde{w}}=(\tilde{s}_1,\dots,\tilde{s}_l)\in\widetilde{S}^l$, $\tilde{g}\in\widetilde{W}$ and g the image of $\tilde{g}$ in W. For $p_1,\dots,p_l\in R$, we have*

$$\sum_{\underline{\tilde{w}}^e=\tilde{g}}\prod_{i=1}^{l}s_1^{e_1}\dots s_{i-1}^{e_{i-1}}\left(\frac{p_i}{\alpha_{s_i}}\right)=\sum_{c\in\{0,1\}^l}a^{\underline{\tilde{w}}^{(c)}}(\tilde{g})g(D_{s_l}^{(c_l)}(p_lD_{s_{l-1}}^{(c_{l-1})}(\dots(p_2D_{s_1}^{(c_1)}(p_1))\dots))).$$

Proof. We prove the lemma by induction on $l=\ell(\underline{\tilde{w}})$. Set $\underline{\tilde{v}}=(\tilde{s}_1,\dots,\tilde{s}_{l-1})$ and $p_{\underline{\tilde{v}}}^{(c)}=D_{s_{l-1}}^{(c_{l-1})}(p_{l-1}D_{s_{l-2}}^{(c_{l-2})}(\dots(p_2D_{s_1}^{(c_1)}(p_1))\dots))$. The $e_l=0$ part of the left-hand side in the lemma is

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$$g\left(\frac{p_l}{\alpha_{s_l}}\right)\sum_{e\in\{0,1\}^{l-1},\underline{\tilde{v}}^e=\tilde{g}}\prod_{i=1}^{l-1}s_1^{e_1}\dots s_{i-1}^{e_{i-1}}\left(\frac{p_i}{\alpha_{s_i}}\right)=g\left(\frac{p_l}{\alpha_{s_l}}\right)\sum_{c\in\{0,1\}^{l-1}}a^{\underline{\tilde{v}}^{(c)}}(\tilde{g})g(p_{\underline{\tilde{v}}}^{(c)})$$

by inductive hypothesis and similarly the $e_l=1$ part is

$$gs_l\left(\frac{p_l}{\alpha_{s_l}}\right)\sum_{c\in\{0,1\}^{l-1}}a^{\underline{\tilde{v}}^{(c)}}(\tilde{g}\tilde{s}_l)gs_l(p_{\underline{\tilde{v}}}^{(c)})=-g\left(\frac{s_l(p_l)}{\alpha_{s_l}}\right)\sum_{c\in\{0,1\}^{l-1}}a^{\underline{\tilde{v}}^{(c)}}(\tilde{g}\tilde{s}_l)gs_l(p_{\underline{\tilde{v}}}^{(c)}).$$

We have

$$\begin{aligned}
&g\left(\frac{p_l}{\alpha_{s_l}}\right)a^{\underline{\tilde{v}}^{(c)}}(\tilde{g})g(p_{\underline{\tilde{v}}}^{(c)})-g\left(\frac{s_l(p_l)}{\alpha_{s_l}}\right)a^{\underline{v}^{(c)}}(\tilde{g}\tilde{s}_l)gs_l(p_{\underline{\tilde{v}}}^{(c)})\\
&=a^{\underline{\tilde{v}}^{(c)}}(\tilde{g})g\left(\frac{p_lp_{\underline{\tilde{v}}}^{(c)}-s_l(p_lp_{\underline{\tilde{v}}}^{(c)})}{\alpha_{s_l}}\right)+\frac{a^{\underline{\tilde{v}}^{(c)}}(\tilde{g})-a^{\underline{\tilde{v}}^{(c)}}(\tilde{g}\tilde{s}_l)}{g(\alpha_{s_l})}gs_l(p_lp_{\underline{\tilde{v}}}^{(c)})\\
&=a^{\underline{\tilde{v}}^{(c)}}(\tilde{g})g(\partial_{s_l}(p_lp_{\underline{\tilde{v}}}^{(c)}))+a^{(\underline{\tilde{v}}^{(c)},s_l)}(\tilde{g})gs_l(p_lp_{\underline{\tilde{v}}}^{(c)}) && \text{by Lemma 2.12}\\
&=\sum_{d=0}^{1}a^{\underline{w}^{(c,d)}}(\tilde{g})g(D_{s_l}^{(d)}(p_lp_{\underline{\tilde{v}}}^{(c)})).
\end{aligned}$$

We get the lemma. ∎

Therefore, we get the following.

COROLLARY 3.13. *Take $s_1,\dots,s_{m_{s,t}}\in S$ such that $\underline{x}=(s_1,\dots,s_{m_{s,t}})$. For $p_1,\dots,p_{m_{s,t}}\in R$, $\varphi_Q(p_1\otimes p_2\otimes\cdots\otimes p_{m_{s,t}}\otimes 1)$ is given by*

$$\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{y}}(f)}\sum_{c\in\{0,1\}^{m_{s,t}}}a^{\underline{\tilde{x}}^{(c)}}(r(\underline{y}^f))\underline{y}^f(D_{s_{m_{s,t}}}^{(c_{m_{s,t}})}(p_{m_{s,t}}D_{s_{m_{s,t}-1}}^{(c_{m_{s,t}-1})}(\dots(p_2D_{s_1}^{(c_1)}(p_1))\dots)))\right)_f.$$

Hence to prove $\varphi_Q(B_{\underline{x}})\subset B_{\underline{y}}$, it is sufficient to prove that $((\pi_{\underline{x}}/\zeta_{\underline{y}}(f))a^{\underline{\tilde{x}}^{(c)}}(r(\underline{y}^f))\underline{y}^f(p))_f$ is in $B_{\underline{y}}$ for any $p\in R$. To proceed the induction, we formulate as follows.

LEMMA 3.14. *Assume Assumption 3.5. Let $p\in R$, $\underline{\tilde{w}}\in S^l$ and $\underline{\tilde{w}}'\in S^{l'}$ such that $l,l'\le m_{s,t}$. We assume that $l<m_{s,t}$ or $(\underline{\tilde{w}}',\underline{\tilde{w}})=(\underline{\tilde{x}},\underline{\tilde{y}})$. Then we have*

$$\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{w}}(f)}a^{\underline{\tilde{w}}'}(r(\underline{w}^f))\underline{w}^f(p)\right)_f\in B_{\underline{w}}.$$

Proof. We prove the lemma by induction on l. If $l=0$, then the lemma means $\pi_{\underline{x}}a^{\underline{\tilde{w}}'}(1)p\in R$. This is Lemma 3.3.

Take $\tilde{s}_1,\dots,\tilde{s}_l\in\widetilde{S}$ such that $\underline{\tilde{w}}=(\tilde{s}_1,\dots,\tilde{s}_l)$. Put $a(g)=a^{\underline{\tilde{w}}'}(g)$ and $\underline{\tilde{v}}=(\tilde{s}_1,\dots,\tilde{s}_{l-1})$. Then, by Lemma 3.9, it is sufficient to prove

$$\left(\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{w}}((f',0))}a(r(\underline{w}^{(f',0)}))\underline{w}^{(f',0)}(p)\right)\cdot\alpha_{s_l}\right)_{f'\in\{0,1\}^{l-1}}\in B_{\underline{v}}\tag{3.2}$$

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and

$$\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{w}}((f',0))}a(r(\underline{w}^{(f',0)}))\underline{w}^{(f',0)}(p)-\frac{\pi_{\underline{x}}}{\zeta_{\underline{w}}((f',1))}a(r(\underline{w}^{(f',1)}))\underline{w}^{(f',1)}(p)\right)_{f'\in\{0,1\}^{l-1}}\in B_{\underline{v}}. \tag{3.3}$$

We have

$$\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{w}}((f',0))}a(r(\underline{w}^{(f',0)}))\underline{w}^{(f',0)}(p)\right)\cdot\alpha_{s_l}=\underline{v}^{f'}(\alpha_{s_l})\frac{\pi_{\underline{x}}}{\zeta_{\underline{w}}((f',0))}a(r(\underline{w}^{(f',0)}))\underline{w}^{(f',0)}(p)$$

and by the definition of $\zeta_{\underline{w}}((f',0))$, we have $\underline{v}^{f'}(\alpha_{s_l})/\zeta_{\underline{w}}(f',0)=1/\zeta_{\underline{v}}(f')$. We also have $\underline{w}^{(f',0)}=\underline{v}^{f'}$. Hence, the left-hand side of (3.2) is

$$\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{v}}(f')}a(r(\underline{v}^{f'}))\underline{v}^{f'}(p)\right)_{f'\in\{0,1\}^{l-1}},$$

which is in $B_{\underline{v}}$ by inductive hypothesis.

Put $g=\underline{v}^{f'}$. Then $\underline{w}^{(f',0)}=g$ and $\underline{w}^{(f',1)}=gs_l$. Since $\zeta_{\underline{w}}((f',0))=\zeta_{\underline{w}}((f',1))=\underline{v}^{f'}(\alpha_{s_l})\zeta_{\underline{v}}(f')$, the f'-component of the left-hand side of (3.3) is

$$\begin{aligned}
&\frac{\pi_{\underline{x}}}{\zeta_{\underline{v}}(f')}\frac{1}{g(\alpha_{s_l})}(a(r(g))g(p)-a(r(gs_l))gs_l(p))\\
&=\frac{\pi_{\underline{x}}}{\zeta_{\underline{v}}(f')}\left(a(r(g))g\left(\frac{p-s_l(p)}{\alpha_{s_l}}\right)+\frac{a(r(g))-a(r(gs_l))}{g(\alpha_{s_l})}gs_l(p)\right)\\
&=\frac{\pi_{\underline{x}}}{\zeta_{\underline{v}}(f')}\left(a(r(g))g(\partial_{s_l}(p))+\frac{a(r(g))-a(r(gs_l))}{g(\alpha_{s_l})}gs_l(p)\right).
\end{aligned}$$

We prove that

$$\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{v}}(f')}a(r(\underline{v}^{f'}))\underline{v}^{f'}(\partial_{s_l}(p))\right)_{f'},\quad\left(\frac{\pi_{\underline{x}}}{\zeta_{\underline{v}}(f')}\frac{a(r(\underline{v}^{f'}))-a(r(\underline{v}^{f'}s_l))}{\underline{v}^{f'}(\alpha_{s_l})}\underline{v}^{f'}(s_l(p))\right)_{f'} \tag{3.4}$$

are in $B_{\underline{v}}$. The first one is in $B_{\underline{v}}$ by inductive hypothesis.

For the second, we divide into two cases.

- First, assume that $l<m_{s,t}$. Then $\ell(\underline{v}^{f'})+\ell(s_l)<m_{s,t}$. Hence $r(\underline{v}^{f'}s_l)=\tilde{\underline{v}}^{f'}\tilde{s}_l$. Therefore, by Lemma 2.12, we have $(a(r(\underline{v}^{f'}))-a(r(\underline{v}^{f'}s_l)))/\underline{v}^{f'}(\alpha_{s_l})=a^{(\tilde{\underline{w}}',\tilde{s}_l)}(r(\underline{v}^{f'}))$. Therefore if $l'<m_{s,t}$ then the second one of (3.4) is in $B_{\underline{v}}$ by inductive hypothesis.If $l'=m_{s,t}$, we have $\ell(\tilde{\underline{w}}',\tilde{s}_l)=m_{s,t}+1$. We also have $\ell(r(\underline{v}^{f'}))\le\ell(\underline{v})=l-1\le m_{s,t}-2=\ell(\tilde{\underline{w}}',\tilde{s}_l)-3$. Hence $\ell(\tilde{\underline{w}}',\tilde{s}_l)-\ell(r(\underline{v}^{f'}))\ge 3$. By Theorem 2.11 and Assumption 3.5, $a^{(\tilde{\underline{w}}',\tilde{s}_l)}(r(\underline{v}^{f'}))=0$. Hence the second one of (3.4) is zero which is in $B_{\underline{v}}$.
- Next assume that $l=m_{s,t}$. Then we have $\tilde{\underline{w}}'=\tilde{\underline{x}}$ and $\tilde{\underline{w}}=\tilde{\underline{y}}$. In this case we prove that $a(r(\underline{v}^{f'}))=a(r(\underline{v}^{f'}s_l))$.If $f'\neq(1,\dots,1)$, then the calculation in the case of $l<m_{s,t}$ is still valid. Hence $(a(r(\underline{v}^{f'}))-a(r(\underline{v}^{f'}s_l)))/\underline{v}^{f'}(\alpha_{s_l})=a^{(\tilde{\underline{x}},\tilde{s}_l)}(r(\underline{v}^{f'}))$. We have $\ell((\tilde{\underline{x}},\tilde{s}_l))=m_{s,t}+1$ and, since $f'\neq(1,\dots,1)$, we have $\ell(r(\underline{v}^{f'}))\le m_{s,t}-2$. Therefore $\ell((\tilde{\underline{x}},\tilde{s}_l))-\ell(r(\underline{v}^{f'}))\ge 3$. By Theorem 2.11 and Assumption 3.5, we have $a^{(\tilde{\underline{x}},\tilde{s}_l)}(r(\underline{v}^{f'}))=0$.We assume that $f'=(1,\dots,1)$. By the definition, $r(\underline{v}^{f'}s_l)=\tilde{x}$. Hence $a^{\tilde{\underline{x}}}(\tilde{x})=1/\pi_{\underline{x}}$ by Theorem 2.11

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and Lemma 3.2. We have $\ell(r(\underline{v}^{f'})) = m_{s,t} - 1 = \ell(\tilde{x}) - 1$. Therefore by Theorem 2.11 and Lemma 3.2, we have $a^{\tilde{\underline{x}}}(r(\underline{v}^{f'})) = 1/\pi_{\underline{x}}$ as $\xi = 1$.

We finish the proof. ∎

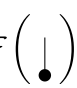
Theorem 3.10 is proved.

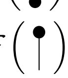
3.5 Relation with the diagrammatic Hecke category

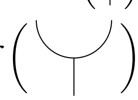
In this subsection assume that $\mathbb{K}$ is a Noetherian integral domain. Let (W, S) be a general Coxeter system such that $\#S < \infty$ (we allow $\#S \neq 2$) and $(V, \{\alpha_u\}_{u\in S}, \{\alpha_u^\vee\}_{u\in S})$ a realization. We assume the following version of Demazure surjectivity: $\alpha_s \neq 0$ and $\alpha_s^\vee \colon V \to \mathbb{K}$ is surjective for any $s \in S$. We also assume that for any $u_1, u_2 \in S$ ($u_1 \neq u_2$) such that the order m_{u_1,u_2} of $u_1 u_2$ is finite, we have $\left[{m_{u_1,u_2} \atop k}\right]_Z = 0$ for any $Z \in \{X, Y\}$ and $1 \le k \le m_{u_1,u_2} - 1$. We can define the category $\mathcal{C}, \mathcal{C}_Q$ by the same way as in §3.3. Let $\mathcal{BS}$ be the full subcategory of $\mathcal{C}$ consisting of objects of the form $B_{s_1} \otimes \cdots \otimes B_{s_l}(n)$. If $u_1, u_2 \in S$, $u_1 \neq u_2$ satisfies $m_{u_1,u_2} < \infty$, then we put $B_{u_1,u_2} = \overbrace{B_{u_1} \otimes B_{u_2} \otimes \cdots}^{m_{u_1,u_2}}$ and $B_{u_2,u_1} = \overbrace{B_{u_2} \otimes B_{u_1} \otimes \cdots}^{m_{u_1,u_2}}$. By Theorem 3.10 there exists a homomorphism $\varphi_{u_1,u_2} \colon B_{u_1,u_2} \to B_{u_2,u_1}$ which sends $(1 \otimes 1) \otimes (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)$ to $(1 \otimes 1) \otimes (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)$.

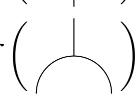
Let $\mathcal{D}$ be the diagrammatic Hecke category defined by Elias–Williamson [7]. Note that this is "well-defined" [9] in the sense of [8, 5.1].

We define a functor $\mathcal{F} \colon \mathcal{D} \to \mathcal{BS}$ as follows. For an object $(s_1, \ldots, s_l) \in \mathcal{D}$, we define $\mathcal{F}(s_1, \ldots, s_l) = B_{s_1} \otimes \cdots \otimes B_{s_l}$. We define $\mathcal{F}$ on morphisms by

$\mathcal{F}$  $= (p \mapsto p\delta_u \otimes 1 - p \otimes u(\delta_u))$,

$\mathcal{F}$  $= (p_1 \otimes p_2 \mapsto p_1 p_2)$,

$\mathcal{F}$  $= (p_1 \otimes p_2 \mapsto p_1 \otimes 1 \otimes p_2)$,

$\mathcal{F}$  $= (p_1 \otimes p_2 \otimes p_3 \mapsto p_1 \partial_u(p_2) \otimes p_3)$,

$$\mathcal{F}(2m_{u_1,u_2} - \text{valent vertex}) = \varphi_{u_1,u_2}.$$

for $u, u_1, u_2 \in S$ and $p, p_1, p_2, p_3 \in R$. Here we regard $B_u \otimes B_u = R \otimes_{R^u} R \otimes_{R^u} R(2)$ and $\delta_u \in V$ is an element satisfying $\langle \alpha_u^\vee, \delta_u \rangle = 1$.

LEMMA 3.15. *The functor $\mathcal{F}$ is well-defined.*

Proof. In [8], a functor $\Lambda \colon \mathcal{D} \to \mathcal{C}_Q$ is defined and it is proved that Λ is well-defined. By the construction, we have $\Lambda = (\cdot)_Q \circ \mathcal{F}$. Therefore $(\cdot)_Q \circ \mathcal{F}$ is well-defined and since $(\cdot)_Q \colon \mathcal{BS} \to \mathcal{C}_Q$ is faithful, $\mathcal{F}$ is also well-defined. ∎

THEOREM 3.16. *The functor $\mathcal{F} \colon \mathcal{D} \to \mathcal{BS}$ gives an equivalence of categories.*

Proof. The proof is the same as that of the corresponding theorem in [1]. It is obviously essentially surjective. In [7], for each object $M, N \in \mathcal{D}$, elements in $\mathrm{Hom}_{\mathcal{D}}(M, N)$ called double leaves are defined and it is proved that they form a basis of $\mathrm{Hom}_{\mathcal{D}}(M, N)$ [7, Th. 6.12]. In [1], the corresponding statement in $\mathcal{BS}$ is proved, namely double leaves in

https://doi.org/10.1017/nmj.2023.38 Published online by Cambridge University Press

$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(M), \mathcal{F}(N))$ are defined and it is proved that they form a basis. By the definition of $\mathcal{F}$, $\mathcal{F}$ sends double leaves to double leaves. Hence $\mathcal{F}$ gives an isomorphism between morphism spaces. ∎

Acknowledgments. The author thanks Syu Kato for giving many helpful comments and Simon Riche for pointing out an error in the previous paper.

Funding. The author was supported by JSPS KAKENHI (Grant Number 18H01107).

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