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THE FOURIER TRANSFORMS OF SMOOTH MEASURES ON HYPERSURFACES OF $\mathbf{R}^{n+1}$

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1. Introduction. The Fourier transform of the surface measure on the unit sphere in $\mathbf{R}^{n+1}$, as is well-known, equals the Bessel function

$$(2\pi)^{(n+1)/2}J_\nu(|\xi|)|\xi|^{-\nu}, \quad \nu = (n-1)/2.$$

Its behaviour at infinity is described by an asymptotic expansion

$$\int_{|x'|=1} e^{-i\xi\cdot x'}dx' = 2(2\pi)^{n/2}\cos\left(|\xi| - \frac{n\pi}{4}\right)|\xi|^{-n/2} + O(|\xi|^{-(n+2)/2}).$$

The purpose of this paper is to obtain such an expression for surfaces Σ other than the unit sphere. If the surface Σ is a sufficiently smooth compact n-surface in $\mathbf{R}^{n+1}$ with strictly positive Gaussian curvature everywhere then with only minor changes in the main term, such an asymptotic expansion exists. This result was proved by E. Hlawka in [**3**]. A similar result concerned with the minimal smoothness of Σ was later obtained by C. Herz [**2**].

Our focus therefore is on surfaces with vanishing curvature. In this case there are the estimates of W. Littman in [**4**]. He showed that if k of the principal curvatures of Σ are bounded away from zero then

$$|\hat{d\mu}(\xi)| = \left|\int_\Sigma e^{-i\xi\cdot x'}d\mu(x')\right| \leqq C(1 + |\xi|)^{-k/2}.$$

More delicate estimates are obtained in [**6**], [**7**] and [**8**]. Their results show that for Σ convex and C the interior of Σ, the radial maximal function

$$U(\xi') = \sup_{0<r<\infty} r^{(n+2)/2}|\hat{\chi}_C(r\xi')|$$

is an L^p function on the unit sphere S^n for some p depending on Σ. In particular Svensson proved that if Σ is smooth, convex and has no tangent of infinite order then U is in L^p if and only if

$$\int_\Sigma \kappa(x)^{(2-p)/2}dx < \infty$$

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where $\kappa(x)$ is the Gaussian curvature at $x \in \Sigma$.

Our goal in studying $\widehat{d\mu}$ is to prove L^p estimates for solutions of certain hyperbolic equations. To obtain the best estimates it is essential to be able to isolate the dominant term in the asymptotic expansion of $\widehat{d\mu}$. It will be seen in [**5**] that estimates on simply the radial maximal function do not yield the best range of L^p spaces. Our results apply to both convex and nonconvex surfaces but we assume that the curvature vanishes in a relatively simple way.

Consider a point x_0 on the surface Σ such that the curvature at x_0 is zero. We will assume that after some translation and orthogonal change of coordinates the surface near x_0 is of the form $x = (y, f(y))$ where $y \in \mathbf{R}^n$ and f is a smooth real-valued function such that $f(0) = 0$ and $\nabla f(0) = 0$. We will assume that f is of the form

$$f(y) = P(y) + h(y).$$

$\mathbf{R}^n$ is the orthogonal direct sum of subspaces $V_1, \dots, V_s$. Let $\pi_1, \dots, \pi_s$ be the corresponding orthogonal projections. P is of the form

$$P(y) = \sum_{j=1}^{s} P_j(\pi_j y)$$

where each of the polynomials P_j is a homogeneous function of $n_j = \dim V_j$ variables, and P is nondegenerate in the sense that for every j,

$$\det d^2 P_j(y) = 0$$

only when $\pi_j y = 0$. Here $d^2 P_j$ is the matrix of second order derivatives of P_j. Fix an orthogonal system of coordinates so that

$$P(y) = P_1(y_1, \dots, y_{j_1}) + P_2(y_{j_1 - 1}, \dots, y_{j_2}) + \dots + P_s(\dots, y_n).$$

If P_m is homogeneous of degree k_m define

$$k_j' = k_m \quad \text{if } j_{m-1} < j \leqq j_m \quad (j_0 = 0, j_s = n)$$

and

$$\alpha = (\alpha_1, \dots, \alpha_n) = (1/k_1', \dots, 1/k_n'), \quad |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

Also assume that $h(y)$ contains only higher order terms; that is,

$$D^\beta h(y) = \left(\frac{\partial}{\partial y_1}\right)^{\beta_1} \dots \left(\frac{\partial}{\partial y_n}\right)^{\beta_n} h(y) \equiv 0$$

if $\beta \notin B$ where

$$B = \{\beta\text{: for every } j = 1, \dots, s\ \pi_j \beta = 0 \text{ or } |\pi_j \beta| \geqq k_j.$$

Also for some j, $|\pi_j\beta| > k_j\}$.

We will describe a critical point x_0 satisfying all these conditions as being of type α, and f is a function of type α.

If f is a function of type α then the Gaussian curvatures of the surfaces $y_{n+1} = f(y)$ and $y_{n+1} = P(y)$ vanish at the same points because

$$\Gamma = \{y:\det d^2 f(y) = 0\} = \{y:\det d^2 P(y) = 0\}.$$

A surface will be called type a if every point x' of the surface is of type $\alpha = \alpha(x')$ for some α and

$$a = \min\{\,|\alpha(x')\,|:x' \in \Sigma\} > 0.$$

A surface is of positive type if such a constant a exists.

If the Gaussian curvature does not vanish then every point is of type $(\frac{1}{2}, \ldots, \frac{1}{2})$ and $a = n/2$.

If $2 \leqq k_1 \leqq \ldots k_{n+1}$ are even positive integers and

$$\Sigma = \{y:y_1^{k_1} + \ldots + y_{n+1}^{k_{n+1}} = 1\}$$

then

$$a = \sum_{j=2}^{n+1} 1/k_j.$$

The function

$$f(y_1, y_2) = y_1^3 - y_1 y_2^2$$

is of type $(\frac{1}{3}, \frac{1}{3})$ but

$$f(y_1, y_2) = y_1^3 + y_1 y_2^2$$

is not.

Define

$$A'(\xi) = \{x' \in \Sigma\text{: the tangent at } x' \text{ is perpendicular to } \xi\}.$$

THEOREM 1. *Suppose that Σ is a compact convex n-dimensional C^∞ submanifold of $\mathbf{R}^{n+1}$ of type a, that $d\sigma$ is surface area on Σ, $g \in C^\infty(\Sigma)$, and $d\mu = gd\sigma$. Suppose that for every ξ, $A'(\xi)$ is a finite set. Then there exists a constant C depending only on Σ and g such that*

$$|\widehat{d\mu}(\xi)\,| \leqq C(1 + |\xi|\,)^{-a} \quad \textit{for all } \xi \in \mathbf{R}^{n+1}.$$

For each $\xi \in \mathbf{R}^{n+1}$, the main part of

$$\widehat{d\mu}(\xi) = \int_\Sigma e^{-i\xi\cdot x'} g(x')d\sigma(x')$$

comes from the points in $A'(\xi)$.

Let $\kappa(x')$ be the absolute Gaussian curvature at x'. The principal part of $\widehat{d\mu}(\xi)$ is

$$\mathscr{I}(\xi) = \sum_{x' \in A'(\xi)} \frac{C_0(x')g(x')}{(\kappa(x'))^{1/2}} e^{-ix'\cdot\xi}$$

where $C_0(x')$ is constant in the components of the set $\{x' \in \Sigma: \kappa(x') \neq 0\}$. Suppose that after a translation and an orthogonal change of coordinates in $\mathbf{R}^{n+1}$ the point x' on Σ is mapped into the origin in $\mathbf{R}^{n+1}$ and the normal vector u at x' that points in the direction of ξ is mapped into $(0, -1) \in \mathbf{R}^n \times \mathbf{R}$. The surface is now given by an equation $y_{n+1} = f(y)$ where $y \in \mathbf{R}^n$. Define ν to be the number of positive eigenvalues of the matrix $d^2f(0)$ minus the number of negative eigenvalues. Then

$$C_0(x') = (2\pi)^{n/2}e^{i\nu\pi/4}.$$

For example, if Σ is the unit sphere $A'(\xi)$ contains two points, $\xi/|\xi|$ and $-\xi/|\xi|$. The corresponding values of ν are n and $-n$ respectively. Therefore

$$\mathscr{I}(\xi) = (2\pi)^{n/2}\{e^{in\pi/4}e^{-i|\xi|} + e^{-in\pi/4}e^{i|\xi|}\}$$
$$= 2(2\pi)^{n/2} \cos\left(|\xi| - \frac{n\pi}{4}\right).$$

If f is of type α then the restrictions imposed on the higher order terms imply that the curvatures of f and P vanish at the same points. This means that

$$\Gamma = \{y: \det d^2f(y) = 0\}$$

is the union of a finite number of linear subspaces $\Gamma_1, \ldots, \Gamma_t$. The subspaces Γ_m are the orthogonal complements of the spaces V_j such that $k_j > 2$. If Γ_m and V_j are orthogonal complements then define

$$\tau_m = n_j/(k_j - 1) \quad \text{where } n_j = \dim V_j.$$

Let

$$\tau = \min\{n_j/(k_j - 1): P_j \text{ is not convex}\}.$$

If every P_j is convex then set $\tau = \infty$. The parameter τ gives an indication of the type of inflection points present on the surface. For example, if $2 \leqq k_1 \leqq \ldots \leqq k_n$,

$$k_m = \max\{k_j: k_j \text{ is odd}\}$$

and

$$f(y) = \pm y_1^{k_1} \pm \ldots \pm y_n^{k_n} \quad k_n > 2$$

then $\tau = 1/(k_m - 1)$. Also let $n_* = \min\{n_j: k_j > 2\}$.

THEOREM 2. *Suppose that Σ is a compact n-dimensional C^∞ submanifold of $\mathbf{R}^{n+1}$ of positive type, that $d\sigma$ is surface area on Σ, $g \in C^\infty(\Sigma)$, and*

$d\mu = g d\sigma$. Assume also that for every ξ the set $A'(\xi)$ is a finite set, and $\tau > 1$.

If $n_* > 1$ then there exists a function $h_*(\xi)$ such that

$$\hat{d\mu}(\xi) = |\xi|^{-n/2}\mathscr{I}(\xi) + h_*(\xi)$$

where

$$r^{-n}\int_{|\xi|=r} |h_*(\xi)|\,d\xi \leqq Cr^{-(n+1)/2} \quad \textit{for all } r \geqq 1.$$

If $n_* = 1$ then the L^1 norm of h_* is replaced by the weak L^1 norm:

$$\sigma_r\{\xi: r^{(n+1)/2}|h_*(\xi)| > \lambda\} \leqq C/\lambda \quad \lambda > 0$$

where σ_r is the uniform probability measure on $\{\,|\xi| = r\}$.

THEOREM 3. *Let Σ and $d\mu$ be as in Theorem 2, except that $0 < \tau \leqq 1$. Then for every $p < \tau$,*

$$r^{-n}\int_{|\xi|=r} |h_*(\xi)|\,d\xi \leqq Cr^{-(n+p)/2} \quad \textit{for all } r \geqq 1.$$

The weaker results in Theorem 3 are caused by the inflection points ($\tau \leqq 1$).

If Σ is orientable then $\mathscr{I}(\xi)$ is closely related to the Gauss map, which maps each point $x' \in \Sigma$ to its outward unit normal vector. In fact, $A'(\xi)$ is the inverse image of the point set $\{\xi, -\xi\}$. Therefore

$$\frac{1}{2}\int_{|\xi|=1} \sum_{x' \in A'(\xi)} \frac{1}{|\kappa(x')|}\,d\xi$$

equals the surface area of Σ. As a result, by the Cauchy-Schwarz inequality, for all $r \geqq 1$

$$\left| r^{-n}\int_{|\xi|=r} \mathscr{I}(\xi)\,d\xi \right| \leqq C\sqrt{\text{Area of } \Sigma} \leqq C.$$

COROLLARY. *If Σ and $d\mu$ are as in Theorem 2 or 3 then there exists a constant C such that*

$$r^{-n}\int_{|\xi|=r} |\hat{d\mu}(\xi)|\,d\xi \leqq Cr^{-n/2} \quad \textit{for } r \geqq 1.$$

Since the main term $\mathscr{I}(\xi)$ is singular where the curvature $\kappa(x')$ vanishes, Theorem 1 is not a consequence of Theorems 2 or 3. The estimate of Theorem 1 is appropriate in directions ξ where the curvature $\kappa(x')$ is zero but in the other directions the decay rate of $\hat{d\mu}$ is $C|\xi|^{-n/2}$.

These estimates are useful in applications. Previously, in describing the behaviour of solutions of hyperbolic partial differential equations one used estimates of the form

$$|\hat{d\mu}(\xi)| \leqq C|\xi|^{-a}$$

as, for example, in [**1**]. The point of Theorems 2, 3 and their corollary, however, is that from the point of view of spherical averages the decay rate of $\widehat{d\mu}$ is like $C|\xi|^{-n/2}$. In this sense the decay of $\widehat{d\mu}$ is the same whether or not the curvature of Σ vanishes. This can be seen similarly in the results of [**6**], [**7**] and [**8**]. In the L^p estimates for wave equations, $\widehat{d\mu}$ is placed into another oscillatory integral and estimated. Since polar coordinates and integration by parts in the radial direction are used, the natural way to approximate $\widehat{d\mu}$ is in terms of averages over spheres, as in Theorems 2 and 3.

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2. Summary. The three theorems will be proven together. This section contains an outline of the whole proof.

The first step is to reduce $\widehat{d\mu}(\xi)$ to an integral on $\mathbf{R}^n$. Fix $\xi_0 \in \mathbf{R}^{n+1}$. The main part of $\widehat{d\mu}(\xi_0)$ comes from the points of $A'(\xi_0)$. If g is a C^∞ function on Σ that is supported away from $A'(\xi_0)$ then $\widehat{d\mu}(\xi_0)$ can be put in the form

$$\widehat{d\mu}(\xi_0) = \int_\Sigma e^{-ix'\cdot\xi_0} g(x')d\mu(x') = \int_{-\infty}^{\infty} e^{-i|\xi_0|s}\tilde{g}(\xi_0', s)ds$$

where $\xi_0' = \xi_0/|\xi_0|$ and $\tilde{g}(\xi_0', s)$ is a C^∞ function of s whose derivatives depend smoothly on ξ_0'. Therefore in this case

$$|\widehat{d\mu}(\xi_0)| \leqq C_N(1 + |\xi_0|)^{-N} \quad \text{for every } N \geqq 0.$$

As a result, by using a C^∞ partition of unity on Σ, we may assume that g is supported in a small neighborhood of a fixed point $x_0 \in A'(\xi_0)$. The set $A'(\xi_0)$ is finite by assumption.

Make a translation and an orthogonal change of coordinates so that $x_0 = (0, 0)$ and $\xi_0 = (0, -1) \in \mathbf{R}^n \times \mathbf{R}$. The surface near x_0 is of the form $x = (y, f(y))$ where $y \in \mathbf{R}^n$ and $f(y) \in \mathbf{R}$. Assume that g is supported in this small neighborhood of x_0. Suppose also that $\xi = R(\theta, -1)$ where $\theta \in \mathbf{R}^n$, $R \geqq 0$. R is not quite the modulus of ξ but $|\xi|/R \to 1$ as $|\theta| \to 0$. Now $\widehat{d\mu}(\xi)$ becomes

$$(1) \quad \widehat{d\mu}(\xi) = \int_\Sigma e^{-ix'\cdot\xi} g(x')d\mu(x') = \int_{\mathbf{R}^n} e^{-iR\varphi(y)}\tilde{g}(y)dy$$

where $\tilde{g}$ is a C^∞ function supported in a neighborhood of the origin and

$$\varphi(y) = \theta \cdot y - f(y).$$

f if a function of type α, as described in the introduction.

If $\nabla P_j(y) = 0$ for some point y such that $\pi_j y \neq 0$ then by the homogeneity $\nabla P_j(y) = 0$ on a ray in V_j. This means that

$$\det d^2P(y) = 0$$

on this ray. Therefore since $\det d^2P(y) = 0$ implies $\pi_j y = 0$, the gradient

$\nabla P_j(y)$ can vanish only when $\pi_j y = 0$. Hence

$$|\nabla P_j(y)| \geqq C|\pi_j y|^{k_j - 1}.$$

Also if $\beta \in B$ then either $|\nabla_j y^\beta| = 0$ or

$$|\nabla_j y^\beta| \leqq C|y|\,|\pi_j y|^{k_j - 1} \quad \text{where } \nabla_j = \pi_j \nabla.$$

Thus for y in a sufficiently small neighborhood of the origin

$$(2) \qquad |\nabla_j h(y)| \leqq \tfrac{1}{2}|\nabla_j P(y)| \quad j = 1, \dots, s.$$

Suppose that $j_{m-1} < j < j_m$. Then

$$|D_i D_j h(y)| \leqq C|y|\,|\pi_m y|^{k_m - 2} \leqq C|y|\,|\det d^2 P_m(y)|.$$

Use the formula expressing a determinant as a sum over permutations. It follows that

$$|\det d^2 h(y)| \leqq C|y|^n \prod_{m=1}^{s} |\det d^2 P_m(y)|^{n_m} \leqq C|y|^n |\det d^2 P(y)|.$$

Similarly,

$$|\det d^2 f(y) - \det d^2 P(y)| \leqq C|y|\,|\det d^2 P(y)|.$$

So for $|y|$ small enough,

$$(3) \qquad |\det d^2 f(y) - \det d^2 P(y)| \leqq \tfrac{1}{2}|\det d^2 P(y)|.$$

We will assume that the support of $\tilde{g}$ is so small that both (2) and (3) hold, and also the estimates of Lemmas 1 and 2 in Section 3 are true.

In estimating the oscillatory integral

$$\widehat{d\mu}(\xi_0) = \int_{\mathbf{R}^n} e^{-iR\varphi(y)} \tilde{g}(y) dy$$

it is natural to first look at the part of $\mathbf{R}^n$ where there is a great deal of cancellation; that is, the set of points y such that $|\nabla \varphi(y)|$ is large. In Section 3 we prove a number of results in a set E_1^c, where $|\nabla \varphi(y)|$ is large. The corresponding results where $|\nabla \varphi|$ is small will be proven in Sections 4 and 5.

Let

$$\alpha = (\alpha_1, \dots, \alpha_n) = (1/k_1', \dots, 1/k_n'),$$

$$|\alpha| = \sum_{j=1}^{n} |\alpha_j|, \quad \text{and}$$

$$\langle \theta \rangle = \sum_{j=1}^{s} |\pi_j \theta|^{k_j/(k_j - 1)}.$$

Also let $b = \max\{R^{-1}, \langle \theta \rangle\}$.

Constants $C_1, \ldots, C_s$ will be chosen so that whenever y is not in the set

$$E_1 = \{y:|\pi_j y| \leqq C_j b^{1/k_j}, j = 1, \ldots, s\}$$

then

$$\nabla f(y) \notin \{w:|\pi_j w| \leqq 2b^{(k_j-1)/k_j}, j = 1, \ldots, s\}.$$

By assumption, if $\nabla P_1(y) = 0$ then $y_1 = y_2 = \ldots = y_{j_1} = 0$ and ∇P_1 is homogeneous of degree $(k_1 - 1)$. Therefore there exists a positive constant C_1' such that

$$|\nabla P_1(y)| \geqq C_1'(|y_1|^2 + \ldots + |y_{j_1}|^2)^{(k_1-1)/2} = C_1'|\pi_1 y|^{k_1-1}.$$

Define $\nabla_j = \pi_j \nabla$. Thus, for example,

$$\nabla_1 f(y) = (D_1 f(y), \ldots, D_{j_1} f(y), 0, \ldots, 0).$$

Since $|\nabla_1 h(y)| \leqq \frac{1}{2}|\nabla P_1(y)|$ then

(4) $\quad |\nabla_1 f(y)| \geqq \frac{1}{2}C_j'|\pi_j y|^{k_j-1}.$

Define C_j' similarly for all $j = 2, \ldots, s$, by considering each V_m separately. Now if

$$\nabla f(y) \in \{w:|\pi_j w| \leqq 2b^{(k_j-1)/k_j}, j = 1, \ldots, s\}$$

then

$$n_j 2b^{(k_j-1)/k_j} \geqq \tfrac{1}{2}C_j'|\pi_j y|^{k_j-1}.$$

Hence

(5) $\quad |\pi_j y| \geqq 2nb^{1/k_j}/\tfrac{1}{2}C_j' \quad$ for $j = 1, \ldots, s$.

Therefore define

$$C_j = 4n/C_j', \quad j = 1, \ldots, s.$$

This shows that in fact $y \in E_1$. Because

$$|\pi_j \theta| \leqq b^{(k_j-1)/k_j} \quad \text{for } j = 1, \ldots, s$$

then if $y \notin E_1$,

$$\nabla \varphi(y) = \theta - \nabla f(y) \notin \{w:|w_j| \leqq b^{(k_j-1)/k_j}, j = 1, \ldots, s\}.$$

Therefore outside E_1 the oscillation is large in the sense that

$$\langle \nabla \varphi(y) \rangle \geqq b = \max\{R^{-1}, \langle \theta \rangle\}.$$

A C^∞ function ψ_0 with compact support can be chosen so that ψ_0 approximates the characteristic function of E_1 in the following way: $\psi_0(y) = 1$ for $y \in E_1$, $\psi_0(y) = 0$ for $y \notin 2E_1$, and for every multi-index β,

$$|D^{\beta}\psi_0(y)| \leqq C_{\beta} b^{-\alpha\cdot\beta}.$$

ESTIMATE 1. *For every* $N > 0$, *there exists a constant* $C_N > 0$ *such that*

$$(6) \qquad |I_1| = \left| \int e^{-iR\varphi(y)}(1 - \psi_0(y))\tilde{g}(y)dy \right| \leqq C_N R^{-|\alpha|}(1 + R\langle\theta\rangle)^{-N}.$$

As in the case of E_1, a set of E_2 of the form

$$E_2 = \{y:|\pi_j y| \leqq C_j|\pi_j\theta|^{1/(k_j-1)}, j = 1, \ldots, s\}$$

can be chosen so that E_2 contains all the points y where

$$|\pi_j \nabla f(y)| \leqq 2|\pi_j\theta| \quad \text{for all } j = 1, \ldots, s.$$

If ψ_2 is a function approximating the characteristic function of E_2 then we will prove the following.

ESTIMATE 2. *For every* $N > 0$ *there exists* $C_N > 0$ *such that*

$$|I_2| = \left| \int e^{-iR\varphi(y)}(1 - \psi_2(y))\tilde{g}(y)dy \right|$$

$$\leqq C_N R^{-|\alpha|} \sum_{j=1}^{s} (R|\pi_j\theta|^{k_j/(k_j-1)})^{-N}.$$

Estimate 2 will be used to prove Theorem 1.

We will show that Estimate 1 is of the right type for Theorems 2 and 3; that is

$$(7) \qquad \int_{|\theta|\leqq 1} |I_1(\theta)|d\theta \leqq CR^{-(n+1)/2}.$$

To prove this it is necessary to estimate

$$\int_{|\theta|\leqq 1} CR^{-|\alpha|}(1 + R\langle\theta\rangle)^{-N}d\theta.$$

If

$$t = (R^{1-\alpha_1}\theta_1, \ldots, R^{1-\alpha_n}\theta_n)$$

then

$$\langle t\rangle = R\langle\theta\rangle \quad \text{and} \quad dt = R^{n-|\alpha|}d\theta.$$

Therefore

$$\int_{|\theta|\leqq 1} CR^{-|\alpha|}(1 + R\langle\theta\rangle)^{-N}d\theta$$

$$= CR^{-n} \int_{\mathbf{R}^n} (1 + \langle t\rangle)^{-N}dt = CR^{-n}$$

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if $N > n$. Since $n \geqq (n + 1)/2$, this completes the proof of (7).

Suppose that $\langle \theta \rangle \leqq R^{-1}$. The measure of

$$F = \{\theta : \langle \theta \rangle \leqq R^{-1}\}$$

is bounded by $CR^{|\alpha|-n}$. Since the measure of E_1 is less than $CR^{-|\alpha|}$,

$$|\widehat{d\mu}(\xi)| \leqq |I_1| + \left| \int e^{-iR\varphi(y)} \psi_0(y) \widetilde{g}(y) dy \right| \leqq CR^{-|\alpha|}$$

and

$$\int_F |\widehat{d\mu}(\xi)| d\theta \leqq CR^{|\alpha|-n} CR^{-|\alpha|} = CR^{-n} \leqq CR^{-(n+1)/2}.$$

This shows that we may assume that $\langle \theta \rangle > R^{-1}$.

This leaves only the integral over $2E_1$:

$$(8) \quad I_1' = \int_{2E_1} e^{-iR\varphi(y)} \psi_0(y) \widetilde{g}(y) dy.$$

This is the set where $|\nabla \varphi|$ is small. It is natural at this point to consider the points

$$z \in A(\theta) = \{z : \nabla \varphi(z) = 0\};$$

That is, the points where $\nabla f(z) = \theta$.

Define, for $j = 1, \ldots, s$,

$$\delta_j(\theta) = \begin{cases} |\pi_j \theta|^{1/(k_j - 1)} & \text{if} \quad k_j > 2 \\ \langle \theta \rangle^{1/k_j} & \text{if} \quad k_j = 2. \end{cases}$$

The set of all θ such that

$$\prod_{j=1}^{s} \delta_j(\theta) = 0$$

is a set of measure zero. In fact it is a union of linear subspaces. From the point of view of Theorems 1, 2 and 3 this set is not important and we may assume that $\Pi \delta_j \neq 0$. Also let

$$\delta_j'(\theta) = \delta_m(\theta) \quad \text{if } j_{m-1} < j \leqq j_m \quad j = 1, \ldots, n; \ m = 1, \ldots, s.$$

Consider $z \in A(\theta)$. Let $Q'(z)$ be the matrix

$$Q'(z) = \tfrac{1}{2} d^2 P(z) \left(\frac{\det d^2 f(z)}{\det d^2 P(z)} \right) \left(1 + \sum_{j=1}^{n} (D_j f(z))^2 \right)^{-(n+2)/2n}$$

where $d^2 f(z)$ is the matrix of second derivatives of f. Notice that since the determinants of the first and second fundamental forms of the surface $y_{n+1} = f(y)$ equal

$$\left(1 + \sum_{j=1}^{n} (D_j f(z))^2 \right) \quad \text{and}$$

$$(\det d^2f(z))\left(1 + \sum_{j=1}^{n} (D_jf(z))^2\right)^{-n/2}$$

respectively, then the absolute value of det $Q'(z)$ equals one half the absolute Gaussian curvature at $(z, f(z))$.

Since Q' is symmetric there is an orthogonal matrix U_z such that $U_zQ'U_z^t$ is diagonal. Let ψ be an even C^∞ function on $\mathbf{R}$ such that $\psi(y) = 1$ if $|y| \leqq 1$ and $\psi(y) = 0$ if $|y| \geqq 2$. Define

$$\psi_3^\theta(y) = \psi(y_1/\delta_1'(\theta)) \ldots \psi(y_n/\delta_n'(\theta))$$

and

$$\psi_z(y) = \psi_3^\theta(U_z(y - z)/C_0)\tilde{g}(y)$$

where C_0 is the constant in Lemma 2 of Section 3. Thus ψ_z is supported in a small neighborhood of z.

If

$$\psi_* = \psi_0\tilde{g} - \sum_{z \in A} \psi_z$$

then the integral I_1' in (8) can be split into parts:

$$I_1' = \int e^{-iR\varphi}\left\{\psi_* + \sum_z \psi_z\right\}dy \equiv I_* + \sum_z I_z.$$

Define

$$(9) \quad J_z = \tilde{g}(z)e^{-iR\varphi(z)} \int_{\mathbf{R}^n} e^{-iRQ(y)}\psi_3^\theta(U_zy)dy$$

where $Q = Q(y) = y^tQ'(z)y$.

Suppose that either Σ is convex or $\tau > 1$ we will prove the following estimates:

$$(10) \quad |I_*| = \left|\int e^{-iR\varphi}\psi_*dy\right| \leqq h_1(R, \theta)$$

and for each $z \in A(\theta)$,

$$|J_z - I_z| = h_2(R, \theta)$$

where

$$(11) \quad \int_{F^c} (h_1 + h_2)d\theta \leqq CR^{-(n+1)/2}$$

and $F^c = \{\theta:\langle\theta\rangle \geqq R^{-1}\}$. If $\tau \leqq 1$ and $p < \tau$ the estimate in (11) is replaced by

$$(12) \quad \int_{F^c} (h_1 + h_2)d\theta \leqq C_pR^{-(n+p)/2} \quad \text{for all } R \geqq 1.$$

The integrals I_* and I_z will be approximated in Sections 4 and 5 respectively.

It is clear that the main part of $\widehat{d\mu}(\xi)$ should come from the points $z \in A$ because

$$\nabla\varphi(z) = \nabla f(z) - \theta = 0.$$

Geometrically, this means that the tangent at $(z, f(z))$ is perpendicular to $\xi = R(\theta, -1)$. The rather poor estimate in (12) shows that the integral I_* can also be important. The gradient $|\nabla\varphi(y)|$ can be very small without ever equalling zero. For example, if $f(y) = y^3$ and $\theta > 0$ then

$$|\nabla\varphi(y)| > 0 \quad \text{for all } y.$$

Equivalently, the graph of f has no tangents perpendicular to $(\theta, -1)$ for $\theta > 0$. On the other hand if $\theta < 0$ there are two such tangents.

All that remains of the proof at this point is to show that the expression J_z in (9) is equivalent to the main term $\mathscr{J}(\xi)$ given in the introduction. At the end of Section 5 we will show that

$$\left|\mathscr{J}(\xi) - \sum_{z \in A(\theta)} J_z\right| \leqq h_3(R, \theta)$$

where h_3 satisfies (11) or (12).

3. The regions of large oscillation.

The proof of Estimate 1. Define

$$||y|| = \sum_{j=1}^{n} |y_j|^{k_j'}.$$

Since P is homogeneous in the first j_1 variables then

$$|D^\beta P_1(y)| = \left|\left(\frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{\beta_n} P_1(y)\right|$$
$$\leqq C|\pi_1 y|^{k_1 - |\pi_1\beta|} \leqq C||y||^{1-\alpha\cdot\beta}$$

for every multi-index β. Since the higher order derivatives of φ are independent of θ and the terms of h are dominated by P then

(13) $$|D^\beta\varphi(y)| \leqq C||y||^{1-\alpha\cdot\beta} \quad |\beta| \geqq 2, y \in \mathbf{R}^n.$$

Define cone-like regions W_j, $j = 1, \ldots, s$ by

$$W_j = \left\{y \in E_1^c : ||\pi_j y|| \geqq \frac{1}{2s}||y||\right\}.$$

Let $\{\eta_j : = 1, \ldots, s\}$ be a C^∞ partition of unity on E_1^c subordinate to the covering $\{W_j\}$ with the homogeneity property:

$$\eta_j(y) = \eta_j(t^{\alpha_1}y_1, \ldots, t^{\alpha_n}y_n) \quad y \in E_1^c, t > 0.$$

For example, $\{\eta_j\}$ could be defined on $\|y\| = 1$ and then extended by homogeneity. If $t = \|y\|^{-1}$ then

$$\|(t^{\alpha_1}y_1, \ldots, t^{\alpha_n}y_n)\| = 1.$$

Thus

(14) $$|D^\beta \eta_j(y)| \leqq C_\beta \|y\|^{-\alpha\cdot\beta} \quad \text{for all } \beta.$$

It follows from the estimate of ψ_0 in Section 2, that ψ_0 also satisfies (14). Therefore

(15) $$|D^\beta((1 - \psi_0)\eta_j\tilde{g})(y)| \leqq C_\beta \|y\|^{-\alpha\cdot\beta} \quad \text{for all } \beta.$$

The integration by parts will involve operators

$$T_j g = \nabla_j \cdot \{(\nabla_j \varphi)g/|\nabla_j\varphi|^2\}, \quad j = 1, \ldots, s.$$

As in the construction of E_1, for $y \in E_1^c$,

$$|\nabla_j\varphi(y)| \geqq C|\pi_j y|^{k_j - 1} = C\|\pi_j y\|^{1 - 1/k_j}.$$

The function η_j is supported in a set where the component $\pi_j y$ is large. Therefore

(16) $$|\nabla_j\varphi(y)| \geqq C\|y\|^{(k_j - 1)/k_j} = C\|y\|^{1 - 1/k_j} \quad y \in W_j.$$

It follows from (15) and (16) that

(17) $$|T_j((1 - \psi_0)\eta_j\tilde{g})(y)| \leqq \max_{\substack{|\beta|=2 \\ \pi_j\beta=\beta}} \frac{\|y\|^{1-\alpha\cdot\beta}}{(\|y\|^{1-1/k_j})^2} = C\|y\|^{-1}$$

and in general,

(18) $$|T_j^m((1 - \psi_0)\eta_j\tilde{g})(y)| \leqq C\|y\|^{-m}.$$

Let

$$g_j^\# = (1 - \psi_0)\eta_j\tilde{g}.$$

The integration by parts formula that we will use is

(19) $$\int_\Omega e^{-iR\varphi}g_j^\# \, dy = \frac{i}{R}\int_{\partial\Omega} e^{-iR\varphi}\frac{\vec{n}\cdot\nabla_j\varphi}{|\nabla_j\varphi|^2}g_j^\# \, dy + \frac{1}{iR}\int_\Omega e^{-iR\varphi}\nabla_j\cdot\left\{\frac{\nabla_j\varphi g_j^\#}{|\nabla_j\varphi|^2}\right\}dy$$

where $\vec{n}$ is the outward unit normal vector on the boundary of Ω. This formula can be derived by applying the divergence theorem to the function

$$F = g_j^{\#} e^{-iR\varphi} \nabla_j \varphi / |\nabla_j \varphi|^2.$$

Let $\Omega = E_1^c$. Because of the cut-off function $(1 - \psi_0)$ and the fact that $\tilde{g}$ is compactly supported, there will be no boundary term in the integration. Integrate by parts N times to get

$$(20) \quad \int_{E_1^c} e^{-iR\varphi} g_j^{\#} dy = (iR)^{-N} \int_{E_1^c} e^{-iR\varphi} T_j^N (g_j^{\#}) dy.$$

By (18),

$$\left| \int_{E_1^c} e^{-iR\varphi} g_j^{\#} dy \right| \leqq CR^{-N} \int_{W_j} \|y\|^{-N} dy$$
$$\leqq CR^{-N} \int_{W_j} \|\pi_j y\|^{-N} dy.$$

Fix $j = 1$. The cross-section of W_1 for a fixed $\pi_1 y = t$ has area

$$C \prod_{j > j_1} |\pi_1 y|^{k_1/k_j} = C |\pi_1 y|^{k_1|\alpha| - j_1}.$$

Therefore if $U = \{z \in \mathbf{R}^{j_1} : b^{\alpha_1} \leqq |z| \leqq 1\}$,

$$\left| \int_{E_1^c} e^{-iR\varphi} g_j^{\#} dy \right| \leqq CR^{-N} \int_U |z|^{k_1|\alpha| - j_1 - k_1 N} dz$$
$$\leqq CR^{-N} \int_{b^{\alpha_1}}^1 r^{k_1(|\alpha| - N) - 1} dr = CR^{-N} b^{|\alpha| - N}$$

if $N > |\alpha|$. After summing over $j = 1, \ldots, s$, then this shows that

$$\left| \int_{E_1^c} e^{-iR\varphi} (1 - \psi_0) \tilde{g} dy \right| \leqq CR^{-|\alpha|} (Rb)^{-N}$$
$$\leqq CR^{-|\alpha|} (1 + R\langle\theta\rangle)^{-N}$$

for every $N > 0$. This proves Estimate 1.

Proof of Estimate 2. The calculation for Estimate 2 is virtually the same except that b is replaced by

$$|\pi_j \theta|^{k_j/(k_j - 1)}.$$

As a result

$$\left| \int e^{-iR\varphi} (1 - \psi_2) \eta_j \tilde{g} dy \right| \leqq CR^{-|\alpha|} (R |\pi_j \theta|^{k_j/(k_j - 1)})^{-N}.$$

Summing over $j = 1, \ldots, s$ completes the proof.

We will finish this section with a number of simple estimates for the function f. Since P is a direct sum of homogeneous polynomials, $d^2 P$ can be diagonalized by a direct sum of orthogonal matrices. Therefore,

$$U_z (d^2 P(z)) U_z^t = \Lambda$$

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where U_z is orthogonal, Λ is diagonal, and the eigenvalues $\lambda_j(z)$ are arranged so that $\lambda_j(z)$ is homogeneous of degree $k_j' - 2$. Note that

$$|\lambda_j(z)| \geqq C|\pi_j z|^{k_j'-2} \quad C > 0.$$

Let $\delta_j'(\theta)$, $j = 1, \ldots, n$, be defined as in Section 2. Also write $i \sim j$ if there exists an m such that $j_{m-1} < i \leqq j_m$, $j_{m-1} < j \leqq j_m$; that is, i and j are associated to the same subspace V_m.

LEMMA 1. *Suppose that f is a function of type α. Then there exists a constant $C_0 > 0$ such that if $|y| \leqq C_0$ then*

(i) $|D_iD_jf(y)| \leqq C|\lambda_i(y)|$ *if* $i \sim j$

(ii) $|D_iD_jf(y)| \leqq \frac{1}{2}|\lambda_i(y)|$ *if* $i \nsim j$

(iii) $|D_iD_jD^\beta f(y)| \leqq C|\lambda_i(y)|\,|\pi_1 y|^{-|\pi_1\beta_1|} \ldots |\pi_s y|^{-|\pi_s\beta_s|}$ *if* $i \sim j$.

Proof. If $j_{m-1} < j \leqq j_m$, $j_{m-1} < i \leqq j_m$ then

$$\begin{aligned}|D_iD_jf(y)| &\leqq |D_iD_jP(y)| + |D_iD_jh(y)| \\ &\leqq C|\pi_m y|^{k_m-2} + C|y|\,|\pi_m y|^{k_m-2} \leqq C|\lambda_i(y)|.\end{aligned}$$

This proves (i). For (ii) notice that $D_iD_jP \equiv 0$ whenever $i \nsim j$. Therefore, as before,

$$|D_iD_jf(y)| \leqq C|y|\,|\pi_m y|^{k_m-2} \leqq C|y|\,|\lambda_i(y)|.$$

Clearly, for $|y|$ sufficiently small (ii) holds.

Estimate (iii) is also clear:

$$|D_iD_jD^\beta f(y)| \leqq C|\pi_m y|^{k_m-2}|\pi_1 y|^{-|\pi_1\beta_1|} \ldots |\pi_s y|^{-|\pi_s\beta_s|}.$$

LEMMA 2. *Let $z \in \mathbf{R}^n$, $\theta = \nabla f(z)$. There is a constant $C_0 > 0$ that is so small that for all*

$$y \in \{y : |\pi_j y| \leqq C_0\delta_j(\theta), j = 1, \ldots, s\} \equiv W$$

the following are true

(i) $|f(y + z) - f(z) - \nabla f(z)y - \frac{1}{2}y^t d^t P(z)y| \leqq \frac{1}{4}|y^t d^2P(z)y|$

(ii) $|\nabla f(y + z) - \nabla f(z) - d^2P(z)y| \leqq \frac{1}{2}|d^2P(z)y|$

(iii) *If e is any unit vector in V_j, for some $j = 1, \ldots, s$ then*

$$|(e \cdot \nabla)^{N+2}f(y + z)| \leqq C|\delta_j(\theta)|^{k_j-2-N}.$$

Proof. The estimate (iii) of Lemma 1 can be improved slightly to show that

$$|D_iD_jD^\beta f(z)| \leqq C|\lambda_i(z)|\,(\delta_1'(\theta))^{-|\beta_1|} \ldots (\delta_n'(\theta))^{-|\beta_n|}$$

because $|\lambda_i(z)| \geqq C$ if $k_i' = 2$. Therefore

$$\begin{aligned}|D_iD_jD^\beta f(z)y_iy_jy^\beta| &\leqq C|\lambda_i(z)|\prod_{m=1}^{s}\left|\frac{\pi_m y}{\delta_m(\theta)}\right|^{|\pi_m\beta|}|y_iy_j| \\ &\leqq CC_0^{|\beta|}|\lambda_i(z)|\,|y_iy_j|.\end{aligned}$$

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Similarly,

$$|D_iD_jD^\beta h(z)y_iy_jy^\beta| \leqq CC_0|\lambda_i(z)|\,|y_iy_j|.$$

Since the roles of i and j can be reversed this shows that the expression in the left hand of (i) is less than

$$CC_0\sqrt{|\lambda_i(z)|}\,|y_i|\sqrt{|\lambda_j(z)|}\,|y_j| \leqq CC_0\sum_{j=1}^{n}|\lambda_j(z)|\,|y_j|^2.$$

Suppose that $j_{m-1} < j \leqq j_m$. Because $|\lambda_j(z)| \geqq |\pi_m z|^{k_m-2}$,

$$\sum_j |\lambda_j(z)|\,|y_j|^2 \leqq C|y^t d^2P(z)y|.$$

As a result the left hand side of (i) is less than

$$CC_0|y^t d^2P(z)y| \leqq \tfrac{1}{4}|y^t d^2P(z)y|$$

for C_0 sufficiently small. The proofs of (ii) and (iii) are similar.

4. The reduction to $A(\theta)$. The purpose of this section is to prove estimates (11) and (12) for h_1. This will take care of the integral

$$I_* = \int e^{-iR\varphi(y)}\psi_*(y)dy.$$

The support of ψ_* is contained in the region

$$\Omega = \mathbf{R}^n - \bigcup_{z\in A}\left\{y + z:|\pi_j U_z y| \leqq \frac{C_0}{4n}\delta_j(\theta), j = 1, \dots, s\right\}.$$

The problems of this section are those associated with inflection points.

We will estimate I_* by integrating by parts using

$$Tg = \nabla\cdot\{\nabla\varphi g|\nabla\varphi|^{-2}\}.$$

This will require in particular an estimate for the minimum value of $\nabla\varphi$ in Ω. This minimum occurs either on the boundary or in the interior of Ω. For $|y|$ sufficiently large, $|\nabla\varphi(y)| \geqq C$ and on the boundary of

$$\left\{y + z:|\pi_j y| \leqq \frac{C_0}{4}\delta_j(\theta), j = 1, \dots, s\right\},$$

$$|\nabla\varphi(y)| = |\nabla f(y) - \nabla f(z)| \geqq \tfrac{1}{2}|d^2P(z)y|$$

because of Lemma 2(ii). Since f is of type α this minimum can be replaced by

$$(21)\quad |\nabla\varphi(y)| \geqq C|d^2P(z)y| \geqq C\min\{|\pi_j\theta|^{(k_j-2)/(k_j-1)}\delta_j(\theta):j = 1, \dots, s\}.$$

Let $\rho(\theta)$ denote the distance from θ to $\nabla f(\Gamma)$:

$$\rho(\theta) = \text{dist}(\theta, \nabla f(\Gamma)) = \inf\{ |\theta - \nabla f(y)|: \det d^2 f(y) = 0\}.$$

Since

$$\rho(\theta) = \min\{ |\pi_j \theta| : j = 1, \dots, s;\ k_j > 2\}$$

the expression in (21) can be bounded below by $C\rho(\theta)$. Hence

$$|\nabla \varphi(y)| \geqq C\rho(\theta).$$

Now consider the interior of Ω. If $|\nabla \varphi|$ has a minimum at y_0 then by differentiating,

$$d^2\varphi(y_0)\nabla\varphi(y_0) = 0.$$

That is,

$$d^2 f(y_0)\{\theta - \nabla f(y_0)\} = 0.$$

Since $y_0 \notin A'$, $\theta - \nabla f(y_0) \neq 0$. Hence

$$\det d^2 f(y_0) = 0.$$

This shows that $y_0 \in \Gamma$. Therefore $|\nabla\varphi(y)| \geqq \rho(\theta)$ at any minimum in Ω.

It follows from the definition of ψ_* in Section 2 that

$$|D^\beta \psi_z(y)| \leqq C_\beta (\delta_1'(\theta))^{-\beta_1} \dots (\delta_n'(\theta))^{-\beta_n}$$

and ψ_* satisfies the same estimate.

Now, since $|\nabla\varphi(y)| \geqq C\rho(\theta)$ for all $y \in \Omega$, and since

$$\rho(\theta) \leqq \min\{\delta_j(\theta) : j = 1, \dots, s\},$$

then

$$(22)\quad |T^N(\psi_*)(y)| \leqq \frac{C}{(\inf|\nabla\varphi|)^{2N}} + \frac{C}{(\inf|\nabla\varphi|)^N(\min \delta_j)^N} \leqq \frac{C}{(\rho(\theta))^{2N}}.$$

This estimate can be improved if the curvature vanishes only at the origin. Integration by parts N times using T shows that

$$(23)\quad |I_*| \leqq CR^{-N}\left| \int e^{-iR\varphi} T^N(\psi_*)(y)dy \right| \leqq CR^{-N}\rho^{-2N}.$$

In the set $F = \{\theta : \rho(\theta) > R^{-b}, |\theta| \leqq 1\}$

$$(24)\quad \int_F |I_*(\theta)|d\theta \leqq C_N R^{-N} R^{2bN} \quad \text{for all } R \geqq 1.$$

As a result, if $b < ½$ then by choosing N large enough we see that

$$\int_F |I_*(\theta)|d\theta \leqq CR^{-(n+1)/2}.$$

As mentioned in the introduction Γ is the union of linear subspaces

$\Gamma_1, \ldots, \Gamma_t$. Consider a fixed surface Γ_m, and let

$$W_m = \{\theta : \text{dist}(\theta, \Gamma) = \text{dist}(\theta, \Gamma_m)\}.$$

Since all points θ are in such a region if suffices to fix Γ_m and W_m, and to consider only $\theta \in W_m$.

The part of the region F in W_1 is sketched in Figure 1. Estimates similar to (24) will eventually also be obtained in the regions $F_1, \ldots, F_N$.

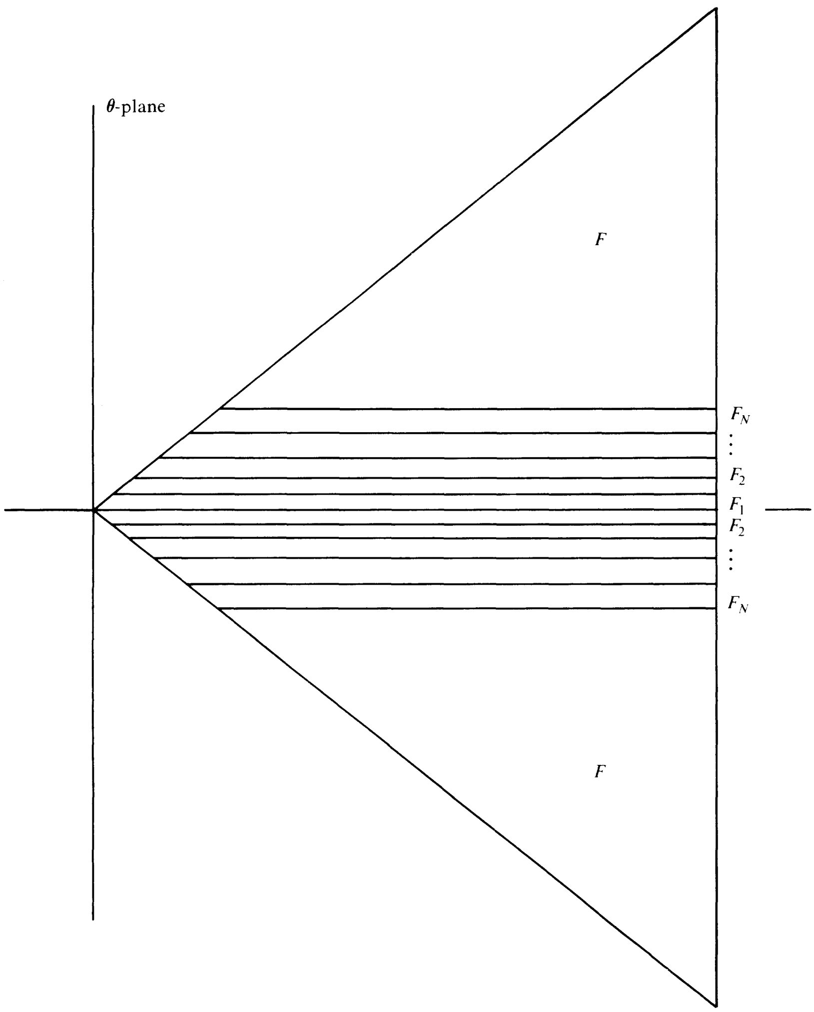

Figure 1

If $\rho^\gamma \geqq \frac{1}{2}|\pi_j\theta|$ and $|\nabla_j\varphi(y)| \leqq \rho^\gamma$ then

$$|\pi_j y| \leqq C_0\rho^{\gamma/(k_j-1)}.$$

Choose a C^∞ function $\eta(y)$ in $\mathbf{R}^{n_j}$ such that $\eta(y) = 1$ if $|y| \leqq 1$ and $\eta(y) = 0$ if $|y| \geqq 2$. Let

$$G_j = \{y \in V_j:|\pi_j y| \leqq C_0\rho^{\gamma/(k_j-1)}\}$$

and

$$\eta_j(y) = \eta(y/C_0\rho^{\gamma/(k_j-1)}).$$

If $\rho^\gamma < \frac{1}{2}|\pi_j\theta|$, G_j and η_j will be different. The fact that det $d^2P_j(y)$ vanishes only at the origin in V_j means that ∇P_j is locally one-to-one from V_j onto itself. Therefore ∇P is locally one-to-one from $\mathbf{R}^n$ to $\mathbf{R}^n$. Because

$$|\det d^2f(y) - \det d^2P(y)| \leqq \tfrac{1}{2}|\det d^2P(y)|$$

then ∇f is also locally one-to-one. If dim $V_j > 1$ then ∇P_j is one-to-one on V_j because the unit sphere in V_j is connected. Suppose for the moment that dim $V_j > 1$. Then for each θ and j there exists a unique w_j in V_j such that

$$\nabla_j f(w_j) = \pi_j\theta.$$

Also

$$C|\pi_j\theta|^{1/(k_j-1)} \leqq |w_j| \leqq C|\pi_j\theta|^{1/(k_j-1)}.$$

Since ∇P_j is homogeneous of degree $k_j - 1$, there exists a set

$$G_j = \{y \in V_j:y \cdot w_j > c|y|\,|w_j| \quad \text{and}$$
$$||y| - |w_j|| \leqq C\rho^\gamma|\pi_j\theta|^{-(k_j-2)/(k_j-1)}\}$$

such that if

$$|(\nabla_j f)(y)| \leqq \rho^\gamma$$

then $y \in G_j$. Let η_j be a C^∞ function on V_j such that $\eta_j(y) = 1$ in G_j and η_j is supported in a set like G_j but with c and C replaced by $\frac{1}{2}c$ and $2C$. Suppose also that

$$|D^\beta\eta_j(y)| \leqq C_\beta\rho^{-\gamma|\beta|} \quad \text{for all } \beta,\ y \in \mathbf{R}^{n_j}.$$

If dim $V_j = 1$ and k_j is even then ∇P_j is still one-to-one and onto and the above construction of G_j and η_j can be used. If k_j is odd then

$$\nabla_j f(w_j) = \pi_j\theta$$

has either two solutions or no solutions in V_j, depending on the sign of $\pi_j\theta$. In this case define G_j to be the union of the two sets corresponding to the two solutions of the equations

$$\nabla_j f(w_j) = \pm \pi_j \theta.$$

$$E_2 = \{y:\pi_j y \in G_j \text{ for } j = 1, \dots, s\} \quad \text{and}$$

$$\psi_2(y) = \eta_1(\pi_1 y) \dots \eta_s(\pi_s y).$$

In this definition each V_j is identified with $\mathbf{R}^{n_j}$ in a natural way.

$$|G_j| \leqq \begin{cases} C\rho^{n_j\gamma/(k_j-1)} & \text{if} \quad \rho^\gamma \geqq \tfrac{1}{2}|\pi_j\theta| \\ C\rho^{n_j\gamma}|\pi_j\theta|^{-(k_j-2)/(k_j-1)} & \text{if} \quad \rho^\gamma \leqq \tfrac{1}{2}|\pi_j\theta| \end{cases}$$
$$\leqq C\rho^{n_j\gamma}(\max\{\rho^\gamma, |\pi_j\theta|\})^{-(k_j-2)/(k_j-1)}.$$

If $J_m = \{j:V_j \subset \Gamma_m\}$ then

$$|E_2| \leqq C\rho^{n\gamma}\left\{\prod_{j\notin J_m} \rho^{-n_j\gamma(k_j-2)/(k_j-1)}\right\}\left\{\prod_{j\in J_m} |\pi_j\theta|^{-(k_j-2)/(k_j-1)}\right\}.$$

This estimate is used to bound part of I_*:

$$(25) \qquad \left|\int e^{-iR\varphi}\psi_2\psi_* dy\right| \leqq C|E_2|.$$

From the construction of E_2 it follows that if y is in the support of $(1 - \psi_2)$ then

$$|\nabla\varphi(y)| \geqq C\rho^\gamma.$$

The estimates for η_j also show that

$$|D^\beta\psi_2(y)| \leqq C_\beta\rho^{-\gamma|\beta|}.$$

Therefore an integration by parts as in (23) shows that

$$(26) \qquad \left|\int e^{-iR\varphi}(1 - \psi_2)\psi_* dy\right| \leqq CR^{-N}\left|\int e^{-iR\varphi}T^N((1 - \psi_2)\psi_*)dy\right|$$
$$\leqq CR^{-N}\rho^{-2\gamma N}.$$

The estimates (25) and (26) will be combined to estimate I_* in a set

$$F = \{\theta:R^{-B} \leqq \rho(\theta) \leqq R^{-b}\}.$$

Hence if $\gamma \leqq 1$,

$$\int_F |I_*(\theta)|d\theta \leqq CR^{-n\gamma b}\left\{\prod_{j\notin J_m} R^{-bn_j+\gamma bn_j(k_j-2)/(k_j-1)}\right\}$$
$$+ CR^{-N}\left\{\prod_{j\notin J_m} R^{-Bn_j+2\gamma NB}\right\}.$$

If $2\gamma B < 1$ then the second term can be made less than $CR^{-(n+1)/2}$ by taking N sufficiently large. Let

$$n\gamma b + \sum_{J_m^c} \left(bn_j - \gamma bn_j \frac{(k_j - 2)}{(k_j - 1)} \right) \equiv b(\sigma_1 + \gamma\sigma_2)$$

where

$$\sigma_1 = n - \dim \Gamma_m \quad \text{and} \quad \sigma_2 = \dim \Gamma_m + \sum_{J_m^c} \frac{n_j}{k_j - 1}.$$

Then if $2\gamma B < 1$,

(27) $$\int_F |I_*(\theta)|\,d\theta \leqq CR^{-b(\sigma_1+\gamma\sigma_2)} + CR^{-(n+1)/2}.$$

If $p > b(\sigma_1 + \sigma_2/2B)$ then γ can be chosen so that

$$2\gamma B < 1 \quad \text{and} \quad p < b(\sigma_1 + \gamma\sigma_2).$$

The problem now is to split the set $\{ |\theta| \leqq 1\}$ into sets of the form F in such a way that the decay rate $b(\sigma_1 + \gamma\sigma_2)$ in (27) is as large as possible. Let

$$F_j = \{\theta : R^{-b_{j-1}} \leqq \rho(\theta) \leqq R^{-b_j}\} \quad \text{for } j = 1, \ldots, N.$$

From the simple estimate, $|I_*(\theta)| \leqq C$ it follows that

$$\int_{\rho \leqq R^{-b_1}} |I_*(\theta)|\,d\theta \leqq CR^{-(n+1)/2}$$

if $b_1 = (n + 1)/2$. It is necessary to maximize

$$b_j(\sigma_1 + \sigma_2/2b_{j-1})$$

for a sequence $b_1 \geqq b_2 \geqq \ldots \geqq b_{N+1}$ such that $b_{N+1} < ½$. If $b_{N+1} < ½$ then (24) shows that $|I_*(\theta)|$ is appropriately bounded in the region

$$\{\rho(\theta) \geqq R^{-b_N}\}.$$

Consider the function

$$g(x) = \frac{p}{(\sigma_1 + \sigma_2/2x)} = \frac{2xp}{(2x\sigma_1 + \sigma_2)}.$$

If $b_1 = (n + 1)/2$ we want to show that for N sufficiently large $g^N(b_1) < ½$. As N increases $g^N(b_1)$ approaches the positive fixed point of the function g. This fixed point is

$$x = \frac{2p - \sigma_2}{2\sigma_1}.$$

Therefore for $g^N(b_1) < ½$ we must have

$$\frac{2p - \sigma_2}{2\sigma_1} < ½.$$

This is equivalent to

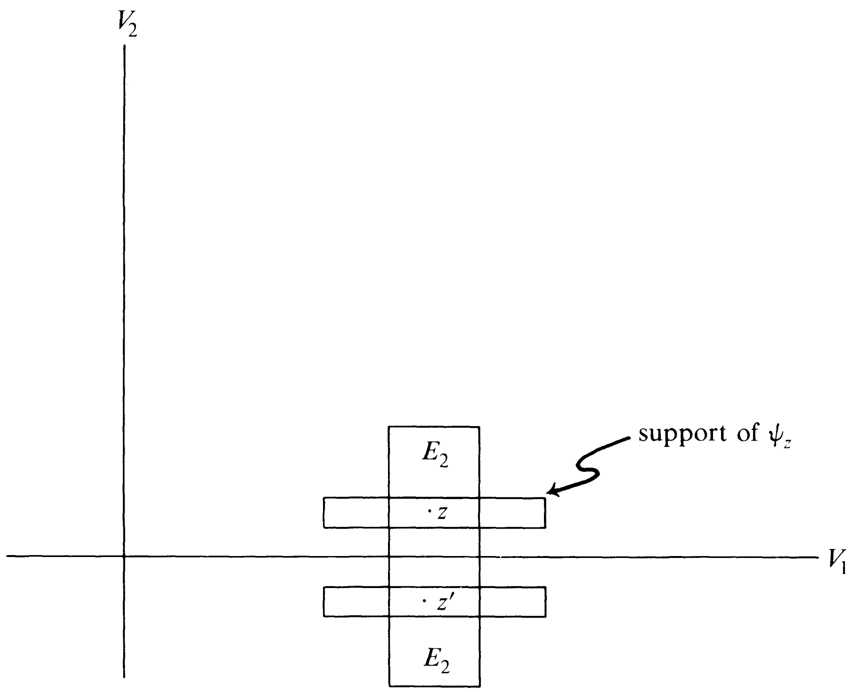

Figure 2

$$p < \tfrac{1}{2}(\sigma_1 + \sigma_2) = \tfrac{1}{2}\left(n + \sum_{J_m^c} \frac{n_j}{k_j - 1}\right) \equiv \tfrac{1}{2}(n + \tau_m).$$

Hence if $b_j = g^{j-1}(b_1)$ for $j = 2, \dots, N + 1$ then we have split W_m into sets F_j such that

$$\int_{F_j} |I_*(\theta)|\, d\theta \leqq CR^{-p} + CR^{-(n+1)/2}.$$

If $\tau_0 = \min\{\tau_m : m = 1, \dots, t\} > 1$ then we can choose $p = (n + 1)/2$ and

$$\int_{|\theta| \leqq 1} |I_*(\theta)|\, d\theta \leqq \sum_{m=1}^{t} \int_{W_m} |I_*| d\theta \leqq CR^{-(n+1)/2} \quad R \geqq 1.$$

If $\tau_0 \leqq 1$ then for any $p < (n + \tau_0)/2$

$$\int_{|\theta| \leqq 1} |I_*(\theta)|\, d\theta \leqq CR^{-p} \quad R \geqq 1.$$

The only estimate that remains for I_* is in the case where Σ is convex. In this case the only points where $|\nabla\varphi(y)| = 0$ are the points of $A'(\theta)$, and there are only two such points. These points are roughly antipodal. Thus in a neighborhood of $y = 0$ there can only be one such point. Since there are no inflection points the problems associated with I_* do not arise. The modifications for the simpler convex case therefore more naturally fit at the end of the next section. Similarly, for subspaces Γ_m associated with convex polynomials P_j, the problem illustrated in Figure 2 does not arise. Therefore we need only be concerned with the distance to those subspaces associated with the nonconvex polynomials. This gives the estimates for the integral of I_* with the parameter τ defined as in the introduction.

5. The points of stationary phase. In this section we estimate the integrals

$$I_z = \int e^{-iR\varphi(y)}\psi_z(y)dy$$

where $z \in A(\theta)$. Fix $z \in A(\theta)$. Since z is fixed the dependence on z will often be suppressed; for example, $Q(y) = y^t(Q'(z))y$. All the constants C, except those identified by C_z or $c_\beta(z)$ are independent of z.

Since z is fixed the coordinate system will be chosen so that $d^2P(z)$ is diagonal. As in Lemmas 1 and 2 we will suppose that the eigenvalues $\{\lambda_j\}$ are arranged so that λ_j is homogeneous of degree k'_j.

Let η_j be a C^∞ partition of unity on the unit sphere such that for some constant $C_1 > 0$

$$\eta_j(y) = \begin{cases} 1 \text{ if } |y_j| \geqq 2C_1|y| \\ 0 \text{ if } |y_j| \leqq C_1|y| \; j = 1, \dots, n. \end{cases}$$

Define

$$\eta_j^{\#}(y) = \eta_j(y/|y|) \quad \text{and} \quad \eta_j^*(y) = \eta_j^{\#}(\sqrt{|\lambda_1|}y_1, \dots, \sqrt{|\lambda_n|}y_n).$$

Then $\eta_j^*(y) = 1$ in the pair of cones defined by

$$\{y:|\lambda_j|\,|y_j|^2 \geqq (2C_1)^2 \sum_{i=1}^{n} |\lambda_i|\,|y_i|^2\}$$

and $\eta_j^*(y) = 0$ in the set

$$\{y:|\lambda_j|\,|y_j|^2 \leqq C_1^2 \sum |\lambda_i|\,|y_i|^2\}.$$

Also it follows from differentiating that

$$|D^\beta\eta_j^{\#}(y)| \leqq C_\beta|y|^{-|\beta|} \quad \text{and}$$
$$|D^\beta\eta_j^*(y)| \leqq C_\beta|\lambda_1^{\beta_1} \dots \lambda_n^{\beta_n}|^{1/2}(\sum |\lambda_i|\,|y_i|^2)^{-|\beta|/2}.$$

However if $|\beta| \neq 0$ then

$$\sum |\lambda_i|\,|y_i|^2 \geqq C|\lambda_j|\,|y_j|^2$$

in the support of $D^\beta\eta_j^*$. Therefore

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(28) $|D^{\beta}\eta_j^*(y)| \leqq C\left(\left|\frac{\lambda_1}{\lambda_j}\right|^{\beta_1} \cdots \left|\frac{\lambda_n}{\lambda_j}\right|^{\beta_n}\right)^{1/2} |y_j|^{-|\beta|}.$

We will use this partition of unity on $\mathbf{R}^n$ to split up I_z. Define

$$L_j = \int e^{-iR\varphi(y+z)}\eta_j^*(y)\psi_z(y + z)dy.$$

Now $I_z = L_1 + \ldots + L_n$. Consider a fixed j between 1 and n. For simplicity

$$\eta(y) = \eta_j^*(y)\psi_z(y + z).$$

We begin by integrating by parts M times using the formula

$$\int_{\Omega} e^{-iR\varphi}\eta = \frac{i}{R}\int_{\partial\Omega} e^{-iR\varphi}\vec{n}_j\left(\frac{\eta}{D_j\varphi}\right) + \frac{1}{iR}\int_{\Omega} e^{-iR\varphi}T_j(\eta)$$

where $T_j(\eta) = D_j(\eta/D_j\varphi)$ and $\vec{n}_j$ is the j-th component of the outward unit normal vector to $\partial\Omega$. Except at the origin $\eta(y) = \eta_j^*(y)\psi_z(y + z)$ is a C^{∞} function with compact support. A natural choice for Ω therefore is

$$\Omega = \{|y| \geqq \epsilon\} = \mathbf{R}^n - B_{\epsilon}.$$

Integrating M times gives

$$(29)\quad L_j = \sum_{m=0}^{M-1}\left(\frac{i}{R}\right)^{m+1}\int_{|y|=\epsilon} e^{-iR\varphi}\vec{n}_j\left(\frac{\eta}{D_j\varphi}\right)T_j^m(\eta)dy$$
$$+ \left(\frac{1}{iR}\right)^M \int_{|y|\geqq\epsilon} e^{-iR\varphi}T_j^M(\eta)dy.$$

We will show that as ϵ approaches zero the boundary terms in (29) disappear.

For ϵ sufficiently small, $|\nabla\varphi| \geqq C_z\epsilon$ because $\det d^2\varphi(z) \neq 0$. Therefore

$$|T_j^m(\eta)| \leqq C_z|\nabla\varphi|^{-2m} \leqq C_z\epsilon^{-2m}.$$
$$\left|\int_{\partial B_{\epsilon}}\right| \leqq C_z\epsilon^{-2m-1}|\partial B_{\epsilon}| \leqq C_z\epsilon^{n-1-2m-1}.$$

Hence the boundary terms in (29) go to zero as $\epsilon \to 0$ if $m < (n - 2)/2$. Therefore

$$(30)\quad L_j = \left(\frac{1}{iR}\right)^M \int_{\mathbf{R}^n} e^{-iR\varphi}T_j^M(\eta)dy$$

if $M < n/2$.

To integrate further it is necessary to obtain better estimates for T_j. It follows from the definitions that

$$(31)\quad |D_j^N(\eta)| = |D_j^N(\eta_j^*\psi_z)| \leqq |y_j|^{-N}.$$

By Lemma 2,

$$(32)\quad |\nabla\varphi(y + z) - d^2P(z)y| \leqq \tfrac{1}{2}|d^2P(z)y|$$

and

$$(33)\quad |D_j^N\varphi(y + z)| \leqq C|\lambda_j(z)|\,(\delta_j'(\theta))^{2-N}.$$

By (32), in the support of η,

$$(34)\quad |\nabla\varphi(y + z)| \geqq \tfrac{1}{2}|d^2P(z)y| \geqq \tfrac{1}{2}|\lambda_j(z)y_j|.$$

It follows from (31), (33), and (34) that

$$(35)\quad |T_j^N(\eta)| \leqq C \sum_{k=0}^{N} \frac{|\lambda_j|^k}{|\lambda_j y_j|^{N+k}}(\delta_j'(\theta))^{k-N} \leqq C|\lambda_j|^{-N}|y_j|^{-2N}.$$

Let

$$S_j(\eta)(y) = D_j\left\{\frac{\eta(y)}{D_jQ(y)}\right\}$$

where

$$Q(y) = \tfrac{1}{2}(y^t d^2f(z)y)(1 + \sum |D_i f(z)|^2)^{-(n+2)/2n}\left\{\frac{\det d^2f(z)}{\det d^2P(z)}\right\}$$

Note that

$$(36)\quad |D_jQ(y) - \lambda_j y_j| \leqq \tfrac{1}{2}|\lambda_j(z)y_j|$$

for z sufficiently small. The following estimates are all for $|\theta| \leqq \epsilon_0$ where ϵ_0 is so small that (36) holds. Then just as in (30)

$$(37)\quad L_j^*(z) \equiv \int_{\mathbf{R}^n} e^{-iR[\varphi(z)+Q(y)]}\eta(y)dy$$

$$= (iR)^{-M}\int_{\mathbf{R}^n} e^{-iR[\varphi(z)+Q(y)]}S_j^M(\eta)dy.$$

As in (35), the estimate

$$|D_j^N\varphi(y + z) - D_j^NQ(y + z)| \leqq C|\lambda_j|\,(\delta_j'(\theta))^{2-N}\left|\frac{y_j}{\delta_j'(\theta)}\right|$$

from Lemma 2 leads to

$$(38)\quad |S_j^N(\eta) - T_j^N(\eta)| \leqq C|\lambda_j|^{-N}|y_j|^{-2N}\left|\frac{y_j}{\delta_j'(\theta)}\right|.$$

The estimate

(39) $|\varphi(y + z) - \varphi(z) - Q(y)| \leqq C|\lambda_j y_j^2| \left|\dfrac{y_j}{\delta_j'(\theta)}\right|$

of Lemma 2 will also be useful. Let

$$Q^{\#}(y) = \varphi(z) + Q(y).$$

Using (35), (38), and (39) it follows that the difference between the integrands in (30) and (39) is less than

(40) $$|e^{-iR\varphi}T_j^N - e^{-iRQ^{\#}}S_j^N| \leqq |e^{-iR\varphi} - e^{-iRQ^{\#}}|\,|T_j^N| + |T_j^N - S_j^N|$$
$$\leqq C(R|\lambda_j|\,|y_j|^2)\left|\frac{y_j}{\delta_j'}\right||\lambda_j|^{-N}|y_j|^{-2N} + C|\lambda_j|^{-N}|y_j|^{-2N}\left|\frac{y_j}{\delta_j'}\right|.$$

Let B_ϵ again be the ball of radius ϵ about z. When $|y|$ is sufficiently small the second term in (40) is the larger. Thus

$$\left|-(iR)^{-M-1}\int_{\partial B_\epsilon} e^{-iR\varphi}\frac{\vec{n}_j}{D_j\varphi}T_j^M - e^{-iRQ^{\#}}\frac{\vec{n}_j}{D_j\varphi}S_j^M dy\right|$$
$$\leqq CR^{-M-1}\int_{\partial B_\epsilon}|\lambda_j|^{-M-1}|y_j|^{-2M-1}\left|\frac{y_j}{\delta_j'}\right|$$
$$\leqq \frac{C\epsilon^{n-1-2M}}{R^{M+1}\delta_j'|\lambda_j|^{M+1}} \to 0 \quad \text{as } \epsilon \to 0$$

if $M < (n - 1)/2$. This gives us an improvement to (30):

(41) $$L_j - L_j^* = (iR)^{-M}\lim_{\epsilon\to 0}\int_{B_\epsilon^c} e^{-iR\varphi}T_j^M - e^{-irQ^{\#}}S_j^M dy$$

if $M < (n + 1)/2$. Now let $U(z)$ be the set of points y in the support of $\eta = \eta_j^*\psi_z$ such that

$$|y_j| \leqq |R\lambda_j(z)|^{-1/2}.$$

In this region $U(z)$, the second term of (40) is the larger. Since the support of η is contained in a set of the form

$$\{y:|\lambda_i|\,|y_i|^2 \leqq C|\lambda_j|\,|y_j|^2 \text{ for } i = 1, \dots, n\}$$

then

(42) $$|(iR)^{-M}\int_{U(z)} e^{-iR\varphi}T_j^M - e^{-iRQ^{\#}}S_j^M dy|$$
$$\leqq CR^{-M}\int_0^{|R\lambda_j|^{-1/2}}|\lambda_j|^{-M}|y_j|^{-2M}\left|\frac{y_j}{\delta_j'}\right|\prod_{i\neq j}\int_0^{cM_i} dy_i dy_j$$

where

$$M_i = \min\left\{ \sqrt{\left|\frac{\lambda_j}{\lambda_i}\right|}|y_j|, \delta_i'\right\}.$$

If

$$M_* = \min\{ \sqrt{|\lambda_i|}\delta_i' : i = 1, \ldots, n\}$$

then

$$M_i \leqq \frac{1}{\sqrt{|\lambda_i|}} \min\{ \sqrt{|\lambda_j|}|y_j|, M_*\}.$$

If $\kappa = |\lambda_1 \ldots \lambda_n|^{1/n}$ then the integral in (42) is less than

$$(43) \quad \frac{c(R|\lambda_j|)^{-M}|\lambda_j|^{1/2}}{\delta_j'\kappa^{1/2}} \int_0^{|R\lambda_j|^{-1/2}} (\min\{ \sqrt{|\lambda_j|}|y_j|, M_*\})^{n-1}|y_j|^{1-2M}dy_j$$
$$\leqq \frac{CR^{-(n+1)/2}}{\kappa^{1/2}\delta_j'|\lambda_j|^{1/2}} \leqq \frac{CR^{-(n+1)/2}}{\kappa^{1/2}M_*}.$$

To approximate the part of (41) over the set $U(z)^c$ we integrate three more times. After one integration the boundary terms will be

$$\left| -(iR)^{-M-1} \int_{\partial U(z)} \left\{ e^{-iR\varphi}\frac{\vec{n}_j}{D_j\varphi}T_j^M - e^{-iRQ^\#}\frac{\vec{n}_j}{D_jQ}S_j^M \right\} \right|$$
$$\leqq CR^{-M-1}\frac{|\lambda_j|^M(R|\lambda_j|)^{-1/2}}{|\lambda_j(R|\lambda_j|)^{-1/2}|^{2M+1}\delta_j'} \prod_{i \neq j} \int_0^{\min|R\lambda_i|^{-1/2}} dy_i$$
$$\leqq \frac{CR^{-(n+1)/2}}{\delta_j'\kappa^{1/2}|\lambda_j|^{1/2}} \leqq \frac{CR^{-(n+1)/2}}{\kappa^{1/2}M_*}.$$

Similarly after the second and third integrations

$$\left| -(iR)^{-M-1} \int_{\partial U} \right| \leqq CR^{-(n+1)/2}\kappa^{-1/2}M_*^{-1}.$$

For the integral in (41) we chose M so that $(n - 1)/2 \leqq M < (n + 1)/2$. In the region outside U the first term of (40) dominates. Therefore

$$(44) \quad \left| \int_{U^c} \{e^{-iR\varphi}T_j^{M+3} - e^{-iRQ^\#}S_j^{M+3}\} \right|$$
$$\leqq \frac{C|\lambda_j|^{(n+1)/2-2(M+3)}}{R^{M+3}\kappa^{1/2}M_*} \int_{|R\lambda_j|^{-1/2}}^{\delta_j'} (R|\lambda_j|)r^{n+2-2(M+3)}dr$$
$$\leqq CR^{-(n+1)/2}\kappa^{-1/2}M_*^{-1}$$

since $n + 2 - 2(M + 3) \leqq -3$.

The calculations from (42) to (44) combine to show that

(45) $$\left| \int_{\mathbf{R}^n} [e^{-iR\varphi} - e^{-iRQ^{\#}}]\eta_j^* \psi_z dy \right| \leqq CR^{-n/2}(\kappa R)^{-1/2} M_*^{-1}.$$

In a similar way it is possible to replace

$$\psi_z(y) = \psi((y_1 - z_1)/C_0\delta_1'(\theta)) \ldots \psi((y_n - z_n)/C_0\delta_n'(\theta))\tilde{g}(y)$$

by

$$\psi_z^{\#}(y) = \psi((y_1 - z_1)/C_0\delta_1'(\theta)) \ldots \psi((y_n - z_n)/C_0\delta_n'(\theta))\tilde{g}(z).$$

Observe that

$$|\psi_z(y) - \psi_z^{\#}(y)| \leqq C|y_j|/\delta_j'$$

and in general

$$|D^\beta(\psi_z - \psi_z^{\#})| \leqq C|y|^{-|\beta|}|y_j|/\delta_j'.$$

It follows therefore that

$$|S(\eta_j^*\psi_z) - S(\eta_j\psi_z^{\#})| \leqq C\frac{|y_j|}{\delta_j'}|\lambda_j|^{-N}|y_j|^{-2N}.$$

With this estimate instead of (40) the integrations proceed as before to show that

(46) $$\left| \int_{\mathbf{R}^n} e^{-iRQ^{\#}}\eta_j^*(\psi_z - \psi_z^{\#})dy \right| \leqq CR^{-n/2}(\kappa R)^{-1/2} M_*^{-1}.$$

Since $\sum \eta_j^* = 1$, then combining (45) and (46) and summing over $j = 1, \ldots, n$ gives the result that

(47) $$|J_z - I_z| \leqq CR^{-n/2}(\kappa R)^{-1/2} M_*^{-1}.$$

The expression in (47) is clearly less than $C(R^n\kappa)^{-1/2}$ when $R^{1/2}M_* \geqq 1$. For the case $R^{1/2}M_* \leqq 1$ is necessary to re-examine the proof. The term $|y_j/\delta_j'|$ in (42), after integrating leads to a factor $R^{-1/2}M_*^{-1}$ in (43). If instead we had used $|y_j/\delta_j'| \leqq 1$ then the expression in (43) would be $C(R^n\kappa)^{-1/2}$. With the same change in the other calculations we get

$$|J_z - I_z| \leqq C(R^n\kappa)^{-1/2}.$$

Therefore

(48) $$|I_z| \leqq C(R^n\kappa)^{-1/2}(1 + RM_*^2)^{-1/2}$$

where

$$M_*^2 = \min\{ |\lambda_i| (\delta_i')^2 : i = 1, \ldots, n\}.$$

To complete the proof of (10) and (11) it is now necessary to examine the integrability of

$$h_2(\theta) = \sum_{z \in A} (\kappa(z))^{-1/2}(M_*)^{-1}$$

in a small neighborhood of the origin: $\{ |\theta| \leqq \epsilon_0\}$.

Since f is of type α

$$(\kappa(z)) \geqq C \prod_{j=1}^{s} |\pi_j\theta|^{n_j(k_j-2)/(k_j-1)}.$$

$|\lambda_j(z)|\,|\delta_j'(\theta)|^2$ will be a minimum in some direction $j = m$ where $k_m > 2$:

$$|\lambda_j(z)|\,|\delta_j'(\theta)|^2 \geqq C|\pi_m\theta|^{k_m/(k_m-1)}.$$

Therefore

$$h_2(\theta) \leqq C|\pi_m\theta|^{-k_m/2(k_m-1)} \prod_{j=1}^{s} |\pi_j\theta|^{-n_j(k_j-2)/2(k_j-1)}.$$

This is integrable over each subspace V_j, $j \neq m$, because

$$n_j(k_j - 2)/2(k_j - 1) < n_j.$$

If $j = m$ then the exponent of $|\pi_m\theta|$ is

$$-\frac{k_m}{2(k_m - 1)} - \frac{n_m(k_m - 2)}{2(k_m - 1)}.$$

When $n_m \neq 1$ this is greater than $-n_m$ and so $h_2(\theta)$ is integrable. When $n_m = 1$, $h_2(\theta)$ contains a factor of $|\pi_m\theta|^{-1}$, which is not integrable. In this case however the Gaussian curvature vanishes on the orthogonal complement of V_m, which is a subspace of dimension $n - 1$. Therefore

$$\tau \leqq \frac{n_m}{k_m - 1} = \frac{1}{k_m - 1} < 1.$$

Although $h_2(\theta)$ is not integrable, it is of weak type L^1 on $\{ |\theta| \leqq \epsilon_0\}$:

(49) $\quad |\{ |h_2(\theta)| > t\}| \leqq C/t.$

Now we must show that $\sum J_z$ approximates the main term $\mathscr{I}(\xi)$. From the definition of J_z in (9)

$$\int_{\mathbf{R}^n} e^{-iRQ(y)}\psi_3^\theta(U_z y)dy = \int_{\mathbf{R}^n} e^{-iRy^t\Lambda y/2}\psi_3^\theta(y)dy$$

where $\Lambda = 2U_zQ'U_z^t$ is a diagonal matrix with eigenvalues $\{\lambda_1', \ldots, \lambda_n'\}$. $2Q'$ was modified so that the absolute value of its determinant is the absolute Gaussian curvature $\kappa(z)$. Thus

$$J_z = \tilde{g}(z)e^{-iR\varphi(z)} \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{iR\lambda_j' y^2/2}\psi(y/\delta_j'(\theta))dy$$

$$= \tilde{g}(z)e^{-iR\varphi(z)}2^{n/2}(R^n\kappa)^{-1/2}$$

$$\times \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-i(\operatorname{sgn} \lambda_j')y^2}\psi(y(R|\lambda_j'|(\delta_j')^2)^{-1/2})dy.$$

A simple integration by parts shows that since $\psi \equiv 1$ near the origin then the integrals equal

$$(50) \quad \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-i(\operatorname{sgn} \lambda_j')y^2}dy + O((1 + RM_*^2)^{-1/2}).$$

The values of the Fresnel integrals at infinity are

$$\int_{-\infty}^{\infty} \sin(x^2)dx = \int_{-\infty}^{\infty} \cos(x^2)dx = \sqrt{\frac{\pi}{2}}.$$

If ν is the number of positive eigenvalues of $d^2f(z)$ minus the number of negative eigenvalues then the integrals in (50) equal

$$\prod_{j=1}^{n} \left(\sqrt{\frac{\pi}{2}} + i(\operatorname{sgn} \lambda_j)\sqrt{\frac{\pi}{2}} \right) = \pi^{n/2}e^{i\nu\pi/4}.$$

This shows that

$$(51) \quad |J_z - \tilde{g}(z)e^{-iR\varphi(z)}(2\pi)^{n/2}(R^n\kappa)^{-1/2}e^{i\nu\pi/4}| \leqq C(R^n\kappa)^{-1/2}(1 + RM_*^2)^{-1/2}.$$

Also R can be replaced by $(R\sqrt{1 + |\theta|^2})$ with the same error.

The error on the right hand side of (51) is acceptable because it is the same as that of (48). If $x' = (z, f(z))$ is a point on the surface where the tangent is perpendicular to $\xi = R(\theta, -1)$ then $\tilde{g}(z) = g(x')$,

$$R\sqrt{1 + |\theta|^2} = |\xi| \quad \text{and} \quad R\varphi(z) = R(z \cdot \theta - f(z)) = x' \cdot \xi.$$

Therefore

$$\sum_{z \in A} \tilde{g}(z)e^{-iR\varphi(z)}(2\pi)^{n/2}(R\sqrt{1 + |\theta|^2})^{-n/2}e^{i\nu\pi/4}\kappa^{-1/2}$$

$$= \sum_{x' \in A'(\theta)} g(x')e^{-ix'\cdot\xi}(2\pi)^{n/2}|\xi|^{-n/2}\kappa(x')^{-1/2}e^{i\nu\pi/4} = |\xi|^{-n/2}\mathscr{I}(\xi).$$

This completes the proofs of Theorems 2 and 3, except when Σ is convex.

Suppose that Σ is convex and consider the region $|y_j| \geqq C|y|$. In the calculations of this section we used the fact that

$$|\nabla\varphi(y + z)| \geqq \tfrac{1}{2}|\lambda_j(z)y_j| \quad |y_j| \geqq C|y|$$

in the support of ψ_z. This gave the estimate in (35) for T_j. Since Σ is convex there are exactly two points where the tangents are perpendicular to $\xi = R(\theta, -1)$ and these points are far apart. Therefore in this case we estimate the whole integral

$$I_1' = \int e^{-iR\varphi(y)}\psi_0(y)\tilde{g}(y)dy$$

rather than splitting it into parts as was done in Sections 4 and 5. Since $|\nabla\varphi(y + z)|$ increases away from the origin we use

$$|\nabla\varphi(y + z)| \geqq C \min(|\lambda_j(z)y_j|, |\lambda_j(z)\delta_j'(\theta)|) \quad |y_j| \geqq C|y|$$

in the support of ψ_0. Now the integral from $|R\lambda_j|^{-1/2}$ to δ_j' in (44) is the same as before but to it we must add an integral

$$\int_{\delta_j'}^{C\langle\theta\rangle^{\alpha_j}} r^{n+2}|\delta_j'(\theta)|^{-2(M+3)}dr \leqq C \int_{|R\lambda_j|^{-1/2}}^{\delta_j'} r^{n+2-2(M+3)}dr$$

if $\delta_j' \geqq |R\lambda_j|^{-1/2}$. If $\delta_j' \leqq |R\lambda_j|^{-1/2}$ then the integral in (44) does not occur and so the estimate in (43) alone suffices. A similar modification to (43) has no effect on the estimate. Therefore the calculations proceed as in (48)

$$(52) \quad |I_1'| \leqq C(R^n\kappa)^{-1/2}(1 + RM_*^2)^{-1/2}.$$

This proves Theorem 2 and 3 when Σ is convex.

Now consider Theorem 1. As in (52)

$$(53) \quad |I_3'| = \left|\int e^{-iR\varphi(y)}\psi_3^\theta(y)\tilde{g}dy\right| \leqq C(R\kappa)^{-n/2}$$
$$\leqq CR^{-n/2}\prod_{j=1}^{s} |\pi_j\theta|^{-n_j(k_j-2)/2(k_j-1)}.$$

If $\delta_j(\theta) \geqq R^{-1/k_j}$ for all j then

$$(54) \quad |I_3'| \leqq CR^{-n/2}\prod_{j=1}^{s} R^{n_j(k_j-2)/2k_j} = CR^{-|\alpha|}.$$

On the other hand the support of ψ_3^θ is contained in the rectangular set

$$E_3 = \{y:|y_j| \leqq \delta_j'(\theta), j = 1, \dots, n\}.$$

If $\delta_j(\theta) \leqq R^{-1/k_j}$ for all j then

$$(55) \quad |I_3'| \leqq C|E_3| \leqq C\prod_{j=1}^{s}(\delta_j(\theta))^{n_j} \leqq CR^{-|\alpha|}.$$

The general case is a combination of these two extremes. The set E_3 can be written as the Cartesian product $E_3 = E_3' \times E_3''$ where E_3' is a rectangular solid with sides of length $\delta_j'(\theta) \geqq R^{-1/k_j}$ and E_3'' has sides of length $\delta_j'(\theta) < R^{-1/k_j}$. Now

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$$I_3' \equiv \int_{E_3''}\left\{\int_{E_3'} e^{-iR\varphi(y)}\psi_3^{\theta}\tilde{g}dy'\right\}dy''.$$

Integrate by parts to get an estimate similar to (53) for the inner integral. The constant obtained will depend continuously on y'', since it depends on the derivatives of f and g. Then a calculation similar to (55) completes the proof that

$$|I_3'| \leqq CR^{-|\alpha|}.$$

Note that the restriction on the higher order terms of f (and φ) plays a role in this step. This proves Theorem 1.

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