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MULTIPLE SOLUTIONS FOR A QUASILINEAR ELLIPTIC VARIATIONAL SYSTEM ON STRIP-LIKE DOMAINS

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Abstract We consider the quasilinear elliptic variational system

$$\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v) + \mu H_u(x,u,v) && \text{in } \Omega, \\ -\Delta_q v &= \lambda F_v(x,u,v) + \mu H_v(x,u,v) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a strip-like domain and λ and μ are positive parameters. Using a recent two-local-minima theorem and the principle of symmetric criticality, existence and multiplicity are proved under suitable conditions on F.

Keywords: Strip-like domain; eigenvalue problem; elliptic systems

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Secondary 35P30

1. Introduction

Very recently, in [**1**] Kristály studied the eigenvalue problem

$$\left.\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v) && \text{in } \Omega, \\ -\Delta_q v &= \lambda F_v(x,u,v) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \qquad (S_\lambda)$$

where $\lambda > 0$ is a parameter and Ω is a strip-like domain in $\mathbb{R}^N$, i.e. $\Omega = \omega \times \mathbb{R}^l$, ω being a bounded open subset of $\mathbb{R}^m$ with smooth boundary, $m \geqslant 1$, $l \geqslant 2$, $1 < p, q < N = m+l$, $F \in C^0(\Omega \times \mathbb{R}^2, \mathbb{R})$, and $\Delta_\alpha w = \operatorname{div}(|\nabla w|^{\alpha-2}\nabla w)$. Here, F_z denotes the partial derivative of F with respect to variable z. He applies a critical point result (see [**5**]) in order to obtain the existence of an open interval $\Lambda \subset (0, +\infty)$ such that, for every $\lambda \in \Lambda$, the system S_λ

* Because of a surprising coincidence of names within our department, we have to point out that the author was born on 4 August 1968.

has at least two distinct non-trivial solutions. Also, he assumes that the nonlinear term F is sub-p,q-linear; that is,

$$(1_F)\quad \lim_{u,v\to 0}\frac{F_u(x,u,v)}{|u|^{p-1}}=\lim_{u,v\to 0}\frac{F_v(x,u,v)}{|v|^{q-1}}=0, \text{ uniformly w.r.t. } x\in\Omega.$$

Inspired by [**1**], we prove two multiplicity theorems, which extend the results contained in [**1**], for the system

$$\left.\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v)+\mu H_u(x,u,v) && \text{in } \Omega,\\ -\Delta_q v &= \lambda F_v(x,u,v)+\mu H_v(x,u,v) && \text{in } \Omega,\\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned}\right\} \qquad (S_{\lambda,\mu})$$

where μ is a positive parameter. Our approach is based on a recent result of Ricceri [**6**, Theorem 4]; in a convenient form for our purposes it can be read as follows.

Theorem 1.1 (Ricceri). *Let X be a reflexive real Banach space, let $I\subseteq\mathbb{R}$ be an interval, and let $\Psi : X\times I\to\mathbb{R}$ be a function such that $\Psi(x,\cdot)$ is concave in I for all $x\in X$, while $\Psi(\cdot\,,\lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in X for all $\lambda\in I$. Further, assume that*

$$\sup_{\lambda\in I}\inf_{x\in X}\Psi(x,\lambda)<\inf_{x\in X}\sup_{\lambda\in I}\Psi(x,\lambda).$$

Then, for each $\rho>\sup_I\inf_X\Psi(x,\lambda)$ there exists a non-empty open set $A\subseteq I$ with the following property: for every $\lambda\in A$ and every sequentially weakly lower semicontinuous functional $\Phi : X\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in\,]0,\delta[$, the functional $\Psi(\cdot\,,\lambda)+\mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x\in X:\Psi(x,\lambda)<\rho\}$.

In the present paper, the function F is assumed to be a $C^0(\Omega\times\mathbb{R}^2,\mathbb{R})$ function such that

(2_F) F is *axially symmetric* in the first variable; that is,

$$F((x_1,x_2),s,t)=F((x_1,gx_2),s,t)\quad \text{for all } x_1\in\omega,\ x_2\in\mathbb{R}^l,\ g\in O(l),\ (s,t)\in\mathbb{R}^2;$$

(3_F) $(s,t)\to F(x,s,t)$ is of class C^1 and $F(x,0,0)=0$ for all $x\in\Omega$.

Moreover, let $\alpha^*=N\alpha/(N-\alpha)$, $\alpha\in\{p,q\}$, be the critical Sobolev exponent and we assume that

(4_F) there exist $\varepsilon>0$, and $r\in\,]p,p^*[$, $s\in\,]q,q^*[$, with $ps=qr$, such that

$$|F_u(x,u,v)|\leqslant\varepsilon(|u|^{p-1}+|v|^{(p-1)q/p}+|u|^{r-1}),$$
$$|F_v(x,u,v)|\leqslant\varepsilon(|v|^{q-1}+|u|^{(q-1)p/q}+|v|^{s-1})$$

for each $x\in\Omega$ and $(u,v)\in\mathbb{R}^2$.

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Throughout this paper, the norm on $W_0^{1,\alpha}(\Omega)$ is defined by

$$\|u\|_\alpha = \left(\int_\Omega |\nabla u|^\alpha\right)^{1/\alpha}, \quad \alpha \in \{p, q\}.$$

As Kristály points out in [**1**], since Ω is unbounded, the loss of compactness of the Sobolev embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$, $\beta \in [\alpha, \alpha^*]$, $\alpha \in \{p, q\}$, makes standard variational techniques more delicate. For this reason, we consider the subgroup G of $O(l)$ defined by $G = \mathrm{id}^m \times O(l)$. The action of G on $W_0^{1,\alpha}(\Omega)$ is defined by

$$gu(x_1, x_2) = u(x_1, g_1^{-1} x_2)$$

for each $(x_1, x_2) \in \omega \times \mathbb{R}^l$, $g = \mathrm{id}^m \times g_1 \in G$ and $u \in W_0^{1,\alpha}(\Omega)$. Let

$$W_{0,G}^{1,\alpha}(\Omega) = \mathrm{Fix}\, W_0^{1,\alpha}(\Omega) = \{u \in W_0^{1,\alpha}(\Omega) : gu = u,\ \forall g \in G\}.$$

Hence, the elements of $W_{0,G}^{1,\alpha}(\Omega)$ are the axially symmetric functions of $W_0^{1,\alpha}(\Omega)$.

Obviously, the action G on $W_{0,G}^{1,\alpha}(\Omega)$ is isometric, that is

$$\|gu\|_\alpha = \|u\|_\alpha, \quad \text{for all } g \in G.$$

Since $l \geqslant 2$, the embedding $W_{0,G}^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$, $\alpha < \beta < \alpha^*$, $\alpha \in \{p, q\}$, is compact [**2**]. In the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, endowed with the norm

$$\|(u, v)\|_{p,q} = \|u\|_p + \|v\|_q,$$

one has

$$\begin{aligned}\mathrm{Fix}_G(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)) &= \{(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : g(u, v) = (u, v),\ \forall g \in G\} \\ &= W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega).\end{aligned}$$

2. Main result

Our main result is the following.

Theorem 2.1. *Let $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function that satisfies conditions (1_F)–(4_F). Furthermore, assume that*

(5) $\displaystyle\limsup_{|(\xi,\eta)| \to +\infty} \frac{F(x, \xi, \eta)}{|\xi|^p + |\eta|^q} \leqslant 0$ *uniformly for every $x \in \Omega$;*

(6) *there exists $(u_0, v_0) \in W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$ such that*

$$\int_\Omega F(x, u_0(x), v_0(x))\, \mathrm{d}x > 0.$$

Then there exist a number σ and a non-degenerate compact interval $C \subseteq [0, +\infty[$ such that, for every continuous function $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying conditions (1_H)–(4_H) and for every $\lambda \in C$, there exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the problem $(S_{\lambda,\mu})$ has at least two solutions, denoted by $(u_{\lambda,\mu}^i, v_{\lambda,\mu}^i)$, $i \in \{1, 2\}$, with $u_{\lambda,\mu}^i$ and $v_{\lambda,\mu}^i$ axially symmetric and with norms less than σ.

Proof. Let $X = W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega)$. We define two functionals Φ and $\mathcal{F}$ by setting, for each $(u,v) \in X$,

$$\Phi(u,v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q,$$
$$\mathcal{F}(u,v) = -\int_\Omega F(x,u(x),v(x))\,\mathrm{d}x.$$

In view of (3_F) and (4_F), and using the Sobolev embeddings, we can prove that $\mathcal{F}$ is a class-C^1 function; its differential is given by

$$\mathcal{F}'(u,v)(w,y) = -\int_\Omega [F_u(x,u,v)w + F_v(x,u,v)y]\,\mathrm{d}x.$$

By the same arguments as used in the proof of [**1**, Theorem 2.2], owing to (1_F), (3_F) and (6) there exists $\rho > 0$ such that the functional

$$\mathcal{G}(u,v,\lambda) = \Phi(u,v) + \lambda\mathcal{F}(u,v) + \lambda\rho$$

satisfies the inequality

$$\sup_{\lambda\in I}\inf_{(u,v)\in X}\mathcal{G}(u,v,\lambda) < \inf_{(u,v)\in X}\sup_{\lambda\in I}\mathcal{G}(u,v,\lambda),$$

where $I = [0,+\infty[$. Now, we wish to apply Theorem 1.1 to the continuous functional $\mathcal{G}$. Clearly, for each $(u,v) \in X$, the functional $\mathcal{G}(u,v,\cdot)$ is concave in I.

Fix $\lambda \in I$. Since $W^{1,\alpha}_G(\Omega) \hookrightarrow L^\alpha(\Omega)$ is continuous, there exist two positive constants, c_1 and c_2, such that

$$\|u\|_{L^p} \leqslant c_1\|u\|_p \quad \text{and} \quad \|v\|_{L^q} \leqslant c_2\|v\|_q.$$

Let

$$a < \min\left\{\frac{1}{\lambda p c_1^p},\ \frac{1}{\lambda q c_2^q}\right\}.$$

Since (5) holds, there exists a function $k_a \in L^1(\Omega)$ such that

$$F(x,\xi,\eta) \leqslant a(|\xi|^p + |\eta|^q) + k_a(x)$$

for all $(\xi,\eta) \in \mathbb{R}^2$ and $x \in \Omega$.

Fix $(u,v) \in X$. From the last inequality we deduce that

$$\int_\Omega F(x,u(x),v(x))\,\mathrm{d}x \leqslant a(c_1^p\|u\|_p^p + c_2^q\|v\|_q^q) + \|k_a\|_{L^1}.$$

So,

$$\mathcal{G}(u,v,\lambda) \geqslant \left(\frac{1}{p} - \lambda c_1^p a\right)\|u\|_p^p + \left(\frac{1}{q} - \lambda c_2^q a\right)\|v\|_q^q - \lambda\|k_a\|_{L^1} + \lambda\rho,$$

i.e. $\mathcal{G}(\cdot\,,\cdot\,,\lambda)$ is coercive.

Fix $\lambda \in I$. In view of (1_F) and (4_F), by [**1**, Lemma 3.4], the functional $\mathcal{F}$ is sequentially weakly continuous on X. Thus, the functional $\mathcal{G}(\cdot\,,\cdot\,,\lambda)$ is sequentially weakly lower semicontinuous in X.

Now, fixing $\gamma > \sup_{\lambda\in I}\inf_{(u,v)\in X}\mathcal{G}(u,v,\lambda)$, Theorem 1.1 ensures that there exists a non-empty open set $A \subseteq I$ with the following property: for every $\lambda \in A$ and every continuous function $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying conditions (1_H)–(4_H), there exists $\delta > 0$ such that, for each $\mu \in \left]0,\delta\right[$, the functional

$$E_{\lambda,\mu}(u,v) = \mathcal{G}(u,v,\lambda) + \mu\mathcal{H}(u,v)$$

has at least two local minima lying in the set $\{(u,v) \in X : \mathcal{G}(u,v,\lambda) < \gamma\}$, namely $(u^i_{\lambda,\mu}, v^i_{\lambda,\mu})$, $i \in \{1,2\}$, where $\mathcal{H}$ is the sequentially weakly continuous functional defined by

$$\mathcal{H}(u,v) = -\int_\Omega H(x,u(x),v(x))\,\mathrm{d}x.$$

Since F and H are axially symmetric in the first variable, and each $g \in G$ is isometric, the function $E_{\lambda,\mu}$ is G-invariant, i.e.

$$E_{\lambda,\mu}(g(u,v)) = E_{\lambda,\mu}(gu,gv) = E_{\lambda,\mu}(u,v)$$

for each $g \in G$, $(u,v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$. As

$$\mathrm{Fix}(W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)) = W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega),$$

by the principle of symmetric criticality of [**3**], we find that $(u^i_{\lambda,\mu}, v^i_{\lambda,\mu})$, $i \in \{1,2\}$, are also the critical points of $E_{\lambda,\mu}$ and then weak solutions of the problem $(S_{\lambda,\mu})$.

Finally, let $[a,b] \subset A$ be any non-degenerate compact interval. Observe that

$$\bigcup_{\lambda\in[a,b]} \{(u,v) \in X : \mathcal{G}(u,v,\lambda) \leqslant \gamma\} \\ \subseteq \{(u,v) \in X : \mathcal{G}(u,v,a) \leqslant \gamma\} \cup \{(u,v) \in X : \mathcal{G}(u,v,b) \leqslant \gamma\}.$$

This implies that the set

$$S := \bigcup_{\lambda\in[a,b]} \{(u,v) \in X : \mathcal{G}(u,v,\lambda) \leqslant \gamma\}$$

is bounded. Hence, the local minima of $E_{\lambda,\mu}$ have norm less than or equal to σ, taking $\sigma = \sup_{(u,v)\in S} \|(u,v)\|_{p,q}$. This concludes the proof. □

Now, we give an example in which the hypotheses of Theorem 2.1 are satisfied.

Example 2.2. Let $\Omega = \omega \times \mathbb{R}^2$, where ω is a bounded open interval in $\mathbb{R}$. Let $\alpha, \beta : \Omega \to \mathbb{R}$ be two continuous, non-negative, not identically zero, axially symmetric functions with compact support in Ω. Then there exist a number σ and a non-degenerate compact interval $C \subseteq [0, +\infty[$ such that, for every $a \in]\frac{3}{2}, 3[$, $b \in]\frac{9}{4}, 9[$, $\lambda \in C$, there exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the system

$$\begin{aligned} -\Delta_{3/2} u &= \tfrac{5}{2}\lambda\alpha(x)|u|^{1/2}u\cos(|u|^{5/2} + |v|^3) + \mu\beta(x)au|u|^{a-2} && \text{in } \Omega, \\ -\Delta_{9/4} v &= 3\lambda\alpha(x)|v|v\cos(|u|^{5/2} + |v|^3) + \mu\beta(x)bv|v|^{b-2} && \text{in } \Omega, \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned}$$

has at least two solutions with the properties from Theorem 2.1.

In this case we have

$$F(x, \xi, \eta) = \alpha(x)\sin(|\xi|^{5/2} + |\eta|^3) \quad \text{and} \quad H(x, \xi, \eta) = \beta(x)(|\xi|^a + |\eta|^b)$$

for each $(x, \xi, \eta) \in \Omega$. It is easy to observe that conditions (1_F)–(3_F) and (1_H)–(3_H) hold immediately, while (4_F) is verified by choosing $r = \frac{11}{4}$, $s = \frac{33}{8}$, and (4_H) is verified choosing $r \in]a, 3[$, $s \in]b, 9[$ with $s = \frac{3}{2}r$. Finally, (5) is obvious and (6) follows by putting $u_0(x) = (\frac{1}{2}\pi)^{2/5}$ for every $x \in \operatorname{supp}\alpha$ and $v_0(x) = 0$ for every $x \in \Omega$.

By the same arguments as used in the proof of Theorem 2.1, but applying also the Palais–Smale properties, we obtain the result below. We recall that a Gâteaux differentiable functional S on a real Banach space X is said to satisfy the Palais–Smale condition if each sequence $\{x_n\}$ in X such that $\sup_{n\in\mathbb{N}} |S(x_n)| < +\infty$ and $\lim_{n\to+\infty} \|S'(x_n)\|_X = 0$ admits a strongly converging subsequence.

Theorem 2.3. *Assume that the hypotheses of Theorem 2.1 hold.*

Then there exists a non-empty open set $A \subseteq [0, +\infty[$ such that, for every $\lambda \in A$ and for every continuous function $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying conditions (1_H)–(4_H) and

(5_H) $\displaystyle\limsup_{|(\xi,\eta)|\to+\infty} \frac{H(x, \xi, \eta)}{|\xi|^p + |\eta|^q} < +\infty$ *uniformly for every $x \in \Omega$,*

there exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the problem $(S_{\lambda,\mu})$ has at least three solutions axially symmetric.

Proof. Let A and $E_{\lambda,\mu}$ have the same meaning as in the proof of Theorem 2.1, $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ being a continuous function satisfying (5_H). Reasoning as in the proof of Theorem 2.1, there exists $\delta_1 > 0$ such that, for each $\mu \in]0, \delta_1[$, the problem $(S_{\lambda,\mu})$ has at least two solutions.

First of all, the functional $E_{\lambda,\mu}$ is coercive. In fact, from (5_H), there exist a positive constant $b \in \mathbb{R}$ and a function $k_b(x) \in L^1(\Omega)$ such that

$$H(x, \xi, \eta) \leqslant b(|\xi|^p + |\eta|^q) + k_b(x)$$

for all $x \in \Omega$ and $(\xi, \eta) \in \mathbb{R}^2$.

Fix $(u, v) \in X$. From the previous inequality we deduce that

$$\mathcal{H}(u, v) = -\int_\Omega H(x, u(x), v(x))\,\mathrm{d}x \geqslant -b(c_1^p\|u\|_p^p + c_2^q\|v\|_q^q) - \|k_b\|_{L^1}.$$

Let

$$\delta < \min\left\{\delta_1, \frac{1}{b}\left(\frac{1}{pc_1^p} - \lambda a\right), \frac{1}{b}\left(\frac{1}{qc_2^q} - \lambda a\right)\right\}.$$

So, for each $\lambda \in A$ and $\mu \in]0, \delta[$, we have

$$\begin{aligned} E_{\lambda,\mu}&(u, v) \\ &= \mathcal{G}(u, v, \lambda) + \mu\mathcal{H}(u, v) \\ &\geqslant \left(\frac{1}{p} - c_1^p(\lambda a + \mu b)\right)\|u\|_p^p + \left(\frac{1}{q} - c_2^q(\lambda a + \mu b)\right)\|v\|_q^q - \lambda\|k_a\|_{L^1} + \lambda\rho - \mu\|k_b\|_{L^1} \end{aligned}$$

for all $(u, v) \in X$. This ensures the coercivity of the functional $E_{\lambda,\mu}$ for each $\lambda \in A$ and $\mu \in]0, \delta[$.

Now, let us check the Palais–Smale condition for $E_{\lambda,\mu}$. To this end, let $\{(u_n, v_n)\}$ be a sequence in X satisfying

$$\sup_{n\in\mathbb{N}} |E_{\lambda,\mu}(u_n, v_n)| \leqslant M, \qquad \lim_{n\to\infty} \|E'_{\lambda,\mu}(u_n, v_n)\|_{X^*} = 0. \tag{2.1}$$

Since the functional $E_{\lambda,\mu}$ is coercive, the sequence $\{(u_n, v_n)\}$ is bounded in X. So, applying [**1**, Lemma 3.5] to the functional $E_{\lambda,\mu}(\cdot\,,\cdot)$ we obtain that $\{(u_n, v_n)\}$ contains a strongly convergent subsequence in X.

Since the functional $E_{\lambda,\mu}$ is C^1 in X, our conclusion follows by [**4**, Corollary 1], which ensures that there exists a third critical point of the functional $E_{\lambda,\mu}$ which is a solution of problem $(S_{\lambda,\mu})$. □

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