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ON PRIMENESS AND NILPOTENCE IN STRUCTURAL MATRIX NEAR-RINGS

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The structure of completely prime ideals in any structural matrix near-rings is determined. Partial descriptions are obtained for prime, nil, nilpotent, and locally nilpotent ideals of structural matrix near-rings. Their associated radicals are also studied in this paper.

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The concept of matrix near-rings was introduced by Meldrum and van der Walt [9]. Many interesting results have been obtained along this line since then. The investigation of "structural" matrix near-rings was initiated by van der Walt and van Wyk. A structural matrix near-ring "$\mathcal{M}_n(B,R)$" is considered as a subnear-ring of the matrix near-ring "$\mathcal{M}_n(R)$" and is determined by virtue of the shape of the Boolean matrix "B". Booth and Groenewald [4], Groenewald [5] have studied certain concepts of primeness in matrix near-rings. Lee [7] studied prime ideals and their associated radicals in structural matrix near-rings and completely determined 1-prime and equiprime ideals of $\mathcal{M}_n(B,R)$. Partial results related to prime and completely prime ideals were obtained there. (Note different notation for 1-prime ideals was used by Groenewald [5].) We continue to investigate various concepts of primeness in this paper. The structure of completely prime ideals of structural matrix near-rings is described completely. Moreover, we study prime, nil, nilpotent, and locally nilpotent ideals and their associated radicals in some class of structural matrix near-rings. (For more on matrix near-rings see [8], where a substantial bibliography on the subject can be found.)

1. Preliminaries and notation

Throughout this paper, the word "near-ring" means a right zero-symmetric near-ring with an identity element 1. Near-rings shall be denoted by the letter R (except where noted). By a *subnear-ring* of a near-ring R, we shall always mean a subnear-ring containing the identity element 1 of R. By an *ideal* in R, we shall always mean a 2-sided ideal in R. Let R^n denote the direct sum of n copies of $(R,+)$ where n is a fixed natural number. Elements of R^n are written as $\bar{u}$, $\bar{v}$, and so on. If $a \in R$ and $\bar{u}=(u_1,\ldots,u_n)\in R^n$, then $\bar{u}\cdot a$ is defined to be $(u_1a,\ldots,u_na)$. Denote the n-tuple with 1 in the i-th component and 0 elsewhere by $\bar{\varepsilon}_i$. A nonempty subset X of R is called *left* or *right invariant* according to whether $RX \subseteq X$ or $XR \subseteq X$. A nonempty subset of R is said to be *2-sided invariant* if it is both left and right invariant.

The $n \times n$ *elementary matrices* are defined as functions from R^n to itself as:

$$f_{ij}^r = \iota_i f^r \pi_j$$

for $1 \leqq i, j \leqq n, r \in R$ where $f^r: R \to R$ is left multiplication by r and ι_i and π_j are the i-th coordinate injection and the j-th coordinate projection, respectively. The subnear-ring of $M_0(R^n)$ generated by:

$$\{f_{ij}^r \mid r \in R, 1 \leqq i, j \leqq n\}$$

is called an $n \times n$ *matrix near-ring* over R, denoted by $\mathcal{M}_n(R)$, and each element of $\mathcal{M}_n(R)$ is called a matrix.

Let B be a *Boolean matrix* of size n where $b_{ij} = 0$ or 1 is the element in the i-th row and j-th column, for $1 \leqq i, j \leqq n$. We assume that B satisfies the following two conditions:

(1) $b_{ii} = 1$ for $1 \leqq i \leqq n$; and

(2) $b_{ik} = 1$ whenever $b_{ij} = b_{jk} = 1$.

We write

$$\bar{u} \sim_i \bar{v} \text{ if and only if } \pi_j \bar{u} = \pi_j \bar{v} \text{ for all } j \text{ such that } b_{ij} = 1.$$

Observe that if $b_{ij} = 0$, then $\bar{\varepsilon}_j \sim_i \bar{0}$ where $\bar{0} = (0, \ldots, 0)$. Let $\mathcal{M}_n(B, R)$ denote the set:

$$\{X \in \mathcal{M}_n(R) \mid (\forall 1 \leqq i \leqq n, \forall \bar{u}, \bar{v} \in R^n)(\bar{u} \sim_i \bar{v} \Rightarrow \pi_i X \bar{u} = \pi_i X \bar{v})\}.$$

In [**11**], van der Walt and van Wyk proved that $\mathcal{M}_n(B, R)$ is a subnear-ring of $\mathcal{M}_n(R)$. We call $\mathcal{M}_n(B, R)$ the $n \times n$ *structural matrix near-ring* over R with respect to B.

Definition 1.1. Let $\mathcal{L} \leqq \mathcal{M}_n(B, R), L \subseteq R, 1 \leqq j \leqq n$. Then

(1) $\prod(R, j) = \{(u_1, \ldots, u_n) \in R^n \mid u_i = 0 \text{ if } b_{ij} = 0\}$;

(2) $\mathcal{L}_{(j)} = \{x \in R \mid (\exists X \in \mathcal{L})(\exists \bar{u} \in \prod(R, j))(x = \pi_j X \bar{u})\}$;

(3) $\coprod(j, L) = \{(u_1, \ldots, u_n) \in R^n \mid u_k \in L \text{ if } b_{jk} = 1\}$;

(4) $L^{(j)} = \{X \in \mathcal{M}_n(B, R) \mid X(\prod(R, j)) \subseteq \coprod(j, L)\}$.

Basic properties concerning the above four sets were developed in [**7**]. The author there showed that $L^{(j)} = (R^n(j, L): R^n(j, R))$ where the right-hand term was studied in [**11**] by van der Walt and van Wyk.

The next definition is necessary to our investigation on primeness and nilpotence.

Definition 1.2. Let $\mathfrak{B}$ denote the set of natural numbers:

$$\{k\in\{1,\dots,n\}\mid \text{if } 1\leqq h\leqq n \text{ and } h\neq k, \text{ then } b_{kh}=0 \text{ or } b_{hk}=0\}$$

where n is the size of the Boolean matrix $B=[b_{ij}]$.

Remark. If B is an upper or a lower triangular matrix, then $\mathfrak{B}=\{1,\dots,n\}$. However $\mathfrak{B}=\emptyset$ if each entry b_{ij} of B is 1.

Example 1.3. Suppose R is a ring with identity and

$$B=\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&0&1&0\\1&1&1&1\end{pmatrix} \text{ is of size 4.}$$

Observe that we can identify $\mathcal{M}_4(B,R)$ with

$$\begin{pmatrix}R&R&R&0\\R&R&R&0\\0&0&R&0\\R&R&R&R\end{pmatrix}.$$

A routine calculation shows that each completely prime ideal of $\mathcal{M}_4(B,R)$ must be either

$$P^{(3)}=\begin{pmatrix}R&R&R&0\\R&R&R&0\\0&0&P&0\\R&R&R&R\end{pmatrix} \text{ or } P^{(4)}=\begin{pmatrix}R&R&R&0\\R&R&R&0\\0&0&R&0\\R&R&R&P\end{pmatrix}$$

where P is a proper completely prime ideal of R. (See Definition 1.1(4) for the meaning of $P^{(3)}$ or $P^{(4)}$.) Note that the corresponding set $\mathfrak{B}$ is equal to $\{3,4\}$.

The next lemma is the starting point to our approach:

Lemma 1.4. *Let $k\in\mathfrak{B}$. Then we have:*

(1) $\pi_h(\prod(R,k))=0$ *whenever* $h\neq k$ *and* $b_{kh}=1$;

(2) $\bar{r}\sim_k(\bar{\varepsilon}_k\cdot r_k)$ *whenever* $\bar{r}=(r_1,\dots,r_n)\in\prod(R,k)$;

(3) $\pi_k XY\bar{\varepsilon}_k=(\pi_k X\bar{\varepsilon}_k)(\pi_k Y\bar{\varepsilon}_k)$ *whenever* X *and* $Y\in\mathcal{M}_n(B,R)$.

Proof. From the assumption that $k\in\mathfrak{B}$ if $b_{kh}=1$ and $h\neq k$, then we have $b_{hk}=0$. This implies $\pi_h(\prod(R,k))=0$. Part (1) follows. Let $\bar{r}=(r_1,\dots,r_n)\in\prod(R,k)$. To show $\bar{r}\sim_k(\bar{\varepsilon}_k\cdot r_k)$, it suffices to show $\pi_h\bar{r}=\pi_h(\bar{\varepsilon}_k\cdot r_k)$ for all h such that $b_{kh}=1$. Assume $b_{kh}=1$. If $h\neq k$, then $b_{hk}=0$. By Definition 1.1(1), we have $\pi_h\bar{r}=0$ and $\pi_h(\bar{\varepsilon}_k\cdot r_k)=0$. If $h=k$, then $\pi_h\bar{r}=r_k$ and $\pi_h(\bar{\varepsilon}_k\cdot r_k)=r_k$. Thus part (2) follows. In the following, we assume X and Y are matrices of $\mathcal{M}_n(B,R)$. (Recall: $Y\prod(R,k)\subseteq\prod(R,k)$ [7, Proposition 2.8].) Since $\bar{\varepsilon}_k\in\prod(R,k)$, we have $Y\bar{\varepsilon}_k\in\prod(R,k)$. Take $\bar{r}=Y\bar{\varepsilon}_k$ and $r_k=\pi_kY\bar{\varepsilon}_k$ in part (2). Then we have $Y\bar{\varepsilon}_k\sim_k(\bar{\varepsilon}_k\cdot(\pi_kY\bar{\varepsilon}_k))$. Use Proposition 2.2 of [**11**] to show that $XY\bar{\varepsilon}_k\sim_k X(\bar{\varepsilon}_k\cdot(\pi_kY\bar{\varepsilon}_k))$ and hence $\pi_kXY\bar{\varepsilon}_k=\pi_kX(\bar{\varepsilon}_k\cdot(\pi_kY\bar{\varepsilon}_k))$. Finally, apply Lemma 2.1 [**4**] to obtain $\pi_kXY\bar{\varepsilon}_k=(\pi_kX\bar{\varepsilon}_k)(\pi_kY\bar{\varepsilon}_k)$. Part (3) is immediate.

Proposition 1.5. *Assume $k\in\mathfrak{B}$.*

(1) *Let L be a right invariant subset of R. Then we have:*

$$X\in L^{(k)} \text{ if and only if } \pi_kX\bar{\varepsilon}_k\in L.$$

(2) *Let L and H be right invariant subsets of R. Then $L^{(k)}H^{(k)}\subseteq(LH)^{(k)}$.*

Proof. (1) Suppose $X\in L^{(k)}$. We then have $\pi_kX\bar{\varepsilon}_k\in L$. Conversely, suppose $\pi_kX\bar{\varepsilon}_k\in L$. To show $X\in L^{(k)}$, it suffices to show $\pi_hX\bar{r}\in L$ for all $\bar{r}\in\prod(R,k)$ and for all h such that $b_{kh}=1$. Therefore we assume $b_{kh}=1$. Part (2) of Lemma 1.4 gives $\bar{r}\sim_k(\bar{\varepsilon}_k\cdot r_k)$. Invoke Proposition 2.2 of [**11**] to obtain $X\bar{r}\sim_k X(\bar{\varepsilon}_k\cdot r_k)$. We then have $\pi_hX\bar{r}=\pi_hX(\bar{\varepsilon}_k\cdot r_k)=(\pi_hX\bar{\varepsilon}_k)r_k$. If $h\neq k$, then $\pi_h(X\bar{\varepsilon}_k)=0$ by Lemma 1.4(1) (note that $X\bar{\varepsilon}_k\in\prod(R,k)$) and hence $\pi_hX\bar{r}=0\in L$. If $h=k$, then $\pi_hX\bar{r}=(\pi_kX\bar{\varepsilon}_k)r_k\in L\cdot r_k$ and so $\pi_hX\bar{r}\in L$. Consequently, $X\in L^{(k)}$.

(2) Let C and D be elements of $L^{(k)}$ and $H^{(k)}$, respectively. Part (1) yields that $\pi_kC\bar{\varepsilon}_k\in L$ and $\pi_kD\bar{\varepsilon}_k\in H$. The previous lemma gives $\pi_kCD\bar{\varepsilon}_k=(\pi_kC\bar{\varepsilon}_k)(\pi_kD\bar{\varepsilon}_k)\in LH$. Using part (1), we then have $CD\in(LH)^{(k)}$.

2. Prime ideals and radicals

Recall that a proper ideal P of R is (1) a *prime ideal* if for any ideals A and B of R such that $AB\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$; (2) a *completely prime ideal* if for any elements a and b of R such that $ab\in P$ implies $a\in P$ or $b\in P$.

We list the following useful results which are Theorems 3.1 and 3.2 of [7], respectively.

Theorem 2.1. *Let P be a prime ideal of R. Then $P^{(i)}$ is a prime ideal of $\mathcal{M}_n(B,R)$ for $1\leqq i\leqq n$.*

Theorem 2.2. *Let $\mathcal{Q}$ be a completely prime ideal of $\mathcal{M}_n(B,R)$. Then $\mathcal{Q}_{(i)}$ is a completely prime ideal of R for $1\leqq i\leqq n$.*

These two results provide partial descriptions of prime and completely prime ideals of any structural matrix near-rings. We shall determine all completely prime ideals of any structural matrix near-rings in this section. Furthermore, a better description is obtained for the prime case.

Theorem 2.3. *Let P be a completely prime ideal of R and $k \in \mathfrak{B}$. Then $P^{(k)}$ is a completely prime ideal of $\mathcal{M}_n(B,R)$.*

Proof. Let X and Y be elements of $\mathcal{M}_n(B,R)$ such that $XY \in P^{(k)}$. Apply Proposition 1.5(1) to obtain $\pi_k XY\bar{\varepsilon}_k \in P$ and then use Lemma 1.4(3) to yield $(\pi_k X\bar{\varepsilon}_k)(\pi_k Y\bar{\varepsilon}_k) \in P$. Since P is completely prime, we have $\pi_k X\bar{\varepsilon}_k \in P$ or $\pi_k Y\bar{\varepsilon}_k \in P$. By Proposition 1.5(1) again, we have $X \in P^{(k)}$ or $Y \in P^{(k)}$.

The following technical lemma will be useful in the sequel.

Lemma 2.4. *Let $\mathcal{L}$ be an ideal of $\mathcal{M}_n(B,R)$ and $k \in \mathfrak{B}$. Then $X \in (\mathcal{L}_{(k)})^{(k)}$ if and only if $f^x_{kk} \in \mathcal{L}$ where $x = \pi_k X\bar{\varepsilon}_k$.*

Proof. Proposition 1.5(1) shows that $X \in (\mathcal{L}_{(k)})^{(k)}$ if and only if $x = \pi_k X\bar{\varepsilon}_k \in \mathcal{L}_{(k)}$. However Proposition 2.6 of [7] shows that $x \in \mathcal{L}_{(k)}$ if and only if $f^x_{kk} \in \mathcal{L}$.

Proposition 2.5. *Let $\mathcal{Q}$ be a proper completely prime ideal of $\mathcal{M}_n(B,R)$. Then there exists $k \in \mathfrak{B}$ such that $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. Hence $\bigcap\{(\mathcal{Q}_{(k)})^{(k)} \mid k \in \mathfrak{B}\} = \mathcal{Q}$.*

Proof. Since $\mathcal{Q}$ is proper, there exists k such that $f^1_{kk} \notin \mathcal{Q}$. Next we prove that $f^1_{kk} \notin \mathcal{Q}$ implies $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. So assume $f^1_{kk} \notin \mathcal{Q}$ for some k and assume $X \notin \mathcal{Q}$. Thus $f^1_{kk} X f^1_{kk} \notin \mathcal{Q}$. A routine calculation shows that $f^1_{kk} X f^1_{kk} = f^x_{kk}$ where $x = \pi_k X\bar{\varepsilon}_k$. Hence $X \notin (\mathcal{Q}_{(k)})^{(k)}$ by the previous lemma. Recall that $\mathcal{Q} \subseteq (\mathcal{Q}_{(i)})^{(i)}$ for $1 \leqq i \leqq n$ [7, Proposition 2.13(2)]. Therefore we obtain $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. Now we want to show that such k must be an element of $\mathfrak{B}$. Assume for purposes of contradiction that $k \notin \mathfrak{B}$. Then there exists h such that $h \neq k$ and $b_{hk} = b_{kh} = 1$. Since $f^1_{kh} f^1_{hh} f^1_{hk} = f^1_{kk} \notin \mathcal{Q}$, we have $f^1_{hh} \notin \mathcal{Q}$. However $f^1_{kk} f^1_{hh} = 0 \in \mathcal{Q}$ implies $f^1_{kk} \in \mathcal{Q}$ or $f^1_{hh} \in \mathcal{Q}$. This is a contradiction! We then have $k \in \mathfrak{B}$. The last part is now an immediate consequence.

We use $\langle X \rangle_R$ to denote the ideal of R generated by the nonempty subset X of R. If there can be no confusion, we write $\langle X \rangle$ for $\langle X \rangle_R$.

Lemma 2.6. *Let $\mathcal{Q}$ be a prime ideal of $\mathcal{M}_n(B,R)$ and $f^1_{kk} \notin \mathcal{Q}$ for some k. Then $f^1_{hh} \in \mathcal{Q}$ if and only if $b_{kh} = 0$ or $b_{hk} = 0$.*

Proof. First we prove that $\langle f^1_{ii} \rangle \cdot \langle f^1_{jj} \rangle = \{0\}$ whenever $b_{ij} = 0$. Assume $b_{ij} = 0$. Note that $f^1_{jj}(R^n) \subseteq \prod(R,j)$, thus $f^1_{jj} \in (\prod(R,j): R^n) = \{X \in \mathcal{M}_n(B,R) \mid X(R^n) \subseteq \prod(R,j)\}$. Note it was shown [7, Lemma 2.10] that the set $(\prod(R,j): R^n)$ is an ideal of $\mathcal{M}_n(B,R)$. We then obtain $\langle f^1_{jj} \rangle \subseteq (\prod(R,j): R^n)$. Since $b_{ij} = 0$, we have $f^1_{ii}(\prod(R,j)) = \{0\}$. This implies $f^1_{ii}\langle f^1_{jj} \rangle = \{0\}$ and hence $\langle f^1_{ii} \rangle \cdot \langle f^1_{jj} \rangle = \{0\}$. Now we are ready to prove our claim that $f^1_{hh} \in \mathcal{Q}$ if and only if $b_{kh} = 0$ or $b_{hk} = 0$. Since $f^1_{kk} \notin \mathcal{Q}$, we have $\langle f^1_{kk} \rangle \nsubseteq \mathcal{Q}$. If $b_{kh} = 0$ or $b_{hk} = 0$, then either $\langle f^1_{kk} \rangle \cdot \langle f^1_{hh} \rangle = \{0\}$ or $\langle f^1_{hh} \rangle \cdot \langle f^1_{kk} \rangle = \{0\}$. In either case we have $f^1_{hh} \in \mathcal{Q}$. If $f^1_{hh} \in \mathcal{Q}$ and if we assume that $b_{kh} = b_{hk} = 1$, then $f^1_{kk} = f^1_{kh} f^1_{hh} f^1_{hk} \in \mathcal{Q}$. This is not possible. So if $f^1_{hh} \in \mathcal{Q}$, then $b_{kh} = 0$ or $b_{hk} = 0$.

Proposition 2.7. *Let $\mathcal{Q}$ be a prime ideal of $\mathcal{M}_n(B,R)$ and $k \in \mathfrak{B}$. If $f_{kk}^1 \notin \mathcal{Q}$, then $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$ and $\mathcal{Q}_{(k)}$ is a prime ideal of R.*

Proof. Assume $f_{kk}^1 \notin \mathcal{Q}$. Let $X \in (\mathcal{Q}_{(k)})^{(k)}$ and $\bar{r} \in \prod(R,k)$. First we want to show $f_{kk}^1 X\bar{r} = f_{kk}^x \bar{r}$ where $x = \pi_k X\bar{\varepsilon}_k$. Since $k \in \mathfrak{B}$, we have $\bar{r} \sim_k (\bar{\varepsilon}_k \cdot r_k)$. Furthermore $X\bar{r} \sim_k (X\bar{\varepsilon}_k)\cdot r_k$. This implies $\pi_k X\bar{r} = (\pi_k X\bar{\varepsilon}_k)\cdot r_k$ and hence $f_{kk}^1 X\bar{r} = (f_{kk}^1 X\bar{\varepsilon}_k)\cdot r_k = f_{kk}^x \bar{r}$ where $x = \pi_k X\bar{\varepsilon}_k$. From the fact that $\langle f_{kk}^1\rangle(R^n) \subseteq \prod(R,k)$, we obtain $f_{kk}^1 X\langle f_{kk}^1\rangle = f_{kk}^x\langle f_{kk}^1\rangle$. The right-hand term is in $\mathcal{Q}$, since $f_{kk}^x \in \mathcal{Q}$ (see Lemma 2.4). Thus we have $f_{kk}^1(\mathcal{Q}_{(k)})^{(k)}\langle f_{kk}^1\rangle$ and then $\langle f_{kk}^1\rangle(\mathcal{Q}_{(k)})^{(k)}\langle f_{kk}^1\rangle$ are subsets of $\mathcal{Q}$. This forces $(\mathcal{Q}_{(k)})^{(k)} \subseteq \mathcal{Q}$. Hence $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. Now let L and H be ideals of R such that $LH \subseteq \mathcal{Q}_{(k)}$. Proposition 1.5 gives that $L^{(k)}H^{(k)} \subseteq (LH)^{(k)} \subseteq (\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. Therefore $L^{(k)} \subseteq \mathcal{Q}$ or $H^{(k)} \subseteq \mathcal{Q}$. Eventually $L \subseteq \mathcal{Q}_{(k)}$ or $H \subseteq \mathcal{Q}_{(k)}$; and $\mathcal{Q}_{(k)}$ is prime.

Let $\mathbf{P}_\nu(R)$ (resp. $\mathbf{Spec}_\nu(R)$) be the intersection (resp. the set) of all proper prime or completely prime ideals of R according to $\nu = 0$ or 2.

Theorems 2.2 and 2.3 and Proposition 2.5 give a complete description of all completely prime ideals of $\mathcal{M}_n(B,R)$. Moreover Theorem 2.1 and Proposition 2.7 describe prime ideals of all those structural matrix near-rings $\mathcal{M}_n(B,R)$ such that $\mathfrak{B} = \{1,\ldots,n\}$. Note if B is an upper or a lower triangular matrix, then the corresponding set $\mathfrak{B}$ is equal to $\{1,\ldots,n\}$.

Theorem 2.8. (1) *If $\mathfrak{B} = \{1,\ldots,n\}$, then:*

$$\mathbf{Spec}_0(\mathcal{M}_n(B,R)) = \{P^{(i)} \mid P \in \mathbf{Spec}_0(R), 1 \leqq i \leqq n\}.$$

(2) $\mathbf{Spec}_2(\mathcal{M}_n(B,R)) = \{P^{(i)} \mid P \in \mathbf{Spec}_2(R), i \in \mathfrak{B}\}$.

Note that if $\mathfrak{B}$ is empty, then $\mathbf{Spec}_2(\mathcal{M}_n(B,R))$ is empty. For example, we have $\mathbf{Spec}_2(\mathcal{M}_n(B,R)) = \mathbf{Spec}_2(\mathcal{M}_n(B,R)) = \emptyset$ whenever each entry b_{ij} of B is equal to 1.

Denote by γ the size of the set $\mathfrak{B}$. We can now describe the size of the sets $\mathbf{Spec}_0(\mathcal{M}_n(B,R))$ and $\mathbf{Spec}_2(\mathcal{M}_n(B,R))$. We write $|X|$ the cardinal of any set X. Then we have:

Theorem 2.9. (1) *If $\mathfrak{B} = \{1,\ldots,n\}$, then:*

$$|\mathbf{Spec}_0(\mathcal{M}_n(B,R))| = n \cdot |\mathbf{Spec}_0(R)|.$$

(2) $|\mathbf{Spec}_2(\mathcal{M}_n(B,R))| = \gamma \cdot |\mathbf{Spec}_2(R)|$.

Proof. Observe that if h and k are in $\mathfrak{B}$ and if $h \neq k$, then it is impossible to have $b_{hk} = b_{kh} = 1$. Now the result follows immediately from Theorem 2.8.

Theorem 2.10. (1) *If $\mathfrak{B} = \{1,\ldots,n\}$, then:*

$$\mathbf{P}_0(\mathcal{M}_n(B,R)) = \bigcap \{(\mathbf{P}_0(R))^{(i)} \mid 1 \leqq i \leqq n\}.$$

(2) $\mathbf{P}_2(\mathcal{M}_n(B,R)) = \bigcap \{(\mathbf{P}_2(R))^{(i)} \mid i \in \mathfrak{B}\}$.

Remark. In part (2), if $\mathfrak{B}$ is empty, then $\mathbf{P}_2(\mathcal{M}_n(B,R))=\mathcal{M}_n(B,R)$.

Example 2.11. (1) Suppose B is one of the following:

$$\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0&1\\1&1&1&1\\1&1&1&1\\1&0&0&1\end{pmatrix},\text{ or }\quad \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}.$$

Then $\mathfrak{B}$ is empty. Hence $\mathbf{P}_2(\mathcal{M}_4(B,R))=\mathcal{M}_4(B,R)$. The structural matrix near-ring with respect to the last Boolean matrix is, in fact, the matrix near-ring $\mathcal{M}_4(R)$. Note that the second Boolean matrix is not even symmetric.

(2) Suppose B is one of the following:

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&1\\1&1&1\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}.$$

Then $\mathfrak{B}=\{1,2,3\}$. Hence $\mathbf{P}_0(\mathcal{M}_3(B,R))=\bigcap\{(\mathbf{P}_0(R))^{(i)}\mid 1\leqq i\leqq 3\}$ and $\mathbf{P}_2(\mathcal{M}_3(B,R))=\bigcap\{(\mathbf{P}_2(R))^{(i)}\mid 1\leqq i\leqq 3\}$. Furthermore we have $|\mathbf{Spec}_0(\mathcal{M}_3(B,R))|=3\cdot|\mathbf{Spec}_0(R)|$ and $|\mathbf{Spec}_2(\mathcal{M}_3(B,R))|=3\cdot|\mathbf{Spec}_2(R)|$.

(3) Suppose

$$B=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}.$$

Then $\mathfrak{B}=\{2\}$. We have $\mathbf{P}_2(\mathcal{M}_3(B,R))=(\mathbf{P}_2(R))^{(2)}$ and $|\mathbf{Spec}_2(\mathcal{M}_3(B,R))|=|\mathbf{Spec}_2(R)|$.

(4) Suppose

$$B=\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&0&1&0\\1&1&1&1\end{pmatrix}.$$

Then $\mathfrak{B}=\{3,4\}$ and $\mathbf{P}_2(\mathcal{M}_4(B,R))=(\mathbf{P}_2(R))^{(3)}\bigcap(\mathbf{P}_2(R))^{(4)}$ and $|\mathbf{Spec}_2(\mathcal{M}_4(B,R))|=2\cdot|\mathbf{Spec}_2(R)|$.

A (zero-symmetric) near-ring R is called *2-primal* if the prime radical, $\mathbf{P}_0(R)$, is equal to the set of all nilpotent elements. We say an ideal I of R is *2-primal* if R/I is a 2-primal near-ring. It was shown in [2] that R is 2-primal if and only if $\mathbf{P}_0(R)=\mathbf{P}_2(R)$. Furthermore, the authors there investigated the following conditions:

(1) every prime ideal of R is a completely prime ideal;

(2) every ideal of R is a 2-primal ideal.

They showed that these two conditions are equivalent. Denote by $\mathfrak{N}_0^2$ the class of all (zero-symmetric) near-rings which satisfy these two conditions. It was shown in [3] that if R is a ring (not necessarily with identity), then R is a 2-primal ring (resp. is in $\mathfrak{N}_0^2$) if and only if the ring of all $n \times n$ upper triangular matrices over R is a 2-primal ring (resp. is in $\mathfrak{N}_0^2$). We extend this result to structural matrix near-rings.

Theorem 2.12. *Let B be a Boolean matrix such that $\mathfrak{B} = \{1, \ldots, n\}$. Then R is a 2-primal near-ring (resp. is in $\mathfrak{N}_0^2$) if and only if $\mathcal{M}_n(B, R)$ is a 2-primal near-ring (resp. is in $\mathfrak{N}_0^2$).*

Proof. Use Theorem 2.10 to obtain $\mathbf{P}_0(\mathcal{M}_n(B,R)) = \bigcap \{(\mathbf{P}_0(R))^{(i)} \mid 1 \leqq i \leqq n\}$ and $\mathbf{P}_2(\mathcal{M}_n(B,R)) = \bigcap \{(\mathbf{P}_2(R))^{(i)} \mid 1 \leqq i \leqq n\}$. Assume R is 2-primal. Therefore $\mathbf{P}_0(R) = \mathbf{P}_2(R)$. Obviously we then have $\mathbf{P}_0(\mathcal{M}_n(B,R)) = \mathbf{P}_2(\mathcal{M}_n(B,R))$ and hence $\mathcal{M}_n(B,R)$ is 2-primal. Conversely, we assume $\mathcal{M}_n(B,R)$ is 2-primal. So $\mathbf{P}_0(\mathcal{M}_n(B,R)) = \mathbf{P}_2(\mathcal{M}_n(B,R))$. (Recall: R can be identified with a subnear-ring of $\mathcal{M}_n(B,R)$. See 3.4 Corollary of [9].) Proposition 3.4 of [2] yields that $\mathbf{P}_0(R) = R \bigcap \mathbf{P}_0(\mathcal{M}_n(B,R)) = R \bigcap \mathbf{P}_2(\mathcal{M}_n(B,R)) = \mathbf{P}_2(R)$. So R is 2-primal. Similarly, we can show R is in $\mathfrak{N}_0^2$ if and only if $\mathcal{M}_n(B,R)$ is in $\mathfrak{N}_0^2$.

3. Nil, nilpotent, and locally nilpotent ideals

In this section we discuss nil, nilpotent, and locally nilpotent ideals of structural matrix near-rings. Nil and Levitzki nil radicals are studied. A subset X of a near-ring is *locally nilpotent* if every finite subset of X is nilpotent. In a near-ring R, the *nil radical* (resp. *Levitzki nil radical*) is the sum of all nil ideals (resp. locally nilpotent ideals).

In [10], van der Walt gave a description of nilpotent ideals in any matrix near-ring, that is:

the ideal I of R is nilpotent if and only if I^+ is nilpotent in $\mathcal{M}_n(R)$.

Note I^+ is the ideal of $\mathcal{M}_n(R)$ generated by the set $\{f_{ij}^a \mid a \in I, 1 \leqq i, j \leqq n\}$. We begin our investigation of nilpotence with the following result.

Proposition 3.1. *Let $\mathcal{K}$ be a nil (resp. nilpotent, locally nilpotent) 2-sided invariant subset of $\mathcal{M}_n(B,R)$. Then $\mathcal{K}_{(i)}$ is a nil (resp. nilpotent, locally nilpotent) 2-sided invariant subset of R for $1 \leqq i \leqq n$.*

Proof. We prove the nil case. The other two cases can be proved in a similar way. Assume $\mathcal{K}$ is a nil 2-sided invariant subset of $\mathcal{M}_n(B,R)$. We then have $x \in \mathcal{K}_{(i)}$ if and only if $f_{ii}^x \in \mathcal{K}$ [7, Proposition 2.6]. Note $f_{ii}^{x^m} = (f_{ii}^x)^m = 0$ for some $m \geqq 1$ since $\mathcal{K}$ is nil. This forces $x^m = 0$. We have the result.

Observe that in the preceding if $\mathscr{K}$ is nilpotent of index m, then $\mathscr{K}_{(i)}$ is nilpotent of index at most m.

Lemma 3.2. *Let H and L be right invariant subsets and let $k \in \mathfrak{B}$. Then $H^{(k)} + L^{(k)} = (H+L)^{(k)}$.*

Proof. Suppose $A \in (H+L)^{(k)}$. Then $\pi_k A\bar{\varepsilon}_k \in H+L$. Write $\pi_k A\bar{\varepsilon}_k = x+y$ where $x \in H$ and $y \in L$. We have $f^y_{kk} \in L^{(k)}$. (Recall: $y \in L$ if and only if $f^y_{kk} \in L^{(k)}$ [7, Lemma 2.12(1)].) Note that $\pi_k(A - f^y_{kk})\bar{\varepsilon}_k = x \in H$. This implies $A - f^y_{kk}$ is in $H^{(k)}$. (See Proposition 1.5(1).) Hence $A \in H^{(k)} + L^{(k)}$. Conversely, we suppose $A \in H^{(k)} + L^{(k)}$. Write $A = X + Y$ where $X \in H^{(k)}$ and $Y \in L^{(k)}$. Then $\pi_k X\bar{\varepsilon}_k \in H$ and $\pi_k Y\bar{\varepsilon}_k \in L$. Moreover we have $\pi_k(X+Y)\bar{\varepsilon}_k \in H + L$. Using Proposition 1.5 again, we obtain that A is in $(H+L)^{(k)}$.

Definition 3.3. Recall $B = [b_{ij}]$ is an $n \times n$ Boolean matrix with $1 \leqq i, j \leqq n$. We will denote by Λ_m subsets of $\{1,\ldots,n\}$ defined inductively for any natural numbers m as follows:

$$\Lambda_1 = \{j \mid \text{if } b_{jk} = 1, \text{ then } k = j\};$$

$$\Lambda_{m+1} = \{j \mid \text{if } b_{jk} = 1 \text{ and } k \neq j, \text{ then } k \in \Lambda_m\} \Big\backslash \bigcup_{i=1}^{m} \Lambda_i.$$

Observe that $\Lambda_\alpha \bigcap \Lambda_\beta = \emptyset$ if $\alpha \neq \beta$. The sets Λ_m may be empty for some m. For instance, if B is the Boolean matrix as described in Example 2.11(4), then $\Lambda_1 = \{3\}$ but $\Lambda_2, \Lambda_3, \ldots$ are empty. However if the Boolean matrix B satisfies the condition that $\mathfrak{B} = \{1,\ldots,n\}$, then the set $\{1,\ldots,n\}$ is equal to a finite, disjoint union of nonempty sets $\Lambda_1,\ldots,\Lambda_\lambda$. (See Proposition 3.4 below.)

Proposition 3.4. *Let B be an $n \times n$ Boolean matrix such that $\mathfrak{B} = \{1,\ldots,n\}$. Then there is a natural number λ less than or equal to n such that $\{1,\ldots,n\}$ is the disjoint union of nonempty sets $\Lambda_1,\ldots,\Lambda_\lambda$.*

Proof. We first show Λ_1 is nonempty. Assume for purpose of contradiction that Λ_1 is empty. For convenience sake we let $j_1, j_2, \ldots$ be elements of the set $\{1,\ldots,n\}$. Since $\mathfrak{B} = \{1,\ldots,n\}$ and $\Lambda_1 = \emptyset$, for any j_1 there exists j_2 such that $j_1 \neq j_2$ and $b_{j_1 j_2} = 1$. Similarly there exists j_3 such that $j_2 \neq j_3$ and $b_{j_2 j_3} = 1$. Moreover we have $j_1 \neq j_3$. (Note: if $j_1 = j_3$, then $b_{j_2 j_1} = b_{j_2 j_3} = 1$. This implies both $b_{j_1 j_2}$ and $b_{j_2 j_1}$ are equal to 1; a contradiction to the assumption of $\mathfrak{B}$.) Continue this process to obtain a collection of natural numbers $j_1, j_2, \ldots$ of $\{1,\ldots,n\}$ such that: (1) $j_h \neq j_k$ whenever $h \neq k$ and; (2) $b_{j_h j_k} = 1$ whenever $h < k$. Since the set $\{1,..,n\}$ is finite, there are j_h, j_k such that $h \neq k$ but $j_h = j_k$. This is not possible. Thus $\Lambda_1 \neq \emptyset$. Now if $\Lambda_1 = \{1,\ldots,n\}$, then $\lambda = 1$ and hence we are done. Assume $\Lambda_1 \neq \{1,\ldots,n\}$. We can show $\Lambda_2 \neq \emptyset$ similarly. Inductively, if we have nonempty sets $\Lambda_i, i = 1,\ldots,m$, such that $\Lambda_1 \bigcup \Lambda_2 \bigcup \cdots \bigcup \Lambda_m \neq \{1,\ldots,n\}$, then $\Lambda_{m+1} \neq \emptyset$. This process must terminate in finitely many steps. Hence the result follows.

Hereafter we stipulate that the Boolean matrix B satisfies the condition that $\mathfrak{B}=\{1,\dots,n\}$, except where noted. Denote by λ the number of nonempty, disjoint sets $\Lambda_1,\dots,\Lambda_\lambda$ such that $\{1,\dots,n\}=\bigcup_{i=1}^{\lambda}\Lambda_i$. From the preceding, λ is uniquely determined by B.

Lemma 3.5. *Let $A_1,\dots,A_\lambda$ be a collection of λ nonzero structural matrices of $\mathcal{M}_n(B,R)$. Assume that $\pi_k A_i \bar{\varepsilon}_k=0$ whenever $1\leqq i\leqq\lambda$ and $k\in\Lambda_i$. Then $A_\lambda\dots A_1=0$.*

Proof. Let $\bar{r}=(r_1,\dots,r_n)\in R^n$. We shall prove a more general result that if $t=i,\dots,\lambda$ and if $k\in\Lambda_i$, then $\pi_k A_t\dots A_1\bar{r}=0$. We use induction on $i=1,\dots,\lambda$. Assume $i=1$. Suppose $k\in\Lambda_1$ and $1\leqq t\leqq\lambda$. From the definition of Λ_1 and that of $\sim_k$, we have $\bar{r}\sim_k\bar{\varepsilon}_k\cdot r_k$. Invoke Proposition 2.2 of [11] to obtain $A_t\dots A_1\bar{r}\sim_k A_t\dots A_1(\bar{\varepsilon}_k\cdot r_k)$. Since $\pi_k A_1\bar{\varepsilon}_k=0$, we have $\pi_k A_t\dots A_1\bar{r}=(\pi_k A_t\dots A_1\bar{\varepsilon}_k)r_k=(\pi_k A_t\bar{\varepsilon}_k)\dots(\pi_k A_1\bar{\varepsilon}_k)r_k=0$. (See Lemma 2.1 of [4] and Lemma 1.4(3).) So the result is true when $i=1$. Now assume it is true for some $i<\lambda$. Suppose $k\in\Lambda_{i+1}$ and $i+1\leqq t\leqq\lambda$. If $j\neq k$ and $b_{kj}=1$, then $j\in\Lambda_i$ (see Definition 3.3). By the induction hypothesis, we have $\pi_j A_i\dots A_1\bar{r}=0$. Therefore $A_i\dots A_1\bar{r}\sim_k\bar{\varepsilon}_k\cdot(\pi_k A_i\dots A_1\bar{r})$. Multiply $A_t\dots A_{i+1}$ to both sides of the relation from the left to obtain $A_t\dots A_1\bar{r}\sim_k A_t\dots A_{i+1}(\bar{\varepsilon}_k\cdot(\pi_k A_i\dots A_1\bar{r}))$. Since $\pi_k A_{i+1}\bar{\varepsilon}_k=0$ by assumption, therefore:

$$\begin{aligned}\pi_k A_t\dots A_1\bar{r}&=(\pi_k A_t\dots A_{i+1}\bar{\varepsilon}_k)\cdot(\pi_k A_i\dots A_1\bar{r})\\&=(\pi_k A_t\bar{\varepsilon}_k)\dots(\pi_k A_{i+1}\bar{\varepsilon}_k)\cdot(\pi_k A_i\dots A_1\bar{r})=0.\end{aligned}$$

Thus, by induction, we have that if $k\in\Lambda_i$ and $i\leqq t\leqq\lambda$, then $\pi_k A_t\dots A_1\bar{r}=0$ for any $\bar{r}\in R^n$. Consequently if we take $t=\lambda$, then our claim that $A_\lambda\dots A_1=0$ follows immediately. (Note that $\{1,\dots,n\}=\bigcup_{i=1}^{\lambda}\Lambda_i$ by Proposition 3.4.)

Lemma 3.6. *Let $\mathscr{H}$ be a nonempty subset of $\mathcal{M}_n(B,R)$ and let X_k be the set $\{\pi_k H\bar{\varepsilon}_k \mid H\in\mathscr{H}\}$ for $k=1,\dots,n$. If there is a natural number m such that $(X_k)^m=0$ for each $k=1,\dots,n$, then $\mathscr{H}^{m\lambda}=0$.*

Proof. Assume there is a natural number m such that $(X_k)^m=0$ for all $k=1,\dots,n$. Let $H_1,\dots,H_{m\lambda}$ be elements of $\mathscr{H}$. Note that we use Lemma 1.4(3) to obtain $\pi_k H_m\dots H_1\bar{\varepsilon}_k=(\pi_k H_m\bar{\varepsilon}_k)\dots(\pi_k H_1\bar{\varepsilon}_k)$ and hence $\pi_k H_m\dots H_1\bar{\varepsilon}_k=0$ for each $k=1,\dots,n$. Similarly, we have:

$$\pi_k H_{2m}\dots H_{m+1}\bar{\varepsilon}_k=\cdots=\pi_k H_{m\lambda}\dots H_{m(\lambda-1)+1}\bar{\varepsilon}_k=0.$$

Take $A_1=H_m\dots H_1, A_2=H_{2m}\dots H_{m+1},\dots,$ and $A_\lambda=H_{m\lambda}\dots H_{m(\lambda-1)+1}$ in the preceding lemma to obtain $H_{m\lambda}\dots H_1=A_\lambda\dots A_1=0$.

Corollary 3.7. *$\mathcal{M}_n(B,R)$ is nil (resp. nilpotent, locally nilpotent) if and only if R is nil (resp. nilpotent, locally nilpotent).*

Proof. Suppose $\mathcal{M}_n(B,R)$ is nilpotent. Since R is isomorphic to a subnear-ring of $\mathcal{M}_n(B,R)$, therefore R is nilpotent. Conversely if R is nilpotent, then Lemma 3.6 shows that $\mathcal{M}_n(B,R)$ is nilpotent. The other two cases can be proved in a similar way.

Similar to the preceding, one can obtain the following:

Proposition 3.8. *Let $I_1,\ldots,I_n$ be nonempty nil (resp. nilpotent, locally nilpotent) subsets of R. Then $\bigcap_{i=1}^n I_i^{(i)}$ is a nil (resp. nilpotent, locally nilpotent) subset of $\mathcal{M}_n(B,R)$.*

Let $\mathbf{N}(R)$ be the sum of all proper nil ideals of R and let $\mathbf{L}(R)$ be the sum of all locally nilpotent ideals of R. Note that $\mathbf{N}$ and $\mathbf{L}$ are radical maps and are called nil radical and Levitzki radical, respectively. (See [1] and [6].)

Theorem 3.9. (1) $\mathbf{N}(\mathcal{M}_n(B,R))=\bigcap_{i=1}^n(\mathbf{N}(R))^{(i)}$.

(2) $\mathbf{L}(\mathcal{M}_n(B,R))=\bigcap_{i=1}^n(\mathbf{L}(R))^{(i)}$.

Proof. Suppose $\mathbf{N}(R)=R$. Then $(\mathbf{N}(R))^{(i)}=\mathcal{M}_n(B,R)$ for $1\leqq i\leqq n$. Corollary 3.7 gives that $\mathbf{N}(\mathcal{M}_n(B,R))=\mathcal{M}_n(B,R)$. Thus we are done. Suppose $\mathbf{N}(R)\neq R$. Then $\mathbf{N}(\mathcal{M}_n(B,R))\neq \mathcal{M}_n(B,R)$. The preceding yields $\bigcap_{i=1}^n(\mathbf{N}(R))^{(i)}\subseteq\mathbf{N}(\mathcal{M}_n(B,R))$. Furthermore Proposition 3.1 shows that $(\mathbf{N}(\mathcal{M}_n(B,R)))_{(i)}\subseteq\mathbf{N}(R)$ for each $i=1,\ldots,n$. A moment's thought, we have $((\mathbf{N}(\mathcal{M}_n(B,R)))_{(i)})^{(i)}\subseteq(\mathbf{N}(R))^{(i)}$ for $i=1,\ldots,n$. However $\mathbf{N}(\mathcal{M}_n(B,R))\subseteq((\mathbf{N}(\mathcal{M}_n(B,R)))_{(i)})^{(i)}$ for $i=1,\ldots,n$. (Recall: $\mathcal{L}\subseteq(\mathcal{L}_{(i)})^{(i)}$ for any left invariant subset $\mathcal{L}$ of $\mathcal{M}_n(B,R)$ [7, Proposition 2.13(2)].) Thus:

$$\mathbf{N}(\mathcal{M}_n(B,R))\subseteq\bigcap_{i=1}^n((\mathbf{N}(\mathcal{M}_n(B,R)))_{(i)})^{(i)}\subseteq\bigcap_{i=1}^n(\mathbf{N}(R))^{(i)}.$$

We have the result. The proof for the Levitzki radical case is similar.

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