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Elliptic hyperlogarithms

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Abstract. Let $\mathcal{E}$ be a complex elliptic curve and S be a non-empty finite subset of $\mathcal{E}$. We show that the functions $\tilde{\Gamma}$ introduced in [BDDT] out of string theory motivations give rise to a basis (as a vector space) of the minimal algebra $A_{\mathcal{E}\backslash S}$ of holomorphic multivalued functions on $\mathcal{E}\backslash S$ which is stable under integration, introduced in [EZ]; this basis is alternative to the basis of $A_{\mathcal{E}\backslash S}$ constructed in *loc. cit.* using elliptic analogs of the hyperlogarithm functions.

1 Introduction

The hyperlogarithms (HLs) are a family of multivalued holomorphic functions on the punctured complex plane, which were first introduced in [LD, Po]. They are natural one-variable specializations of the family of multiple polylogarithms, which have found numerous applications in mathematics (see [BB] for a survey, as well as [Br]). The HLs also found applications in particle physics (see [Pa] and the references therein). If $S \subset \mathbb{C}$ is a finite subset, the HLs on $\mathbb{C}\backslash S$ are defined by iterated integration of a flat connection on a trivial bundle over $\mathbb{C}\backslash S$ with values in a suitable free Lie algebra ([Br], Section 5.1), which corresponds to a Maurer–Cartan element on $\mathbb{C}\backslash S$ with values in this Lie algebra, and which is closely related with the KZ connection (see [EZ], Remark 2.20).

Both the algebra $\mathcal{H}_{\mathbb{C}\backslash S}$ generated by the HLs on $\mathbb{C}\backslash S$ and the algebra $\mathcal{O}(\mathbb{C}\backslash S)$ of regular functions on $\mathbb{C}\backslash S$ are subalgebras of the algebra of holomorphic multivalued functions on $\mathbb{C}\backslash S$. Their product $\mathcal{H}_{\mathbb{C}\backslash S} \cdot \mathcal{O}(\mathbb{C}\backslash S)$ within this algebra is stable under the operators $\mathrm{int}_\omega : f \mapsto (z \mapsto \int_{z_0}^{z} f\omega)$ for ω in the space $\Omega(\mathbb{C}\backslash S)$ of regular differentials on $\mathbb{C}\backslash S$, where z_0 is a fixed point of $\mathbb{C}\backslash S$ ([Br], Cor 5.6), and it coincides with the minimal algebra with these properties $A_{\mathbb{C}\backslash S}$ (see [EZ], Theorem A). It is proven in [Br], Section 5.2, that $\mathcal{H}_{\mathbb{C}\backslash S} \cdot \mathcal{O}(\mathbb{C}\backslash S)$ is isomorphic to $\mathcal{O}(\mathbb{C}\backslash S) \otimes \mathcal{H}_{\mathbb{C}\backslash S}$, and that $\mathcal{H}_{\mathbb{C}\backslash S}$ is isomorphic to an explicit shuffle algebra; this enables to prove that this product is a free module over $\mathcal{O}(\mathbb{C}\backslash S)$, and that a basis is given by the HLs (see [Br] Corollary 5.6, and also [DDMS]).

When $\mathbb{C}\backslash S$ is replaced by an arbitrary affine curve C, one can similarly attach to any Maurer–Cartan element J, non-degenerate in the sense of [EZ], an algebra $\mathcal{H}_C(J)$ of multivalued holomorphic functions on C (see [EZ], Section 2.3). The above result for the punctured complex plane extends to this case, namely the product of the algebra $\mathcal{H}_C(J)$ with the algebra $\mathcal{O}(C)$ of regular functions on C coincides with the minimal subalgebra A_C of the algebra of multivalued functions on C that is stable under the

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analogs int_ω of the above operators, where ω runs over the space $\Omega(C)$ of regular differentials on C ([EZ], Theorem A(b)), and is therefore independent of J. The algebra A_C can also be characterised as the set of multivalued holomorphic functions on C which have moderate growth at the cusps of C and have unipotent monodromy, in the sense that the representation of the fundamental group of C generated by such a function is a finite iterated extension of the trivial representation ([EZ], Theorem C). In [EZ], Theorem A(b), one shows the isomorphisms of A_C with $\mathcal{O}(C) \otimes \mathcal{H}_C(J)$ and of $\mathcal{H}_C(J)$ with an explicit shuffle algebra; this enables us to attach to a family of $\Omega(C)$, whose image is a basis of $\Omega(C)/d\mathcal{O}(C)$, a basis of A_C as an $\mathcal{O}(C)$-module (Definition 2.6 and Lemams 2.5 and 2.7), whose elements are analogs of the classical HLs.

In this article, we will be interested in the particular case where C has genus one; more precisely, we take it to coincide with the affine curve $\mathcal{E}_S := \mathcal{E}\backslash S$, for $\mathcal{E}$ a complex elliptic curve and $S \subset \mathcal{E}$ a finite non-empty subset. In §2.5, we exhibit an explicit family of $\Omega(\mathcal{E}_S)$ whose image is a basis of $\Omega(\mathcal{E}_S)/d\mathcal{O}(\mathcal{E}_S)$; according to Lemmas 2.5 and 2.7, this gives rise to a basis of $A_{\mathcal{E}_S}$ as an $\mathcal{O}(\mathcal{E}_S)$-module, whose elements will be called *elliptic hyperlogarithms*.

Recent development in particle physics led to the introduction of two (closely related) classes of multivalued functions $\tilde{\Gamma}$ and E_3 on $\mathcal{E}_S$ (see [BDDT], and [BK, BMMS] for applications in string theory). The functions $\tilde{\Gamma}$ are defined as iterated integrals of multivalued holomorphic differential forms on $\mathbb{C}\backslash pr^{-1}(S)$, $pr : \mathbb{C} \to \mathcal{E}$ being a universal covering map. One can show that the functions $\tilde{\Gamma}$ may also be defined, using iterated integration, out of a holomorphic flat connection on a non-trivial bundle over $\mathcal{E}_S$, which can be identified with a connection obtained upon restriction from the "universal KZB" connection of [CEE] (or [LR] if $|S| = 1$).

The main result of this article is a proof that the collection of functions $\tilde{\Gamma}$ gives rise to a basis of $A_{\mathcal{E}_S}$ over $\mathcal{O}(\mathcal{E}_S)$ (see Theorem 4.27), alternative to the family of elliptic HLs from §2.5. This relies on two main steps:

(a) We prove that $A_{\mathcal{E}_S}$ is equal to an algebra $\mathcal{G}$ generated by the functions $\tilde{\Gamma}$ (Theorem 4.20): the inclusion $\mathcal{G} \subset A_{\mathcal{E}_S}$ (Corollary 4.4) is based on the characterization of $A_{\mathcal{E}_S}$ as the set of multivalued functions sharing certain differential properties, and the inclusion $\mathcal{G} \supset A_{\mathcal{E}_S}$ is based on the characterization of $A_{\mathcal{E}_S}$ as the minimal algebra of multivalued functions which are stable under the endomorphisms int_ω (see Theorem 4.20).

(b) We prove a linear independence result for the functions $\tilde{\Gamma}$ (see Proposition 4.26), based on a criterion of [DDMS] and on a precise analysis of differential algebras attached to $\mathcal{E}_S$.

The definition of the functions E_3 relies on the additional datum of a degree 2 covering map $\pi : \mathcal{E} \to \mathbb{P}^1_{\mathbb{C}}$ and a finite subset $S_0 \subset \mathbb{P}^1_{\mathbb{C}}$. It is shown in [BDDT], §5 (see also Proposition 5.4 in this paper) that the vector spaces of multivalued functions generated by the $\tilde{\Gamma}$ and by the E_3 coincide if $S = \pi^{-1}(S_0)$. In [BDDT] it also shown that an algebra constructed out of the functions E_3 is stable under integration (Section 6, based on explicit computation). A consequence of (a) is an alternative proof of this stability result (see Proposition 5.6).

1.1 History and outlook

The first appearance of elliptic analogs of hyperlogarithms dates back to [BL], Section 10.1, where the authors construct functions called "elliptic Debye hyperlogarithm" via averaging of classical polylogarithm functions, generalizing analogous constructions of [Bl, Le, Z1]. These functions are holomorphic and multivalued, and we expect them to be contained in the algebra $A_{\mathcal{E}_S}$ (work in progress). They are closely related (see [BL], Section 10.3) to certain real-analytic functions defined via iterated integration out of a real-analytic version of the KZB connection of [CEE]; such functions may be seen as real-analytic analogs of elliptic hyperlogarithms, and were recently generalized to higher-genus Riemann surfaces in [DHS].

1.2 Organisation of the article

In Section 2, we recall from [EZ] the construction and properties of the algebra A_C attached to a curve C, and we specialize these results to the situation of an elliptic curve; in particular, we exhibit a basis of $A_{\mathcal{E}_S}$ over $\mathcal{O}(\mathcal{E}_S)$ which consists of elliptic HLs (§2.5). We then recall in Section 3 the definition of the functions $\tilde{\Gamma}$ ([BDDT]), paying special attention to regularization issues. Section 4 is devoted to proving our main result: the family of functions $\tilde{\Gamma}$ gives rise to a basis of $A_{\mathcal{E}_S}$ over $\mathcal{O}(\mathcal{E}_S)$ (Theorem 4.27), alternative to the basis from Section 2.5. In Section 5, we apply results from Section 4 to deduce a new proof of a result of [BDDT] on the stability under integration of an algebra $\mathcal{A}_3$ constructed out of the functions E_3.

1.3 Conventions, notation

In this paper, all vector spaces and algebras will be understood to be with base field $\mathbb{C}$.

1.3.1 Filtrations

A *filtered vector space* $F_\bullet V$ is the data of a vector space V and a collection of subspaces $(F_n V)_{n\geq 0}$ such that $F_n V \subset F_{n+1} V$ for any $n \geq 0$; by convention, $F_{-1} V = 0$. A morphism of filtered vector spaces $F_\bullet V \to F_\bullet W$ is a linear map $f : V \to W$, such that $f(F_n V) \subset F_n W$ for any $n \geq 0$. The *associated graded* of the filtered vector space $F_\bullet V$ is the graded vector space $\mathrm{gr}(V) := \oplus_{n\geq 0} \mathrm{gr}_n(V)$, where $\mathrm{gr}_n(V) := F_n V/F_{n-1} V$. If $F_\bullet V$ is a filtration of a vector space V, we denote by $x \mapsto \overline{x}$ the projection map $F_n V \to \mathrm{gr}_n(V)$ for any $n \geq 0$. If $F_\bullet V$ and $F_\bullet W$ are filtered vector spaces, their tensor product $F_\bullet(V \otimes W)$ is the data of $V \otimes W$, equipped with the filtration given by $F_n(V \otimes W) := \sum_{p=0}^n F_p V \otimes F_{n-p} W$. A *filtered algebra* is an algebra A equipped with a filtration $F_\bullet A$, such that the product of A is a morphism of filtered vector spaces $F_\bullet(A \otimes A) \to F_\bullet A$.

The *total space* of a filtered vector space $F_\bullet V$ is the subspace $F_\infty V := \cup_{n\geq 0} F_n V$ of V. The total space $F_\infty A$ of a filtered algebra $F_\bullet A$ is then a subalgebra of A.

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1.3.2 Meromorphic functions

For $\Sigma \subset \mathbb{C}$ a discrete subset, let $\mathcal{O}_{mer}(\mathbb{C}, \Sigma)$ be the algebra of meromorphic functions on $\mathbb{C}$ with sets of poles contained in Σ. It is contained in the algebra $\mathcal{O}_{hol}(\mathbb{C}\backslash\Sigma)$ of holomorphic functions on $\mathbb{C}\backslash\Sigma$, and it is equipped with a derivation $\partial := \partial/\partial z$; we use the notation $f' := \partial(f)$. For $a \in \mathbb{C}$, we denote by T_a the automorphism of the algebra of meromorphic functions on $\mathbb{C}$ (equal to $\cup_\Sigma \mathcal{O}_{mer}(\mathbb{C}, \Sigma)$) defined by $T_a f := (z \mapsto f(z-a))$.

2 Minimal stable subalgebras and elliptic hyperlogarithms (based on [EZ])

In §§2.1–2.3, we recall material from [EZ]. More precisely, to each smooth affine complex curve C, we attach a subalgebra A_C of the algebra of holomorphic functions $\mathcal{O}_{hol}(\tilde{C})$ on a universal cover of C (see Lem-Definition 2.2), called the minimal stable subalgebra (Section 2.1); we define an algebra filtration $F^\delta_\bullet\mathcal{O}_{hol}(\tilde{C})$ of $\mathcal{O}_{hol}(\tilde{C})$ with $A_C = F^\delta_\infty\mathcal{O}_{hol}(\tilde{C})$ (see Theorem 2.3(b)) (see Section 2.2); we express an $\mathcal{O}(C)$-basis of A_C in terms of HL functions associated with C (Section 2.3). In §2.4, we introduce the elliptic setup needed for the main results of the paper; then $C = \mathcal{E}_S$. In §2.5, we formulate a result of [EZ] in the elliptic context, namely the expression of a $\mathcal{O}(\mathcal{E}_S)$-basis of $A_{\mathcal{E}_S}$ in terms of elliptic HL functions.

2.1 The minimal stable subalgebra A_C associated with a curve C

Let C be a smooth affine complex curve, which we view as a Riemann surface. Let us fix a universal cover $p : \tilde{C} \to C$ and let $\mathcal{O}_{hol}(\tilde{C})$ be the algebra of holomorphic functions on $\tilde{C}$.

Let $\Omega(C)$ be the space of regular differentials on C. For any $\omega \in \Omega(C)$, the map $f \mapsto f \cdot p^*(\omega)$ is a linear map $\mathcal{O}_{hol}(\tilde{C}) \to \Omega_{hol}(\tilde{C})$, where $\Omega_{hol}(V)$ is the space of holomorphic differential 1-forms on a complex manifold V. For any $z_0 \in \tilde{C}$ and any $\alpha \in \Omega_{hol}(\tilde{C})$, the assignment $\alpha \mapsto [z \mapsto \int_{z_0}^z \alpha]$ is well-defined as $\tilde{C}$ is simply-connected, and defines a linear map $\Omega_{hol}(\tilde{C}) \to \mathcal{O}_{hol}(\tilde{C})$. For $\omega \in \Omega(C)$, let int_ω be the linear endomorphism of $\mathcal{O}_{hol}(\tilde{C})$ given by $f \mapsto [z \mapsto \int_{z_0}^z f \cdot p^*(\omega)]$.

Definition 2.1 A *stable subalgebra* of $\mathcal{O}_{hol}(\tilde{C})$ is a subalgebra with unit, which is stable under the endomorphism int_ω for any $\omega \in \Omega(C)$.

Although the definition of the operators int_ω depends on a choice of z_0, one checks that the notion of stable subalgebra is independent of such a choice.

Lemma-Definition 2.2 (see [EZ], Section 5.3) *If $A_C := \cap_{A \text{ stable subalgebra of } \mathcal{O}_{hol}(\tilde{C})} A$, then A_C is a stable subalgebra of $\mathcal{O}_{hol}(\tilde{C})$, which is minimal for the inclusion; it is called the minimal stable subalgebra of $\mathcal{O}_{hol}(\tilde{C})$.*

2.2 Relation of A_C with the differential filtration

Define inductively a collection of subspaces $(F^\delta_n\mathcal{O}_{hol}(\tilde{C}))_{n\geq 0}$ of $\mathcal{O}_{hol}(\tilde{C})$ by $F^\delta_0\mathcal{O}_{hol}(\tilde{C}) := \mathbb{C}$, and $F^\delta_{n+1}\mathcal{O}_{hol}(\tilde{C}) := \{f \in \mathcal{O}_{hol}(\tilde{C}) \,|\, df \in F^\delta_n\mathcal{O}_{hol}(\tilde{C}) \cdot p^*\Omega(C)\}$for $n \geq 0$. Define also the subspace $F^\delta_\infty\mathcal{O}_{hol}(\tilde{C}) := \cup_{n\geq 0} F^\delta_n\mathcal{O}_{hol}(\tilde{C})$ of $\mathcal{O}_{hol}(\tilde{C})$.

Let us denote by $\mathcal{O}(C)$ the algebra of regular functions on C.

Theorem 2.3 (see [EZ]) *(a) $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ is an algebra filtration of $\mathcal{O}_{hol}(\tilde{C})$, such that $p^*\mathcal{O}(C) \subset F_1^\delta \mathcal{O}_{hol}(\tilde{C})$.*

(b) $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) = A_C$.

Proof The fact that $F_\bullet^\delta$ is an algebra filtration is proven in [EZ], Prop 5.5(a). The inclusion $p^*\mathcal{O}(C) \subset F_1^\delta \mathcal{O}_{hol}(\tilde{C})$ follows from the $n = 0$ case of Prop 5.5(b) in [EZ]. Statement (b) follows from [EZ], Theorem C. ■

2.3 Analogs of HLs on a curve C

For W a complex vector space, denote by $\mathrm{Sh}(W)$ the shuffle Hopf algebra of W. This is the vector space $\oplus_{k\geq 0} W^{\otimes k}$ (the element $w_1 \otimes \cdots \otimes w_k \in W^{\otimes k}$ is denoted $[w_1|\cdots|w_k]$), equipped with the shuffle product $\sqcup\!\sqcup$, which is commutative, the deconcatenation coproduct Δ given by $[w_1|\cdots|w_k] \mapsto \sum_{j=0}^k [w_1|\ldots|w_j] \otimes [w_{j+1}|\ldots|w_k]$, the counit ε given by $1 \mapsto 1$ and $[w_1|\cdots|w_k] \mapsto 0$ for $k > 0$, and the antipode given by $[w_1|\ldots|w_k] \mapsto (-1)^k [w_k|\ldots|w_1]$. A linear map $\phi : W \to W'$ induces a Hopf algebra morphism $\mathrm{Sh}(W) \to \mathrm{Sh}(W')$, also denoted ϕ.

Let $z_0 \in \tilde{C}$. There is a unique linear map

$$I_{z_0} : \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{hol}(\tilde{C}) \tag{2.3.1}$$

satisfying the identities $I_{z_0}(a)(z_0) = \varepsilon(a)$ and $d(I_{z_0}([\omega_1|\cdots|\omega_n]) = I_{z_0}([\omega_1|\cdots|\omega_{n-1}]) \cdot \omega_n$ for any $n \geq 0$ and $\omega_1, \ldots, \omega_n \in \Omega(C)$, so that $I_{z_0}([\omega_1|\cdots|\omega_n]) = \mathrm{int}_{\omega_n}(I_{z_0}([\omega_1|\cdots|\omega_{n-1}]))$; such a map I_{z_0} is an algebra morphism, given by iterated integration based at z_0 (see [EZ], Lemma 2.2).

Set $\mathrm{H}_C := \Omega(C)/d\mathcal{O}(C)$. Let $\sigma : \mathrm{H}_C \to \Omega(C)$ be a section of the projection map $\Omega(C) \to \mathrm{H}_C$.

Lemma 2.4 ([EZ]) *(a) Let $\sigma_* : \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$ be algebra morphism induced by σ, then $\tilde{f}_{\sigma,z_0} := I_{z_0} \circ \sigma : \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{hol}(\tilde{C})$ is an injective algebra morphism.*

(b) The image of $\tilde{f}_{\sigma,z_0}$ is independent of z_0; it is a subalgebra of $\mathcal{O}_{hol}(\tilde{C})$, denoted $\mathcal{H}_C(\sigma)$.

(c) The composition $\mathcal{O}(C) \otimes \mathcal{H}_C(\sigma) \hookrightarrow \mathcal{O}_{hol}(\tilde{C}) \otimes \mathcal{O}_{hol}(\tilde{C}) \to \mathcal{O}_{hol}(\tilde{C})$ of the canonical injections with the product map of $\mathcal{O}_{hol}(\tilde{C})$ yields an algebra isomorphism $\mathcal{O}(C) \otimes \mathcal{H}_C(\sigma) \simeq A_C$.

Proof Statements (a) and (c) are consequences of Theorem A(b) in [EZ] with J equal to the element J_σ defined in Definition 1.2 of [EZ]. Statement (b) follows from [EZ], Theorem A(a). ■

Lemma 2.5 *Let $d := \dim(\mathrm{H}_C)$ and $(h_i)_{i\in[\![1,d]\!]}$ be*[1] *a $\mathbb{C}$-basis of H_C. Then the family*

$$(\tilde{f}_{\sigma,z_0}([h_{i_1}|\ldots|h_{i_s}]))_{s\geq 0,(i_1,\ldots,i_s)\in[\![1,d]\!]^s}$$

is a $\mathbb{C}$-basis of $\mathcal{H}_C(\sigma)$, as well as an $\mathcal{O}(C)$-basis of A_C.

Proof The first statement follows from [EZ], Proposition 5.10, and the second statement follows from its combination with Lemma 2.4(c). ■

[1] We set $[\![1,d]\!] := \{1, \ldots, d\}$.

Definition 2.6 If $(\alpha_i)_{i\in[[1,d]]}$ is a family of $\Omega(C)$ whose image in $\mathrm{H}_C = \Omega(C)/d(\mathcal{O}(C))$ is a $\mathbb{C}$-basis, set

$$L_{\alpha_{i_1},\dots,\alpha_{i_s}} := I_{z_0}([\alpha_{i_1}|\dots|\alpha_{i_s}]) \tag{2.3.2}$$

for any $s \geq 0$ and $i_1, \dots, i_s \in [[1, d]]$.

We call (2.3.2) the *hyperlogarithm functions associated with the family* $(\alpha_i)_{i\in[[1,d]]}$.

Lemma 2.7 *If $(\alpha_i)_{i\in[[1,d]]}$ is a family of $\Omega(C)$ as in Definition 2.6, then there exists a unique pair $(\sigma, (h_i)_i)$ where $\sigma : \mathrm{H}_C \to \Omega(C)$ is a section of the projection $\Omega(C) \to \mathrm{H}_C$ and $(h_i)_{i\in[[1,d]]}$ is a $\mathbb{C}$-basis of H_C, such that $\alpha_i = \sigma(h_i)$ for any i. One then has the identity*

$$L_{\alpha_{i_1},\dots,\alpha_{i_s}} = \tilde{f}_{\sigma,z_0}([h_{i_1}|\dots|h_{i_s}]). \tag{2.3.3}$$

Proof This is straightforward. ∎

2.4 Background on elliptic curves

2.4.1 Elliptic curves and coverings

Let us denote by $\mathfrak{H}$ the complex upper half-plane. Until the end of §4, an element τ in $\mathfrak{H}$ is fixed. Let $\Lambda := \mathbb{Z} + \mathbb{Z}\tau\subset\mathbb{C}$.

Definition 2.8 We set $\mathcal{E} := \mathbb{C}/\Lambda$, and we denote $pr : \mathbb{C} \to \mathbb{C}/\Lambda = \mathcal{E}$ the canonical projection.

$\mathcal{E}$ is a compact genus-one Riemann surface, and pr is a universal cover map for $\mathcal{E}$.

Definition 2.9 Let S be a finite subset of $\mathcal{E}$ containing $pr(0)$. We set $\mathcal{E}_S := \mathcal{E}\backslash S$, and we fix a universal cover $p : \tilde{\mathcal{E}}_S \to \mathcal{E}_S$.

The restriction of pr is a covering map $\mathbb{C}\backslash pr^{-1}(S) \to \mathcal{E}_S$. The universal cover map p then admits a factorization $p = \varpi \circ pr$, where $\varpi : \tilde{\mathcal{E}}_S \to \mathbb{C}\backslash pr^{-1}(S)$ is a universal cover map, so that the following diagram commutes

$$\begin{tikzcd} \tilde{\mathcal{E}}_S \arrow[rr, "\varpi"] \arrow[dr, "p"'] & & \mathbb{C}\backslash pr^{-1}(S) \arrow[dl, "pr"] \\ & \mathcal{E}_S & \end{tikzcd}$$

Definition 2.10 (a) Let $s \mapsto \tilde{s}$ be a section of the map $pr : pr^{-1}(S) \to S$ such that $pr(0) \mapsto 0$.

(b) We denote by $\tilde{S}$ the subset of $pr^{-1}(S)$ defined as the image of the map $s \mapsto \tilde{s}$.

One has $pr^{-1}(S) = \cup_{s\in S}(\tilde{s} + \Lambda)$ (equality of subsets of $\mathbb{C}$).

2.4.2 Functions in $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$

For $r \in \mathbb{Z}_{\geq 1}$ and $z \in \mathbb{C}\backslash\Lambda$, let

$$E_r(z) := \sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^r} \in \mathbb{C},$$

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where the series is absolutely convergent for $r > 2$, and is defined by Eisenstein summation for $r = 1, 2$ (see [BMMS], footnote 8), and let E_r be the function $z \mapsto E_r(z)$; then E_r belongs to $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$. If $r \geq 2$, then the function E_r is elliptic, i.e., Λ-invariant; on the other hand, E_1 satisfies the identities[2]

$$T_1 E_1 = E_1, \qquad T_\tau E_1 = E_1 + 2\pi \mathrm{i}. \tag{2.4.1}$$

For every $r \geq 1$, the function E_r satisfies the identity

$$E_r' = -r\, E_{r+1}. \tag{2.4.2}$$

Moreover, for every $r \geq 1$ consider also the Eisenstein series

$$e_r := \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^r} \in \mathbb{C}, \tag{2.4.3}$$

which is again defined via the Eisenstein summation for $r \leq 2$, and vanishes when r is odd.

The Weierstrass $\wp$-function attached to Λ is given by $\wp := E_2 - e_2$. One then has

$$\forall r \geq 0, \quad \wp^{(r)} = (-1)^r (r+1)! E_{r+2} - e_2 \delta_{r,0}. \tag{2.4.4}$$

Definition 2.11 We denote by $(g_n)_{n \geq 0}$ the family of meromorphic functions[3] in $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$ defined via the following identity of generating series in the formal variable α:

$$\sum_{n \geq 0} g_n\, \alpha^{n-1} := \frac{1}{\alpha} \exp\Big(-\sum_{r \geq 1} \frac{(-\alpha)^r}{r} (E_r - e_r) \Big). \tag{2.4.5}$$

In particular, $g_0 = 1$ and $g_1 = E_1$ has a simple pole with residue 1 at all points of Λ. Fourier expansions for the functions g_n can be found in [BMMS], Equations (3.35) and (3.36).

Lemma 2.12 (see [Ma], Prop 2.1.3) *The function g_n is regular at 0 for any $n \geq 2$.*

The left-hand side of (2.4.5) is an element of the ring $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)((\alpha))$ of formal Laurent series with coefficients in $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$, which we denote by F.

Remark 2.13 F is the expansion at $\alpha = 0$ of a meromorphic function of two variables, known as the Kronecker function ([W, Z2]).

Lemma-Definition 2.14 *(a) For every $a \in pr^{-1}(S)$ and every $n \geq 0$, the function $T_a g_n = (z \mapsto g_n(z - a))$ belongs to $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$.*

(b) Set $\omega_{n,a} := T_a(g_n) \cdot dz$, then $\omega_{n,a}$ belongs to $\Omega_{hol}(\mathbb{C} \setminus pr^{-1}(S))$.

Proof The first statement follows from $T_a(\mathcal{O}_{mer}(\mathbb{C}, \Lambda)) = \mathcal{O}_{mer}(\mathbb{C}, a + \Lambda)$ and $a + \Lambda \subset pr^{-1}(S)$. The second statement follows from the module structure of $\Omega_{hol}(\mathbb{C} \setminus pr^{-1}(S))$ over $\mathcal{O}_{hol}(\mathbb{C} \setminus pr^{-1}(S))$, and the inclusion $\mathcal{O}_{mer}(\mathbb{C}, \Lambda) \subset \mathcal{O}_{hol}(\mathbb{C} \setminus pr^{-1}(S))$. ■

[2] We set $\mathrm{i} := \sqrt{-1}$.

[3] Notice the slight change of notation with respect to [BMMS] and [BDDT], where these functions are denoted $g^{(n)}$, in order to avoid confusion with n-th derivatives.

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2.4.3 Functional equations

Lemma 2.15 *For every $n \geq 1$, one has*[4] *the following equalities in $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$:*

$$T_1 g_n = g_n\,, \qquad (T_\tau - id)(g_n) = \sum_{k=0}^{n-1} \frac{(2\pi \mathrm{i})^{n-k}}{(n-k)!} g_k\,, \tag{2.4.6}$$

$$g_n' = \sum_{k=0}^{n-1} (-1)^{n-k} E_{n-k+1} g_k\,. \tag{2.4.7}$$

Proof Applying (2.4.1) and (2.4.2) to the right-hand side of (2.4.5), we obtain that

$$F(z+1, \alpha) = F(z, \alpha)\,, \qquad F(z+\tau, \alpha) = e^{-2\pi i \alpha} F(z, \alpha)\,,$$
$$\frac{\partial}{\partial z} F(z, \alpha) = \Big(\sum_{r \geq 1} (-\alpha)^r E_{r+1}(z) \Big) F(z, \alpha)\,,$$

(the first line is (3.13) in [BMMS]). The statement follows by comparing these identities with the left-hand side of (2.4.5). ∎

Corollary 2.16 *For any $n \geq 0$ and $a \in pr^{-1}(S)$, one has*

$$\omega_{n,a+1} = \omega_{n,a}, \quad \omega_{n,a+\tau} = \sum_{k=0}^{n} \frac{(2\pi \mathrm{i})^{n-k}}{(n-k)!} \omega_{k,a}. \tag{2.4.8}$$

Proof Follows from (2.4.6) and from the invariance of dz under T_a, $a \in pr^{-1}(S)$. ∎

2.5 An $\mathcal{O}(\mathcal{E}_S)$-basis of $A_{\mathcal{E}_S}$ arising from elliptic HLs

We introduce the shorthand notation $O_S := \mathcal{O}(\mathcal{E}_S)$; then O_S is a subalgebra of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$.

Definition 2.17 ∂ is the derivation of O_S arising from the restriction of the action of ∂ on $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$ (see Section 1.3.2).

Set $S_\star := S \sqcup \{\star\}$. Define a map $S_\star \to \Omega(\mathcal{E}_S)$, $s \mapsto \alpha_s$ by $\alpha_s := (T_s - id)(g_1) \cdot dz$ if $s \in S \backslash \{0\}$, $\alpha_\star := dz$, $\alpha_0 := E_2 \cdot dz$.

Lemma 2.18 *The family $(\alpha_s)_{s \in S_\star}$ is a family of $\Omega(\mathcal{E}_S)$ whose image in $\mathrm{H}_{\mathcal{E}_S} = \Omega(\mathcal{E}_S)/d(O_S)$ is a $\mathbb{C}$-basis.*

Proof The residue map induces a linear map $O_S/\partial(O_S) \to (\mathbb{C}^S)_0$, where $(\mathbb{C}^S)_0 \subset \mathbb{C}^S$ is the set of tuples whose sum is equal to 0.

If $f \in O_S$ is a representative of an element of the kernel of $O_S/\partial(O_S) \to (\mathbb{C}^S)_0$, then the residue of $f dz$ at any $s \in S$ is zero, so that the function $z \mapsto \int_{z_0}^z f(u) du$ is single-valued around each $s \in S$. It follows that there exist $a, b \in \mathbb{C}$ such that $z \mapsto \int_{z_0}^z f(u) du - az - bg_1(z)$ belongs to O_S, and therefore $f \equiv a + bg_1' = a - bE_2$ mod $\partial(O_S)$. This implies that the kernel of $O_S/\partial(O_S) \to (\mathbb{C}^S)_0$ is spanned by 1 and E_2.

[4] Equation (2.4.6) shows that g_n is not regular on the whole of $\mathbb{C}$, contrary to the statement of [BDDT], three lines after (4.24).

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The map $O_S/\partial(O_S) \to (\mathbb{C}^S)_0$ is surjective, because a preimage of $(a_s)_s \in (\mathbb{C}^S)_0$ is $\sum_{s\in S} a_s T_{\tilde{s}}(g_1)$. Therefore $O_S/\partial(O_S)$ fits in an exact sequence

$$0 \to \mathrm{Span}_{\mathbb{C}}(1, E_2) \to O_S/\partial(O_S) \to (\mathbb{C}^S)_0 \to 0.$$

A basis of $O_S/\partial(O_S)$ is therefore the union of a basis of $\mathrm{Span}_{\mathbb{C}}(1, E_2)$ and of the preimage of a basis of $(\mathbb{C}^S)_0$. Since $(1, E_2)$ is linearly independent and since $((T_{\tilde{s}} - id)(g_1))_{s\in S\setminus\{0\}}$ is such a preimage, the image of

$$1, E_2, ((T_{\tilde{s}} - id)(g_1))_{s\in S\setminus\{0\}} \tag{2.5.1}$$

in $O_S/\partial(O_S)$ forms a $\mathbb{C}$-basis of $O_S/\partial(O_S)$.

The linear isomorphism $-\cdot dz : O_S \to \Omega(\mathcal{E}_S), f \mapsto f\cdot dz$ is such that $(-\cdot dz)\circ\partial = d$ (equality of linear maps $O_S \to \Omega(\mathcal{E}_S)$). This linear isomorphism therefore induces a linear isomorphism $-\cdot dz : O_S/\partial(O_S) \to \Omega(\mathcal{E}_S)/d(O_S)$. The statement follows from this and from the fact that the claimed family is the image of (2.5.1) by $-\cdot dz$. ■

Corollary 2.19 *The family*

$$L_{\alpha_{s_1},\dots,\alpha_{s_k}} = I_{z_0}([\alpha_{s_1}|\cdots|\alpha_{s_k}]), \quad k \geq 0, \quad (s_1,\dots,s_k) \in S_\star^k. \tag{2.5.2}$$

is an O_S-basis of the O_S-module $A_{\mathcal{E}_S}$.

Proof This follows from Lemmas 2.5 and (2.3.3). ■

3 The multivalued functions $\tilde{\Gamma}$ from [BDDT]

The article [BDDT] introduces a family of functions $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r\end{smallmatrix}; -\right)$, with $r \geq 0$, $n_i \geq 0$ and $a_i \in pr^{-1}(S)$ for[5] $i \in [[1, r]]$, which are multivalued functions on $\mathbb{C}\setminus pr^{-1}(S)$ defined as regularized iterated integrals of the forms $(\omega_{n,a})_{n\geq 0, a\in pr^{-1}(S)}$ from Section 2.4, where the integration path starts at 0. It follows from the functional equation (2.4.8) that this vector space is spanned by the collection of the same functions, where the parameters satisfy the restricted condition $a_i \in \tilde{S}$ for any $i \in [[1, r]]$. Since one of the possible forms, namely $\omega_{1,0}$, has a pole at this point, regularisation is needed for the definition of the functions $\tilde{\Gamma}$ even in the restricted case.

The regularization procedure indicated in [BDDT] relies on replacing the endpoint 0 by a small ε and extracting a value from the asymptotic expansion in ε; this procedure was rigourously carried out in a similar situation in [BGF], Section 3.8, which is based on the "tangential base points" ideas of [De], Section 15. On the other hand, the regularization procedure indicated in [BK] for the definition of the functions $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r\end{smallmatrix}; -\right)$ is an analog of the procedure from [Br], §5.1 (see also [Pa], Section 3.3), for the definition of hyperlogarithm functions in genus zero, which relies on a decomposition result for the shuffle algebra of a vector space (see Lem 3.13).

We show that both the "tangential base point" and the "shuffle" regularization procedures amount to the construction of algebra morphisms $k_{\vec{0}}, k_0^{\sqcup\!\sqcup} : \mathrm{Sh}(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ (Section 3.1, Lemma 3.10 and Section 3.2, Lemma 3.14) where V is a suitable vector space, which we show to coincide (Section 3.2, Lem 3.15). In Section 3.3, we then

[5] For $a \leq b \in \mathbb{Z}$, we set $[[a, b]] := \{a, a+1, \dots, b\}$.

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apply the construction of these two equal morphisms to the definition of the functions $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)$ and to the proof of their properties.

3.1 Tangential base point regularisation

For each $z_0 \in \tilde{\mathcal{E}}_S$, the linear map I_{z_0} defined in (2.3.1) extends to a unique linear map

$$I_{z_0} : \mathrm{Sh}(\Omega_{hol}(\tilde{\mathcal{E}}_S)) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$$

satisfying the identities $I_{z_0}(a)(z_0) = \varepsilon(a)$ and $d(I_{z_0}([\omega_1|\cdots|\omega_n]) = I_{z_0}([\omega_1|\cdots|\omega_{n-1}]) \cdot \omega_n$ for any $n \geq 0$ and $\omega_1, \ldots, \omega_n \in \Omega_{hol}(\tilde{\mathcal{E}}_S)$; then I_{z_0} is an algebra morphism, given by iterated integration based at z_0.

Definition 3.1 (a) V is the complex vector space freely generated by the symbols $\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right)$, where $(n, a) \in \mathbb{Z}_{\geq 0} \times \tilde{S}$, where $\tilde{S}$ is as in Definition 2.10(b).

(b) $V_+ \subset V$ is the vector subspace generated by $\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right)$, where $(n, a) \neq (1, 0)$.

(c) $\mathrm{Sh}^*(V)$ is the subspace $\mathbb{C}1 \oplus [V_+|\mathrm{Sh}(V)]$ of $\mathrm{Sh}(V)$.

One checks that $\mathrm{Sh}^*(V)$ is a subalgebra of $\mathrm{Sh}(V)$.

Definition 3.2 (a) $\kappa : V \to \Omega_{hol}(\tilde{\mathcal{E}}_S)$ is the linear map induced by $\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right) \mapsto \omega_{n,a}$ for any $(n, a) \in \mathbb{Z}_{\geq 0} \times \tilde{S}$.

(b) We denote by $\kappa_* : \mathrm{Sh}(V) \to \mathrm{Sh}(\Omega_{hol}(\tilde{\mathcal{E}}_S))$ the induced Hopf algebra morphism.

(c) For $z_0 \in \tilde{\mathcal{E}}_S$, $k_{z_0} : \mathrm{Sh}(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ is the composed algebra morphism $I_{z_0} \circ \kappa_*$.

We fix $\delta > 0$ such that $]0, \delta[\subset \mathbb{C}\backslash pr^{-1}(S)$; we fix a continuous map $\mathrm{sect} :]0, \delta[\to \tilde{\mathcal{E}}_S$ such that $\varpi \circ \mathrm{sect}$ is the canonical inclusion.

Definition 3.3 $\mathcal{F} \subset \mathbb{C}^{]0,\delta[}$ is the set of functions $\phi :]0, \delta[\to \mathbb{C}$, such that for some $\alpha > 0$ one has $\phi(t) = O(t^{1-\alpha})$ when $t \to 0$.

Lemma 3.4 *(a) The map $\mathbb{C}[X] \otimes \mathcal{F} \to \mathcal{F}$ given by $P \otimes \phi \mapsto P \bullet \phi := (t \mapsto P(\log(t))\phi(t))$ is well-defined and makes $\mathcal{F}$ into a $\mathbb{C}[X]$-module.*

(b) The direct sum $\mathbb{C}[X] \oplus \mathcal{F}$, equipped with the product given by $(P, \phi) \cdot (Q, \psi) = (PQ, P \bullet \psi + Q \bullet \phi + \phi\psi)$, is a commutative and associative algebra.

(c) The algebra from (b) is equipped with a pair of algebra morphisms

$$\mathbb{C} \xleftarrow{\mathrm{ev}} \mathbb{C}[X] \oplus \mathcal{F} \overset{\mathrm{can}}{\hookrightarrow} \mathbb{C}^{]0,\delta[}$$

where $\mathrm{ev}(P, \phi) := \phi(0)$ *and* $\mathrm{can}(P, \phi) := (t \mapsto P(\log(t)) + \phi(t))$, *and the algebra morphism* can *is injective.*

Proof (a) follows from the fact that for any $r \geq 0$, one has $(\log(t))^r = O(t^{-\alpha})$ at the neighborhood of 0 for any $\alpha > 0$. (b) is an immediate direct check. Let us prove (c). The fact that ev and can are algebra morphisms can be verified directly. Let us prove the injectivity of can (see also [BGF], Lemma 3.350). Assume that $(P, \phi) \in \mathbb{C}[X] \oplus \mathcal{F}$ and $\mathrm{can}(P, \phi) = 0$. If P is nonzero, let $d \geq 0$ be its degree and $a_d X^d$ be its dominant term. Then $\mathrm{can}(P, \phi)(t) \sim a_d(\log(t))^d$ at the neighborhood of 0, which contradicts $\mathrm{can}(P, \phi) = 0$, therefore $P = 0$. Then $\mathrm{can}(0, \phi) = \phi$, which implies $\phi = 0$. ∎

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In what follows, we will view $\mathbb{C}[X]\oplus\mathcal{F}$ as a subalgebra of $\mathbb{C}^{]0,\delta[}$, making use of the injectivity of the morphism can.

Lemma 3.5 *For $a\in \mathrm{Sh}(V)$ and $z\in\tilde{\mathcal{E}}_S$, denote by $k(a,z)$ the assignment $]0,\delta[\ni t\mapsto k_{\mathrm{sect}(t)}(a)(z)\in\mathbb{C}$; then $k(a,z)\in\mathbb{C}^{]0,\delta[}$. For any $z\in\tilde{\mathcal{E}}_S$, the map $a\mapsto k(a,z)$ is an algebra morphism $\mathrm{Sh}(V)\to\mathbb{C}^{]0,\delta[}$.*

Proof This follows from the fact that for any $z\in\tilde{\mathcal{E}}_S$ and $t\in]0,\delta[$, the map $a\mapsto k_{\mathrm{sect}(t)}(a)(z)$ is an algebra morphism $\mathrm{Sh}(V)\to\mathbb{C}$ (see Definition 4.2(c)). ∎

Lemma 3.6 *For each $z\in\tilde{\mathcal{E}}_S$, there exists a complex number $G(z)$ such that the function $]0,\delta[\ni t\mapsto\int_{\mathrm{sect}(t)}^{z}\omega_{1,0}+\log(t)$ belongs to $G(z)+\mathcal{F}$. Then $G\in\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ and $(t\mapsto G(\mathrm{sect}(t)))\in(t\mapsto\log(t))+\mathcal{F}$.*

Proof Since $\omega_{1,0}$ has a pole of order 1 at 0, for any $z\in\tilde{\mathcal{E}}_S$, the map $t\mapsto\int_{\mathrm{sect}(t)}^{z}\omega_{1,0}+\log(t)$ is an analytic function; denoting by $G(z)$ its value at 0, it therefore belongs to $G(z)+\mathcal{F}$. One then has $G(z)=G(\mathrm{sect}(\delta/2))+\int_{\mathrm{sect}(\delta/2)}^{z}\omega_{1,0}$ for any $z\in\tilde{\mathcal{E}}_S$, which implies the holomorphicity statement. ∎

Notice that one has

$$\mathrm{Sh}(V)=\oplus_{r\geq 0}[\underbrace{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)|\dots|\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}_{r\text{ factors}}|\mathrm{Sh}^*(V)]. \tag{3.1.1}$$

Definition 3.7 (a) The degree of an element $a\in\mathrm{Sh}(V)$ is $-\infty$ if $a=0$, and is the maximum of the set of integers $r\geq 0$ such that the projection of a in the r-th component of the decomposition (3.1.1) is nonzero if $a\neq 0$.

(b) The degree of an element of $\mathbb{C}[X]\oplus\mathcal{F}$ is the polynomial degree of the first component.

Lemma 3.8 *(a) For $a\in\mathrm{Sh}(V)$ and $z\in\tilde{\mathcal{E}}_S$, one has $k(a,z)\in\mathbb{C}[X]\oplus\mathcal{F}$, and $\deg(k(a,z))\leq\deg(a)$.*

(b) For any $z\in\tilde{\mathcal{E}}_S$, the map $\mathrm{Sh(V)}\ni a\mapsto k(a,z)\in\mathbb{C}[X]\oplus\mathcal{F}$ is an algebra morphism.

Proof (a) By (3.1.1), it suffices to prove that for any $r\geq 0$ and $a\in[\underbrace{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)|\dots|\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}_{r\text{ factors}}|a^*]$, with $a^*\in\mathrm{Sh}^*(V)$, one has $k(a,z)\in\mathbb{C}[X]_{\leq r}\oplus\mathcal{F}$. The element $k(a,z)\in\mathbb{C}^{]0,\delta[}$ is the map taking $t\in]0,\delta[$ to the first term of the sequence of equalities

$$\begin{aligned}
I_{\mathrm{sect}(t)}([\underbrace{\omega_{1,0}|\dots|\omega_{1,0}}_{r\text{ factors}}|\kappa(a^*)])(z) &= I_{\mathrm{sect}(t)}([\underbrace{dG|\dots|dG}_{r\text{ factors}}|\kappa_*(a^*)])(z)\\
&=\frac{1}{r!}I_{\mathrm{sect}(t)}\big([(G-G(\mathrm{sect}(t)))^r\odot\kappa_*(a^*)]\big)(z)\\
&=\sum_{l=0}^{r}\frac{1}{l!(r-l)!}(-G(\mathrm{sect}(t))^l I_{\mathrm{sect}(t)}\big([G^{r-l}\odot\kappa_*(a^*)]\big)(z)
\end{aligned}$$

where $\odot$ is the linear map $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)\otimes\mathrm{Sh}(\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S))\to\mathrm{Sh}(\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S))$ given by $f\odot(a_1\otimes\cdots\otimes a_n):=(fa_1)\otimes\cdots\otimes a_n$. One shows that for any $l\in[[0,r]]$, the map $t \mapsto I_{\mathrm{sect}(t)}[G^{r-l} \odot \kappa_*(a^*)]$ belongs to $\mathbb{C} + \mathcal{F}$. It follows from Lemma 3.6 that for any $l \in [[0, r]]$, the function $t \mapsto (-G(\mathrm{sect}(t))^l$ belongs to $(t \mapsto (\log(t))^l) + \mathcal{F}$. Since $((t \mapsto (\log(t))^l) + \mathcal{F}) \cdot (\mathbb{C} + \mathcal{F}) \subset \mathbb{C}(t \mapsto (\log(t))^l) + \mathcal{F}$, $k(a, z)$ belongs to $\sum_{l=0}^r \mathbb{C}(t \mapsto (\log(t))^l) + \mathcal{F}$, which is $\mathbb{C}[X]_{\leq r} + \mathcal{F}$.

Statement (b) follows from (a), Lemma 3.5 and the injectivity of can (see Lemma 3.4(c)). ∎

Definition 3.9 For $a \in \mathrm{Sh}(V)$, $k_{\vec{0}}(a) \in \mathbb{C}^{\tilde{\mathcal{E}}_S}$ is the function $\tilde{\mathcal{E}}_S \to \mathbb{C}$ given by $z \mapsto k_{\vec{0}}(a)(z) := \mathrm{ev}(k(a, z))$.

Lemma 3.10 *(a) The map* $\mathrm{Sh}(V) \to \mathbb{C}^{\tilde{\mathcal{E}}_S}$ *given by* $a \mapsto k_{\vec{0}}(a)$ *is an algebra morphism.*

(b) For $z \in \tilde{\mathcal{E}}_S$, $a \in \mathrm{Sh}(V)$, *one has*

$$k(a, z) = k(a^{(1)}, \mathrm{sect}(\delta/2)) k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z),$$

where we denote $\Delta(a)$ *as* $a^{(1)} \otimes a^{(2)}$ *(equality in* $\mathbb{C}[X] \oplus \mathcal{F}$*).*

(c) For any $a \in \mathrm{Sh}(V)$, $k_{\vec{0}}(a)$ *belongs to the subalgebra* $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S) \subset \mathbb{C}^{\tilde{\mathcal{E}}_S}$.

(d) The map $k_{\vec{0}} : \mathrm{Sh}(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ *is an algebra morphism.*

Proof (a) For each $z \in \tilde{\mathcal{E}}_S$, the composition $\mathrm{Sh}(V) \overset{k(-,z)}{\to} \mathbb{C}[X] \oplus \mathcal{F} \overset{\mathrm{ev}}{\to} \mathbb{C}$ is an algebra morphism as it is a composition of such morphisms (see Lemma 3.8(b) and Lemma 3.4(c)). The map $k_{\vec{0}} : \mathrm{Sh}(V) \to \mathbb{C}^{\tilde{\mathcal{E}}_S}$ is the composition $\mathrm{Sh}(V) \to \mathrm{Sh}(V)^{\tilde{\mathcal{E}}_S} \to \mathbb{C}^{\tilde{\mathcal{E}}_S}$, where the first map is the diagonal morphism and the second map is the product $\prod_{z \in \tilde{\mathcal{E}}_S} (\mathrm{ev} \circ k(-, z))$. Since both the diagonal morphism and $\mathrm{ev} \circ k(-, z)$ are algebra morphisms for each $z \in \tilde{\mathcal{E}}_S$, $k_{\vec{0}}$ is an algebra morphism, which proves (a).

(b) Both sides belong to $\mathbb{C}[X] \oplus \mathcal{F}$, which is contained in $\mathbb{C}^{]0,\delta[}$, so it suffices the prove that they coincide as functions on $]0, \delta[$. For $t \in]0, \delta[$, one has

$$\begin{aligned} k(a, z)(t) &= k_{\mathrm{sect}(t)}(a)(z) = k_{\mathrm{sect}(t)}(a^{(1)})(\mathrm{sect}(\delta/2)) k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z) \\ &= k(a^{(1)}, \mathrm{sect}(\delta/2))(t) \cdot k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z) \end{aligned}$$

where the first and last equalities follow from definitions, and the middle equality follows from [EZ], Lemma 2.5.

(c) Applying ev to the identity of (b), for any $a \in \mathrm{Sh}(V)$ and any $z \in \tilde{\mathcal{E}}_S$ one finds

$$k_{\vec{0}}(a)(z) = k_{\vec{0}}(a^{(1)})(\mathrm{sect}(\delta/2)) k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z)$$

(equality in $\mathbb{C}$). Since, for any $b \in \mathrm{Sh}(V)$, the map $z \mapsto k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z)$ belongs to $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S) \subset \mathbb{C}^{\tilde{\mathcal{E}}_S}$, it follows that the map $z \mapsto k_{\vec{0}}(a)(z)$ also belongs to $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$.

(d) This statement follows from (a) and (c). ∎

3.2 Shuffle regularization

We now present the construction of the algebra morphism $k_0^{\sqcup\!\sqcup} : \mathrm{Sh}(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. In this approach, Lemma 3.4 is replaced by:

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Lemma 3.11 *Let us equip $\mathbb{C} \oplus \mathcal{F}$ with the algebra structure such that $1 \in \mathbb{C}$ is the unity and $\mathcal{F}$ is a nonunital subalgebra. There is a diagram of algebra morphisms*

$$\mathbb{C} \overset{\mathrm{ev}_0}{\leftarrow} \mathbb{C} \oplus \mathcal{F} \overset{\mathrm{can}_0}{\hookrightarrow} \mathbb{C}^{]0,\delta[}$$

where ev_0 *is the projection* $\mathbb{C} \oplus \mathcal{F} \to \mathbb{C}$ *and* $\mathrm{can}_0(\lambda, \phi) = \lambda + \phi$.

Proof Obvious. ■

Remark 3.12 The diagram from Lemma 3.11 is the pull-back of the diagram $\mathbb{C} \leftarrow C([0,\delta[) \hookrightarrow \mathbb{C}^{]0,\delta[}$ by the corestriction $\mathbb{C} \oplus \mathcal{F} \to C([0,\delta[)$ of can_0, where $C([0,\delta[)$ is the space of continuous complex functions on $[0,\delta[$, where $C([0,\delta[) \to \mathbb{C}$ is the evaluation at 0 and where $C([0,\delta[) \hookrightarrow \mathbb{C}^{]0,\delta[}$ is the natural inclusion.

One also uses the following.

Lemma 3.13 (see [Pa], Lemma 3.2.4) *There exists a unique algebra isomorphism* $\mathrm{Sh}^*(V)[X] \to \mathrm{Sh}(V)$ *given by* $a \mapsto a$ *for* $a \in \mathrm{Sh}^*(V)$ *and* $X \mapsto \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

Lemma-Definition 3.14 *(a) For* $a \in \mathrm{Sh}^*(V)$ *and* $z \in \tilde{\mathcal{E}}_S$, *the function* $]0,\delta[\ni t \mapsto k_{\mathrm{sect}(t)}(a)(z)$ *belongs to* $\mathbb{C} \oplus \mathcal{F}$; *denote by* $k_0^{\sqcup\!\sqcup}(a)(z) \in \mathbb{C}$ *its image by* ev_0. *Then, for any* $z \in \tilde{\mathcal{E}}_S$, $a \mapsto (t \mapsto k_{\mathrm{sect}(t)}(a)(z))$ *is an algebra morphism* $\mathrm{Sh}^*(V) \to \mathbb{C} \oplus \mathcal{F}$, *and* $a \mapsto k_0^{\sqcup\!\sqcup}(a)(z)$ *is an algebra morphism* $\mathrm{Sh}^*(V) \to \mathbb{C}$.

(b) For $a \in \mathrm{Sh}^*(V)$, *the function* $z \mapsto k_0^{\sqcup\!\sqcup}(a)(z)$ *belongs to* $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$; *we denote by* $k_0^{\sqcup\!\sqcup} : \mathrm{Sh}^*(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ *the map taking* a *to this function. Then* $k_0^{\sqcup\!\sqcup}$ *is an algebra morphism.*

(c) There is a unique algebra morphism $k_0^{\sqcup\!\sqcup} : \mathrm{Sh}(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ *that extends* $k_0^{\sqcup\!\sqcup} : \mathrm{Sh}^*(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ *and such that* $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \mapsto G$.

Proof (a) The first statement follows from the specialization of Lemma 3.8(a) for $\deg(a) \le 0$. The second statement follows from the fact that for any pair (z,t), the map $a \mapsto k_{\mathrm{sect}(t)}(a)(z)$ is an algebra morphism $\mathrm{Sh}^*(V) \to \mathbb{C}$, and from the first statement; the last statement follows from the second statement by composing with ev_0.

(b) Let $a \in \mathrm{Sh}^*(V)$. Since $\Delta(\mathrm{Sh}^*(V)) \subset \mathrm{Sh}^*(V) \otimes \mathrm{Sh}(V)$, the equality from Lemma 3.10(b) takes place in $\mathbb{C} \oplus \mathcal{F}$. Applying ev_0 to it, one obtains that for every $z \in \tilde{\mathcal{E}}_S$

$$k_0^{\sqcup\!\sqcup}(a)(z) = k_0^{\sqcup\!\sqcup}(a^{(1)})(\mathrm{sect}(\delta/2))k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z)$$

(equality in $\mathbb{C}$). The first part of (b) then follows from the fact that for any $a \in \mathrm{Sh}(V)$, the map $z \mapsto k_{\mathrm{sect}(\delta/2)}(a^{(2)})(z)$ belongs to $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S) \subset \mathbb{C}^{\tilde{\mathcal{E}}_S}$. The second part of (b) follows from its first part, combined with statement that the maps $a \mapsto (z \mapsto k_0^{\sqcup\!\sqcup}(a))$ is an algebra morphism $\mathrm{Sh}^*(V) \to \mathbb{C}^{\tilde{\mathcal{E}}_S}$, which follows from (a).

(c) This follows from (b) and from Lemma 3.13. ■

Proposition 3.15 *One has* $k_{\vec{0}} = k_0^{\sqcup\!\sqcup}$.

Proof One checks that the restrictions of $k_{\vec{0}}$ and $k_0^{\sqcup\!\sqcup}$ to $\mathrm{Sh}^*(V)$ coincide. Let us compute $k_{\vec{0}}(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$. For $z \in \tilde{\mathcal{E}}_S$, $k(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), z)$ is the function $]0,\delta[\ni t \mapsto k_{\mathrm{sect}(t)}(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))(z) = \int_{\mathrm{sect}(t)}^{z} \omega_{1,0}$. By Lemma 3.6, this function belongs to $(t \mapsto -\log(t) + G(z)) + \mathcal{F} = P(\log(t)) + \mathcal{F}$, where $P(X) = G(z) - X \in \mathbb{C}[X]$. It

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follows that $k_{\vec{0}}((\begin{smallmatrix}1\\0\end{smallmatrix}))(z) = G(z) = k_0^{\sqcup\!\sqcup}((\begin{smallmatrix}1\\0\end{smallmatrix}))(z)$, hence that $k_{\vec{0}}((\begin{smallmatrix}1\\0\end{smallmatrix})) = k_0^{\sqcup\!\sqcup}((\begin{smallmatrix}1\\0\end{smallmatrix}))$. By Lemma 3.13, this implies the statement. ∎

Lemma 3.16 *(a) For any $z_0 \in \tilde{\mathcal{E}}_S$ and any $a \in \mathrm{Sh}(V)$ one has the equality (in $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$)*

$$k_{\vec{0}}(a) = k_{\vec{0}}(a^{(1)})(z_0) \cdot k_{z_0}(a^{(2)}). \tag{3.2.1}$$

(b) For any $z_0 \in \tilde{\mathcal{E}}_S$, the images of the morphisms k_{z_0} and $k_{\vec{0}} : \mathrm{Sh}(V) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ coincide.

Proof (a) Fix $a \in \mathrm{Sh}(V)$ and $z \in \tilde{\mathcal{E}}_S$. By [EZ], Lemma 2.5, one has the identity $k_{\mathrm{sect}(t)}(a)(z) = k_{\mathrm{sect}(t)}(a^{(1)})(z_0) \cdot k_{z_0}(a^{(2)})(z)$ for any $t \in]0, \delta[$. One derives from this the equality (in $\mathbb{C}^{]0,\delta[}$) $(t \mapsto k_{\mathrm{sect}(t)}(a)(z)) = (t \mapsto k_{\mathrm{sect}(t)}(a^{(1)})(z_0)) \cdot k_{z_0}(a^{(2)})(z)$. Since both sides actually belong to the subalgebra $\mathbb{C}[X] \oplus \mathcal{F} \subset \mathbb{C}^{]0,\delta[}$, then this may be viewed as an equality in $\mathbb{C}[X] \oplus \mathcal{F}$. Applying to both sides the morphism $\mathrm{ev} : \mathbb{C}[X] \oplus \mathcal{F} \to \mathbb{C}$, one derives the equality (in $\mathbb{C}$) $k_{\vec{0}}(a)(z) = k_{\vec{0}}(a^{(1)})(z_0) \cdot k_{z_0}(a^{(2)})(z)$.

It follows that, for any $a \in \mathrm{Sh}(V)$, one has $(z \mapsto k_{\vec{0}}(a)(z)) = (z \mapsto k_{\vec{0}}(a^{(1)})(z_0)) \cdot k_{z_0}(a^{(2)})(z)$ (equality in $\mathbb{C}^{\tilde{\mathcal{E}}_S}$). Both sides actually belong to $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$, and are respectively equal to the two sides of (3.2.1), which proves (a).

(b) Equation (3.2.1) implies the inclusion $k_{\vec{0}}(\mathrm{Sh}(V)) \subset k_{z_0}(\mathrm{Sh}(V))$, and also implies the relation $k_{z_0}(a) = k_{\vec{0}}(S(a^{(1)}))(z_0) \cdot k_{\vec{0}}(a^{(2)})$ for any $a \in \mathrm{Sh}(V)$, where S is the antipode of $\mathrm{Sh}(V)$, which implies the opposite inclusion $k_{z_0}(\mathrm{Sh}(V)) \subset k_{\vec{0}}(\mathrm{Sh}(V))$. ∎

3.3 The functions $\tilde{\Gamma}$

Definition 3.17 (see [BDDT], Equation (4.13)) For any $r \geq 1$, any $n_1, \ldots, n_r \geq 0$ and any $a_1, \ldots, a_r \in \tilde{S}$, one defines the function $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right) \in \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ by

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right) := k_{\vec{0}}([(\begin{smallmatrix} n_1 \\ a_1 \end{smallmatrix})|\ldots|(\begin{smallmatrix} n_r \\ a_r \end{smallmatrix})]).$$

One also sets $\tilde{\Gamma}(\ ; -) := 1$ for $r = 0$.

It follows from Proposition 3.15 that $\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; -) = G$.

Lemma 3.18 (see also [Pa], Lemma 3.3.4) *For any $t \in \mathrm{Sh}(V)$ and $(n, a) \in \mathbb{Z}_{\geq 0} \times \tilde{S}$, one has*

$$d(k_{\vec{0}}([t|(\begin{smallmatrix} n \\ a \end{smallmatrix})]) = k_{\vec{0}}(t) \cdot \omega_{n,a}$$

(equality in $\Omega_{hol}(\tilde{\mathcal{E}}_S)$). Equivalent formulation: for any $n_1, \ldots, n_r \geq 0$ and $a_1, \ldots, a_r \in \tilde{S}$, one has

$$d(\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)) = T_{a_r}(g_{n_r}) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right) \cdot dz \tag{3.3.1}$$

Proof The identity

$$d(k_{\mathrm{sect}(\delta/2)}([t|(\begin{smallmatrix} n \\ a \end{smallmatrix})]) = k_{\mathrm{sect}(\delta/2)}(t) \cdot \omega_{n,a} \tag{3.3.2}$$

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for any t and (n, a) as above follows from [EZ], Equation (2.1.1). Let us define the map $\partial^V : \mathrm{Sh}(V) \to \mathrm{Sh}(V) \otimes V$ by setting $\partial^V(1) = 0$ and $\partial^V([t|v]) := t \otimes v$ for any $t \in \mathrm{Sh}(V)$ and $v \in V$. For $t \in \mathrm{Sh}(V)$, denote $\partial^V(t)$ by $t^{[1]} \otimes t^{[2]}$. Then (3.3.2) implies the identity

$$d(k_{\mathrm{sect}(\delta/2)}(t)) = k_{\mathrm{sect}(\delta/2)}(t^{[1]}) \cdot \omega(t^{[2]}), \tag{3.3.3}$$

where $\omega : V \to \Omega_{hol}(\tilde{\mathcal{E}}_S)$ is the map given by $\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right) \mapsto \omega_{n,a}$.

Applying Lemma 3.16 with $z_0 := \mathrm{sect}(\delta/2)$, one finds that, for any $t \in \mathrm{Sh}(V)$,

$$k_{\vec{0}}(t) = k_{\vec{0}}(t^{(1)})(\mathrm{sect}(\delta/2)) \cdot k_{\mathrm{sect}(\delta/2)}(t^{(2)}) \tag{3.3.4}$$

(equality in $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$), which gives rise to the first equality in

$$\begin{aligned} d(k_{\vec{0}}(t)) &= k_{\vec{0}}(t^{(1)})(\mathrm{sect}(\delta/2)) \cdot d(k_{\mathrm{sect}(\delta/2)}(t^{(2)})) \\ &= k_{\vec{0}}(t^{(1)})(\mathrm{sect}(\delta/2)) \cdot k_{\mathrm{sect}(\delta/2)}((t^{(2)})^{[1]}) \cdot \omega((t^{(2)})^{[2]}) \\ &= k_{\vec{0}}((t^{[1]})^{(1)})(\mathrm{sect}(\delta/2)) \cdot k_{\mathrm{sect}(\delta/2)}((t^{[1]})^{(2)}) \cdot \omega(t^{[2]}) \\ &= k_{\vec{0}}(t^{[1]}) \cdot \omega(t^{[2]}), \end{aligned}$$

where the second equality follows from (3.3.3), the third equality follows from $(id \otimes \partial^V) \circ \Delta = (\Delta \otimes id) \circ \partial^V$, and the fourth equality follows from (3.3.4). One derives $d(k_{\vec{0}}(t)) = k_{\vec{0}}(t^{[1]}) \cdot \omega(t^{[2]})$, from which the claimed identity follows. ∎

4 Expression of a basis of the algebra $A_{\mathcal{E}_S}$ in terms of the functions $\tilde{\Gamma}$

Recall the notation $O_S := \mathcal{O}(\mathcal{E}_S)$ from Section 2.5. The purpose of this section is to show that the collection of functions $\tilde{\Gamma}$ gives rise to a basis of $A_{\mathcal{E}_S}$ over O_S (Theorem 4.27), alternative to the family of elliptic HLs (see §2.5). For this, we introduce an algebra $\mathcal{G}$ generated by the functions $\tilde{\Gamma}$ and show its inclusion in $A_{\mathcal{E}_S}$ (§4.1). In §§4.2 and 4.3, we carry out a precise analysis of differential algebras related to $\mathcal{E}_S$, which enables us in §4.4 to prove the equality $\mathcal{G} = A_{\mathcal{E}_S}$ (see Theorem 4.20). In §4.5, we prove a linear independence result for the functions $\tilde{\Gamma}$ (see Proposition 4.26), based on the criterion of [DDMS], thereby proving Theorem 4.27. In §4.6, we give examples of the relations implied by Theorem 4.27 between the functions $\tilde{\Gamma}$ and the elliptic HLs from Section 2.5.

4.1 The algebra $\mathcal{G}$ and its inclusion in $A_{\mathcal{E}_S}$

Definition 4.1 $\mathcal{G}$ is the subalgebra of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ generated by $O_S[g_1]$ and by the functions $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)$, where $r \geq 0$, $n_1, \ldots, n_r \geq 0$ and $a_1, \ldots, a_r \in \tilde{S}$ (see Definition 3.17), namely

$$\mathcal{G} := O_S[g_1]\left[\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right) \,\middle|\, r \geq 0,\ n_1, \ldots, n_r \geq 0,\ a_1, \ldots, a_r \in \tilde{S}\right].$$

In this definition, the subalgebra $O_S[g_1]$ of $\mathcal{O}_{hol}(\mathbb{C} \backslash pr^{-1}(S))$ is viewed as a subalgebra of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ via the morphism $\varpi^* : \mathcal{O}_{hol}(\mathbb{C} \backslash pr^{-1}(S)) \to \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$.

Recall now the filtration $F^\delta_\bullet \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ of the algebra $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ defined in Section 2.2.

Lemma 4.2 *For any $n \geq 0$ and $\tilde{s} \in \tilde{S}$, one has $T_{\tilde{s}} g_n \in F^\delta_n \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$.*

Proof Let $\tilde{s} \in \tilde{S}$. It follows from (2.4.7) that the collection of functions $(T_{\tilde{s}} g_n)_{n\geq 0}$ satisfies the relation $d(T_{\tilde{s}} g_n) = \sum_{k=0}^{n-1} (-1)^{n-k} T_{\tilde{s}} g_k \cdot T_{\tilde{s}}(E_{n-k+1}) \cdot dz$ for any $n \geq 1$. Let us prove the statement by induction on $n \geq 0$. For $n = 0$, one has $T_{\tilde{s}} g_0 = 1 \in F^\delta_0 \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. Assume that $T_{\tilde{s}} g_k \in F^\delta_k \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ for $k < n$. Since $T_{\tilde{s}} E_l \in \mathcal{O}(\mathcal{E}_S)$ for $l \geq 2$, one has $T_{\tilde{s}}(E_l) \cdot dz \in \Omega(\mathcal{E}_S)$ for these values of l, hence $d(T_{\tilde{s}} g_n) = \sum_{k=0}^{n-1} (-1)^{n-k} T_{\tilde{s}} g_k \cdot T_{\tilde{s}}(E_{n-k+1}) \cdot dz \in F^\delta_{n-1} \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S) \cdot p^* \Omega(\mathcal{E}_S)$, and therefore $T_{\tilde{s}} g_n \in F^\delta_n \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. ■

Proposition 4.3 *For any $r \geq 0$, any $n_1, \ldots, n_r \geq 0$ and any $a_1, \ldots, a_r \in \tilde{S}$, the function $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)$ belongs to $F^\delta_{r+n_1+\cdots+n_r} \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$.*

Proof Let us prove the statement by induction on r. If $r = 0$, the statement is clear. If $r > 0$, assume the statement to be true until $r-1$, so that $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right)$ belongs to $F^\delta_{r-1+n_1+\cdots+n_{r-1}} \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. By Lemma 4.2(b), one has $T_{a_r} g_{n_r} \in F^\delta_{n_r} \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. The facts that $F^\delta_\bullet \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ is an algebra filtration and that $dz \in p^* \Omega(\mathcal{E}_S)$ then imply that the right-hand side of (3.3.1) belongs to $F^\delta_{r-1+n_1+\cdots+n_r} \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S) \cdot p^* \Omega(\mathcal{E}_S)$, which implies that $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)$ belongs to $F^\delta_{r+n_1+\cdots+n_r} \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. This concludes the proof by induction. ■

Corollary 4.4 *One has $\mathcal{G} \subset A_{\mathcal{E}_S}$.*

Proof One has $O_S = \mathcal{O}(\mathcal{E}_S) \subset F^\delta_1 \mathcal{O}(\tilde{\mathcal{E}}_S) \subset F^\delta_\infty \mathcal{O}(\tilde{\mathcal{E}}_S)$. Moreover, Lemma 4.2 with $\tilde{s} = 0$ and $n = 1$ implies $g_1 \in F^\delta_1 \mathcal{O}(\tilde{\mathcal{E}}_S) \subset F^\delta_\infty \mathcal{O}(\mathcal{E}_S)$. Together with Proposition 4.3 and the fact that $F^\delta_\infty \mathcal{O}(\mathcal{E}_S)$ is an algebra, this implies that $\mathcal{G} \subset F^\delta_\infty \mathcal{O}(\mathcal{E}_S)$. By [EZ], Theorem C, one has $F^\delta_\infty \mathcal{O}(\mathcal{E}_S) = A_{\mathcal{E}_S}$, which implies the result. ■

Proposition 4.3, together with the functional characterization of the algebra $F^\delta_\infty \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ in Theorem C from [EZ], implies that the functions $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)$ have moderate growth at the cusps in the sense explained in [EZ], Section 3.1, and unipotent monodromy in the sense of the introduction.

4.2 The differential algebra $\mathcal{O}(\mathcal{E}_0)[g_1]$

Let us set $\mathcal{E}_0 := \mathcal{E}_{\{pr(0)\}} (= \mathcal{E} \setminus \{pr(0)\})$, and $O := O_{\{pr(0)\}} (= \mathcal{O}(\mathcal{E}_0))$. Then O is the algebra of regular functions on $\mathcal{E}_0$, which coincides with the subalgebra of Λ-invariant functions in $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$. The derivation $\partial = \partial/\partial z$ of $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$ restricts to a derivation of O.

It is well-known that a linear $\mathbb{C}$-basis of O is $\{1, \wp^{(k)} \mid k \geq 0\}$. It follows from (2.4.4) that an alternative linear $\mathbb{C}$-basis of O is $\{E_n \mid n \in \mathbb{Z}_{\geq 0} \setminus \{1\}\}$, where we set $E_0 := 1$. Let us define an algebra filtration on O by the order of poles at $pr(0)$. One then has $F_n O := \sum_{k \in [\![0,n]\!] \setminus \{1\}} \mathbb{C} E_k$. The associated graded algebra $\mathrm{gr}(O)$ is isomorphic to the subalgebra $\mathbb{C} \oplus Y^2 \mathbb{C}[Y]$ of the polynomial algebra $\mathbb{C}[Y]$, where Y has degree 1; more precisely, one has $\mathrm{gr}_n(O) \simeq \mathbb{C} \cdot Y^n$ for any $n \in \mathbb{Z}_{\geq 0} \setminus \{1\}$, with $\overline{E}_n \simeq Y^n$.

Let us equip $\mathbb{C}[X]$ with the degree filtration, given by $F_n \mathbb{C}[X] = \oplus_{k \in [\![0,n]\!]} \mathbb{C} X^k$ for any $n \geq 0$; then $\mathrm{gr}(\mathbb{C}[X]) \simeq \mathbb{C}[X]$, where X has degree 1.

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Moreover, let us equip the tensor product algebra $O \otimes \mathbb{C}[X]$ with the tensor product filtration, so that $F_n(O \otimes \mathbb{C}[X]) = \sum_{k=0}^{n} F_k O \otimes F_{n-k}\mathbb{C}[X]$. The associated graded algebra $\mathrm{gr}(O \otimes \mathbb{C}[X])$ is then isomorphic to $\mathrm{gr}(O) \otimes \mathrm{gr}(\mathbb{C}[X])$, which is isomorphic to the graded subalgebra $\mathbb{C}[X] \oplus Y^2\mathbb{C}[X,Y]$ of $\mathbb{C}[X,Y]$, where X, Y have degree 1.

Recall that $g_1 = E_1 \in \mathcal{O}_{mer}(\mathbb{C}, \Lambda)$, and $g_1' = -E_2$. Let $O[g_1]$ be the subalgebra of $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$ generated by O and g_1. It follows from $g_1' = -E_2$ that the subalgebra $O[g_1]$ of $\mathcal{O}_{mer}(\mathbb{C}, \Lambda)$ is stable under ∂.

Lemma 4.5 *One has $O[g_1] \cap \mathcal{O}_{hol}(\mathbb{C}) = \mathbb{C}$.*

Proof Let us prove inductively on $d \geq 0$ that if $a_0, \ldots, a_d \in O$ and $\sum_i a_i g_1^i \in \mathcal{O}_{hol}(\mathbb{C})$, then $a_0 \in \mathbb{C}$ and $a_1 = \ldots = a_d = 0$. If $d = 0$, then $a_0 \in O \cap \mathcal{O}_{hol}(\mathbb{C}) = \mathbb{C}$, which proves the statement for $d = 0$. Assume the statement at step $d-1$ and let us prove it at step d. Let $a_0, \ldots, a_d \in O$ be such that $f := \sum_i a_i g_1^i \in \mathcal{O}_{hol}(\mathbb{C})$. Then

$$\sum_{j=0}^{d-1} g_1^j \left(\sum_{i>j} a_i \binom{i}{j} (2\pi \mathrm{i})^{i-j} \right) = \sum_{(i,j)|j<i} a_i \binom{i}{j} g_1^j (2\pi \mathrm{i})^{i-j} = \sum_i a_i \left((g_1 + 2\pi \mathrm{i})^i - g_1^i \right)$$
$$= (T_\tau - id)(f) \in \mathcal{O}_{hol}(\mathbb{C}),$$

which by the induction assumption implies that $\sum_{i>0} a_i (2\pi \mathrm{i})^i \in \mathbb{C}$ and for any $j = 1, \ldots, d-1$, $\sum_{i>j} a_i \binom{i}{j} (2\pi \mathrm{i})^{i-j} = 0$. This implies $a_2 = \ldots = a_d = 0$ and $a_1 \in \mathbb{C}$. Then $a_0 + a_1 g_1 \in \mathcal{O}_{hol}(\mathbb{C})$. Assume $a_1 \neq 0$. Since g_1 has a simple pole at 0 with residue 1, and since $a_0 + a_1 g_1 \in \mathcal{O}_{hol}(\mathbb{C})$, a_0 has a simple pole at 0 with residue $-a_1$, which contradicts $a_0 \in O$. Therefore $a_1 = 0$. Then $a_0 \in \mathcal{O}_{hol}(\mathbb{C}) \cap O = \mathbb{C}$, which implies the statement at step d. ∎

Lemma 4.6 *The algebra morphism $O_S \otimes \mathbb{C}[X] \to O_S[g_1]$ induced by $f \otimes 1 \mapsto f$ for $f \in O_S$ and $1 \otimes X \mapsto g_1$ is an isomorphism.*

Proof Let P belong to the kernel of this morphism and let d be its degree as a polynomial in X. If $P \neq 0$, then there exist $f_0, \ldots, f_d \in O_S$ with $f_d \neq 0$, such that $P = \sum_{i=0}^{d} f_i \otimes X^i$. Then $\sum_{i=0}^{d} f_i g_1^i = 0$. Applying $(T_\tau - id)^d$ to this equality, using the invariance of f_i under T_τ and the relation $(T_\tau - id)^d (g_1^i) = \delta_{i,d} (2\pi \mathrm{i})^d d!$ for $i \in [\![0, d]\!]$, one obtains $f_d = 0$, a contradiction. ∎

Let us equip $O[g_1]$ with the image $F_\bullet(O[g_1])$ of the algebra filtration $F_\bullet(O \otimes \mathbb{C}[X])$ by the algebra isomorphism of Lemma 4.6, specialized to $S = \{0\}$. One then has

(4.2.1)

$$\forall n \geq 0, \quad F_n(O[g_1]) = \sum_{k=0}^{n} (F_k O) \cdot g_1^{n-k}, \quad \mathrm{gr}_n(O[g_1]) \simeq \mathbb{C} \cdot X^n \oplus Y^2 \cdot \mathbb{C}[X,Y]_{n-2},$$

where the notation $\mathbb{C}[X,Y]_{n-2}$ indicates the subspace of $\mathbb{C}[X,Y]$ of polynomials which are homogeneous of degree $n-2$.

Lemma 4.7 *If A is an algebra equipped with a filtration $F_\bullet A$ and if $(a_n)_{n\geq 1}$ is a sequence such that $a_n \in F_n A$ for $n \geq 1$, then:*

(a) the sequence $(b_n)_{n\geq 1}$ of elements of A defined by $1+\sum_{n\geq 1} b_n\alpha^n = \exp(\sum_{n\geq 1} a_n\alpha^n)$ (equality in $A[[\alpha]]$)) is such that $b_n \in F_nA$ for any $n \geq 1$;

(b) one has $1+\sum_{n\geq 1}\overline{b}_n = \exp(\sum_{n\geq 1}\overline{a}_n)$ (equality in $\hat{\oplus}_{n\geq 0}\mathrm{gr}_n(A)$).

The combination of (b) with the specialization of Lemma 4.5 to $S = \{0\}$ enables one to prove a strengthening of (b), namely the image of the family $(g_n)_{n\geq 0}$ is a $\mathbb{C}$-basis of the cokernel $O[g_1]/\partial(O[g_1])$.

Proof One has for any $n \geq 1$ the identity

$$b_n = \sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{n_1+\cdots+n_k=n,\\ (n_1,\ldots,n_k)\in\mathbb{Z}^k_{\geq 1}}} a_{n_1}\cdots a_{n_k},$$

which together with the algebra filtration properties implies (a). This identity implies that, for any $n \geq 1$,

$$\overline{b}_n = \sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{n_1+\cdots+n_k=n,\\ (n_1,\ldots,n_k)\in\mathbb{Z}^k_{\geq 1}}} \overline{a}_{n_1}\cdots \overline{a}_{n_k},$$

where $\overline{a}_i, \overline{b}_i$ are the classes of a_i, b_i in $\mathrm{gr}_i(A)$, from which one derives (b). ∎

Lemma 4.8 *(a) For every $n \geq 0$ one has $g_n \in F_n(O[g_1])$. Moreover, the degree n polynomial $(X-Y)^{n-1}(X+(n-1)Y)/n!$ (by convention equal to 1 when $n = 0$) belongs to $\mathbb{C}[X]\oplus Y^2\mathbb{C}[X,Y]$ and is equal to the image of $\overline{g}_n \in \mathrm{gr}_n(O[g_1])$ by the isomorphism in (4.2.1).*

(b) The family $(g_n)_{n\geq 0}$ of elements of $O[g_1]$ is $\mathbb{C}$-linearly independent.

(c) There is a direct sum decomposition

$$\mathrm{Span}_{\mathbb{C}}\{g_n \mid n \geq 0\} \oplus \partial(O[g_1]) = O[g_1]; \tag{4.2.2}$$

in particular, $\{g_n \mid n \geq 0\}$ is a $\mathbb{C}$-basis of the cokernel of the endomorphism ∂ of $O[g_1]$.

Proof Equation (2.4.5) implies the identity

$$1+\sum_{n\geq 1} g_n\,\alpha^n = \exp\Big((1\otimes X)\,\alpha - \sum_{r\geq 2}\frac{(-1)^r}{r}\,((E_r-e_r)\otimes 1)\,\alpha^r\Big),$$

(equality in $(O\otimes\mathbb{C}[X])[[\alpha]]$) where for $n \geq 1$, g_n is identified with its image in $O\otimes\mathbb{C}[X]$. The first statement in (a) follows from this equality, combined with Lemma 4.7(a) and the relations $X \in F_1\mathbb{C}[X]$ and $E_r - e_r \in F_rO$ for any $r \geq 2$, which imply $1\otimes X \in F_1(O\otimes\mathbb{C}[X])$ and $(E_r-e_r)\otimes 1 \in F_r(O\otimes\mathbb{C}[X])$ for $r \geq 2$. The second statement in (a) follows from Lemma 4.7(b), which implies that

$$\begin{aligned}1+\sum_{n\geq 1}\overline{g}_n &= \exp\Big(\overline{1\otimes X} - \sum_{r\geq 2}\frac{(-1)^r}{r}\,\overline{(E_r-e_r)\otimes 1}\Big)\\ &= \exp\Big(X - \sum_{r\geq 2}\frac{(-1)^r}{r}Y^r\Big) = e^{X-Y}(1+Y),\end{aligned}$$

(equality in the subalgebra $\mathbb{C}[[X]]\oplus Y^2\mathbb{C}[[X,Y]]$ of $\mathbb{C}[[X,Y]]$), which in turn implies that $\overline{g}_n = (X-Y)^n/n! + Y(X-Y)^{n-1}/(n-1)! = (X-Y)^{n-1}(X+(n-1)Y)/n!$.

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(b) If $\sum_n \lambda_n g_n = 0$ is a nontrivial linear dependence relation, let $i := \max\{i|\lambda_i \neq 0\}$. Then $0 = \sum_n \lambda_n g_n \in F_i(\mathcal{O}[g_1])$, and the image of this element in $\mathrm{gr}_i(\mathcal{O}[g_1])$ is $\lambda_i \overline{g_i}$, which is nonzero, this yielding a contradiction.

(c) Let D be the endomorphism of $\mathcal{O} \otimes \mathbb{C}[X]$ defined by $D(f \otimes P) := \partial f \otimes P - fE_2 \otimes P'$. Then D is a derivation of $\mathcal{O} \otimes \mathbb{C}[X]$, which is interwined with ∂ under the specialization of the isomorphism from Lemma 4.6 to $S = \{0\}$.

The derivation ∂ of $\mathcal{O}$ is of filtration degree 1, i.e., $\partial(F_k\mathcal{O}) \subset F_{k+1}\mathcal{O}$ for any $k \geq 0$. The associated graded endomorphism $\mathrm{gr}(\partial)$ is then a degree 1 derivation of $\mathbb{C} \oplus Y^2\mathbb{C}[Y]$. The image of the derivation $Y^2\partial/\partial Y$ of $\mathbb{C}[Y]$ is contained in $Y^2\mathbb{C}[Y]$, therefore $Y^2\partial/\partial Y$ induces a derivation of $\mathbb{C} \oplus Y^2\mathbb{C}[Y]$ of degree 1, which can be seen to coincide with $\mathrm{gr}(\partial)$. One deduces from this that the derivation $\partial \otimes \mathrm{id}$ of $\mathcal{O} \otimes \mathbb{C}[X]$ is of filtration degree 1, and that its associated graded is the derivation $Y^2\partial/\partial Y$ of $\mathbb{C}[X] \oplus Y^2\mathbb{C}[X, Y]$.

The endomorphism $f \otimes P \mapsto -E_2 f \otimes P'$ is a derivation of $\mathcal{O} \otimes \mathbb{C}[X]$, which is also of filtration degree 1, as E_2 has filtration degree 2 and $P \mapsto P'$ has filtration degree -1. The associated graded of the endomorphism $f \mapsto E_2 f$ of $\mathcal{O}$ (resp. $P \mapsto P'$ of $\mathbb{C}[X]$) is the endomorphism $f \mapsto Y^2 f$ of $\mathbb{C} \oplus Y^2\mathbb{C}[Y]$ (resp. $\partial/\partial X$ of $\mathbb{C}[X]$), therefore the associated graded of $f \otimes P \mapsto -E_2 f \otimes P'$ is the degree 1 derivation of $\mathbb{C}[X] \oplus Y^2\mathbb{C}[X, Y]$ given by $-Y^2\partial/\partial X$.

It follows that D has filtration degree 1, and that its associated graded is the derivation of $\mathbb{C}[X] \oplus Y^2\mathbb{C}[X, Y]$ induced by the derivation $Y^2(\frac{\partial}{\partial Y} - \frac{\partial}{\partial X})$ of $\mathbb{C}[X, Y]$.

We now prove by induction on $k \geq 0$ the equality of subspaces of $\mathcal{O}[g_1]$

$$F_{k+1}(\mathcal{O}[g_1]) = \partial(F_k(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \ldots, g_{k+1}). \tag{4.2.3}$$

One has $F_0(\mathcal{O}[g_1]) = \mathbb{C}$ and $F_1(\mathcal{O}[g_1]) = \mathrm{Span}_{\mathbb{C}}(1, g_1)$, which implies (4.2.3) for $k = 0$. Assume $k > 0$ and $F_k(\mathcal{O}[g_1]) = \partial(F_{k-1}(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \ldots, g_k)$, and let us show (4.2.3). The inclusion of the right-hand side in the left-hand side is obvious, so one has to show that this inclusion is an equality.

Recall that the space $\partial(F_k(\mathcal{O}[g_1]))$ is a subspace of $F_{k+1}(\mathcal{O}[g_1])$. It follows from the commutativity of

$$\begin{tikzcd}
F_k(\mathcal{O}[g_1]) \arrow[r, "\partial"] \arrow[d] & F_{k+1}(\mathcal{O}[g_1]) \arrow[d] \\
\mathrm{gr}_k(\mathcal{O}[g_1]) \arrow[r, "\mathrm{gr}_k(\partial)"'] & \mathrm{gr}_{k+1}(\mathcal{O}[g_1])
\end{tikzcd}$$

and from the surjectivity of $F_k(\mathcal{O}[g_1]) \to \mathrm{gr}_k(\mathcal{O}[g_1])$ that the image of this subspace in $\mathrm{gr}_{k+1}(\mathcal{O}[g_1])$ is equal to the image of $\mathrm{gr}_k(\partial) : \mathrm{gr}_k(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$, which is identified with the map

$$Y^2\left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}\right) : \mathbb{C}X^k \oplus Y^2 \cdot \mathbb{C}[X, Y]_{k-2} \to \mathbb{C}X^{k+1} \oplus Y^2 \cdot \mathbb{C}[X, Y]_{k-1}. \tag{4.2.4}$$

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The image of (4.2.4) is obviously contained in $Y^2 \cdot \mathbb{C}[X,Y]_{k-1}$, which induces a linear map

$$Y^2\left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}\right) : \mathbb{C}X^k \oplus Y^2 \cdot \mathbb{C}[X,Y]_{k-2} \to Y^2 \cdot \mathbb{C}[X,Y]_{k-1}. \tag{4.2.5}$$

The kernel of the linear endomorphism $Y^2(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y})$ of $\mathbb{C}[X,Y]$ is $\mathbb{C}[X+Y]$, which implies that the kernel of the linear map $Y^2(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}) : \mathbb{C}[X,Y]_k \to \mathbb{C}[X,Y]_{k+1}$ is the one-dimensional vector space $\mathbb{C} \cdot (X+Y)^k$. The kernel of (4.2.5) is the intersection of this vector space with the source of (4.2.5), which is zero as the coefficient of YX^{k-1} in $(X+Y)^k$ is nonzero; this implies that the map (4.2.5) is injective. The source and target of (4.2.5) both have dimension k; the equality of these dimensions, together with the injectivity of (4.2.5), implies that (4.2.5) is a linear isomorphism. All this implies that the image of (4.2.4) is equal to $Y^2 \cdot \mathbb{C}[X,Y]_{k-1}$, therefore that the image of $\partial(F_k(\mathcal{O}[g_1]))$ by $F_{k+1}(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1]) \simeq \mathbb{C}X^{k+1} \oplus Y^2 \cdot \mathbb{C}[X,Y]_{k-1}$ is $Y^2 \cdot \mathbb{C}[X,Y]_{k-1}$.

On the other hand, $\mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_k) \subset F_k(\mathcal{O}[g_1])$, which implies that the image of $\mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1})$ under $F_{k+1}(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$ is $\mathbb{C}\,\overline{g}_{k+1}$, whose image in $\mathbb{C}X^{k+1} \oplus Y^2 \cdot \mathbb{C}[X,Y]_{k-1}$ is given by (a).

Then the image of $\partial(F_k(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1})$ under $F_{k+1}(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$ is the sum of the images of $\partial(F_k(\mathcal{O}[g_1]))$ and $\mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1})$ under this map, which is equal to the sum $Y^2 \cdot \mathbb{C}[X,Y]_{k-1} + \mathbb{C}(X-Y)^k(X + kY)/(k+1)!$, that can be simplified into $\mathbb{C} \cdot X^k \oplus Y^2 \cdot \mathbb{C}[X,Y]_{k-1} \simeq \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$.

It follows that the image of $\partial(F_k(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1})$ under the projection $F_{k+1}(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$ is equal to its target. The space $\partial(F_k(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1})$ also contains $\partial(F_{k-1}(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_k)$, which by the induction hypothesis is equal to $F_k(\mathcal{O}[g_1])$, and therefore also to the kernel of the projection $F_{k+1}(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$. All this implies the equality $\partial(F_k(\mathcal{O}[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1}) = F_{k+1}(\mathcal{O}[g_1])$, which proves the induction step. This proves (4.2.3) for any $k \geq 0$.

Let us now prove by induction on $k \geq 0$ the equality of subspaces of $F_{k+1}(\mathcal{O}[g_1])$

$$\partial(F_k(\mathcal{O}[g_1])) \cap \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1}) = \{0\}. \tag{4.2.6}$$

One has $F_0(\mathcal{O}[g_1]) = \mathbb{C}$ hence $\partial(F_0(\mathcal{O}[g_1])) = 0$, which implies (4.2.6) for $k = 0$. Assume $k > 0$ and $\partial(F_{k-1}(\mathcal{O}[g_1])) \cap \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_k) = \{0\}$, and let us show (4.2.6). Let $P \in F_k(\mathcal{O}[g_1])$, $(\lambda_i)_{i \in [0,k+1]} \in \mathbb{C}^{k+2}$ be such that $\partial(P) = \sum_i \lambda_i g_i$. The image of this equality under the projection $F_{k+1}(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$ is $\lambda_{k+1}\overline{g_{k+1}} = \mathrm{gr}_k(\partial)(\overline{P})$, where $\overline{P}$ is the image of P in $\mathrm{gr}_k(\mathcal{O}[g_1])$. The map $\mathrm{gr}_k(\partial) : \mathrm{gr}_k(\mathcal{O}[g_1]) \to \mathrm{gr}_{k+1}(\mathcal{O}[g_1])$ is injective (as this maps can be identified with (4.2.5) which has been proved to be injective), and its image does not contain $\overline{g_{k+1}}$ (as this image has been proved to be identified to $Y^2\mathbb{C}[X,Y]_{k-1}$ and in view of the identification of $\overline{g_{k+1}}$ in (a)). It follows that $\lambda_{k+1} = 0$ and $\overline{P} = 0$, therefore in the equality $\partial(P) = \sum_i \lambda_i g_i$ the left-hand side belongs to $\partial(F_{k-1}(\mathcal{O}[g_1]))$ and the right-hand side to $\mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_k)$, which by the induction assumption implies $\partial(P) = 0$ and $\lambda_0 = \ldots = \lambda_k = 0$. This proves the induction step, therefore (4.2.6) holds for any $k \geq 0$.

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Then

$$\begin{aligned} O[g_1] &= \sum_{k\geq 0} F_{k+1}(O[g_1]) = \sum_{k\geq 0} \partial(F_k(O[g_1])) + \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1}) \\ &= \sum_{k\geq 0} \partial(F_k(O[g_1])) + \sum_{k\geq 0} \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{k+1}) \\ &= \partial\left(\sum_{k\geq 0} F_k(O[g_1])\right) + \mathrm{Span}_{\mathbb{C}}(g_i, i \geq 0) \\ &= \partial(O[g_1]) + \mathrm{Span}_{\mathbb{C}}(g_i, i\geq 0), \end{aligned} \tag{4.2.7}$$

where the second equality follows from (4.2.3).

For any $k, l \geq 0$, one has

$$\begin{aligned} \partial(F_k O[g_1]) \cap \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{l+1}) \subset{}& \partial(F_{\max(k,l)} O[g_1]) \\ &\cap \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{\max(k,l)+1}) = 0, \end{aligned}$$

where the last equality follows from (4.2.6). This implies

$$\forall k, l \geq 0, \quad \partial(F_k O[g_1]) \cap \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{l+1}) = 0. \tag{4.2.8}$$

Then

$$\begin{aligned} \partial(O[g_1]) \cap \mathrm{Span}_{\mathbb{C}}(g_i, i\geq 0) &= (\cup_{k\geq 0} \partial(F_k O[g_1])) \cap (\cup_{l\geq 0} \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{l+1}) \\ &= \cup_{k,l\geq 0} \partial(F_k O[g_1]) \cap \mathrm{Span}_{\mathbb{C}}(1, g_1, \dots, g_{l+1}) = 0, \end{aligned} \tag{4.2.9}$$

where the last equality follows from (4.2.8).

The statement then follows from the combination of (4.2.7) and (4.2.9). ■

4.3 The differential algebra $\mathcal{O}(\mathcal{E}_S)[g_1]$

Let us denote by $\mathbb{C}(\mathcal{E})$ the field of rational functions on $\mathcal{E}$. As an algebra, it can be identified with the algebra of Λ-invariant meromorphic functions on $\mathbb{C}$. Recall that, for $a \in \mathbb{C}$, the translation T_a is an automorphism of the algebra of meromorphic functions on $\mathbb{C}$; it induces an automorphism of $\mathbb{C}(\mathcal{E})$, which moreover depends only on the class of a in $\mathcal{E}$.

Recall the map $s \mapsto \tilde{s}$ from Definition 2.10(a).

Lemma 4.9 *(a) For any $s \in S$, one has $(T_{\tilde{s}} - id)(g_1) \in O_S$.*

(b) $O_S = \sum_{s\in S} T_s(O) + \sum_{s\in S\setminus\{pr(0)\}} \mathbb{C}\cdot(T_{\tilde{s}} - id)(g_1)$ (equality of subspaces of $\mathbb{C}(\mathcal{E})$).

Proof The relations $T_1 g_1 = g_1$ and $T_\tau g_1 = g_1 + 2\pi\mathrm{i}$, together with the commutativity of the various T_a, $a \in \mathbb{C}$, implies that for any $s \in S$ the function $(T_{\tilde{s}} - 1)(g_1)$ is Λ-invariant. It has simple poles at 0 and s, which implies that it belongs to O_S, which proves (a). One clearly has $T_s(O) \subset O_S$ for any $s \in S$, so that (a) implies the inclusion $O_S \supset \sum_{s\in S} T_s(O) + \sum_{s\in S\setminus\{pr(0)\}} \mathbb{C}\cdot(T_{\tilde{s}} - 1)(g_1)$. If now $f \in O_S$, let $(f_s)_{s\in S}$ be the collection where $f_s \in \mathbb{C}((z - \tilde{s}))$ is the local expansion of f at $\tilde{s}$. The map $O \to \mathbb{C}((z)) \to \mathbb{C}((z))/z^{-1}\mathbb{C}[[z]]$ is surjective, so for any $s \in S$ one can find $o_s \in O$ whose image by this map coincides with the image of f_s by the map $\mathbb{C}((z - \tilde{s})) \simeq \mathbb{C}((z)) \to \mathbb{C}((z))/z^{-1}\mathbb{C}[[z]]$. Then $g := f - \sum_{s\in S} T_s(o_s)$ belongs to O_S and has at

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most simple poles at S. Then $g - \sum_{s\in S\backslash pr(0)} \mathrm{res}_s(g \cdot dz) \cdot (T_{\tilde{s}} - id)(g_1)$ belongs to O_S and is regular on $S\backslash pr(0)$; hence it belongs to O and has at most a simple pole at $pr(0)$, which implies that it is constant. This proves the desired opposite inclusion. ■

Recall that $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$ is the algebra of meromorphic functions on $\mathbb{C}$ with set of poles contained in $pr^{-1}(S)$. Then Λ acts by translation on $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$, and $O_S = \mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))^{\Lambda}$. The operator $\partial = d/dz$ defines a derivation on $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$, which restricts to a derivation of O_S.

Consider the subalgebra $O_S[g_1]$ of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$. Then $\partial(g_1) = -E_2 \in O_S$ implies that $O_S[g_1]$ is stable under ∂.

Lemma 4.10 *Let $a \in \mathbb{C}\backslash\Lambda$.*

(a) If $f \in \mathcal{O}_{mer}(\mathbb{C}, \Lambda \cup (a + \Lambda))$ is such that $T_1(f) = f$ and $(T_\tau - id)(f) \in O[g_1] + T_a(O[g_1])$, then $f \in O[g_1] + T_a(O[g_1])$.

(b) If $f \in \mathcal{O}_{mer}(\mathbb{C}, \Lambda \cup (a + \Lambda))$ is such that $T_1(f) = f$ and that for some $n \geq 1$ one has $(T_\tau - id)^n(f) \in O[g_1] + T_a(O[g_1])$, then $f \in O[g_1] + T_a(O[g_1])$.

(c) For any $n \geq 0$, one has

$$g_1^n \cdot T_a(g_1) \subset O[g_1] + T_a(O[g_1]). \tag{4.3.1}$$

Proof (a) The automorphism T_τ of $\mathcal{O}_{mer}(\mathbb{C}, \Lambda \cup (a + \Lambda))$ restricts to automorphisms of the subalgebras $O[g_1]$ and $T_a(O[g_1])$, therefore to an automorphism of the vector subspace $O[g_1] + T_a(O[g_1])$.

The O-algebra morphism $O[X] \to O[g_1]$ induced by $X \mapsto g_1$ intertwines the linear endomorphism $T_\tau - id$ of $O[g_1]$ with the linear endomorphism $\exp(2\pi\mathrm{i}\partial_X) - 1$ of $O[X]$, where ∂ is the derivation of $O[X]$ given by $O \mapsto 0$ and $X \mapsto 1$. On the other hand, $\exp(2\pi\mathrm{i}\partial_X) - 1 = \frac{\exp(2\pi\mathrm{i}\partial_X)-1}{\partial_X} \circ \partial_X$, where $\frac{\exp(2\pi\mathrm{i}\partial_X)-1}{\partial_X}$ is a linear automorphism of $O[X]$ as ∂_X is locally nilpotent, and ∂_X is surjective. It follows that the linear endomorphism $\exp(2\pi\mathrm{i}\partial_X) - 1$ of $O[X]$ is surjective. It follows from this surjectivity, from the interwtining of this operator with $T_\tau - id$ and from the surjectivity of the map $O[X] \to O[g_1]$ that the linear endomorphism $T_\tau - id$ of $O[g_1]$ is surjective. This implies the surjectivity of the linear endomorphism $T_\tau - id$ of $T_a(O[g_1])$, and therefore that of the linear endomorphism $T_\tau - id$ of $O[g_1] + T_a(O[g_1])$.

Let now f be as in the (a). It follows from the surjectivity of the linear endomorphism $T_\tau - id$ of $O[g_1] + T_a(O[g_1])$ that there exists $\tilde{f} \in O[g_1] + T_a(O[g_1])$, such that $(T_\tau - id)(\tilde{f}) = (T_\tau - id)(f)$ (equality in $O[g_1] + T_a(O[g_1])$). Then $f - \tilde{f}$ belongs to $\mathcal{O}_{mer}(\mathbb{C}, \Lambda \cup (a + \Lambda))$ and is invariant both under T_1 and T_τ, which implies that it belongs to $O_{\{pr(0),pr(a)\}}$, which by Lemma 4.9(b) for $S = \{pr(0), pr(a)\}$ is equal to $O + T_a(O) + \mathbb{C}(T_a - id)(g_1)$, and is therefore contained in $O[g_1] + T_a(O[g_1])$. The statement then follows from $\tilde{f} \in O[g_1] + T_a(O[g_1])$ and $f - \tilde{f} \in O[g_1] + T_a(O[g_1])$.

(b) The proof is by induction on $n \geq 1$. For $n = 1$, the statement follows from (a). Assume the statement at step n and let us prove it at step $n + 1$. Let $g \in \mathcal{O}_{mer}(\mathbb{C}, \Lambda \cup (a + \Lambda))$ be such that $T_1(g) = g$ and $(T_\tau - id)^{n+1}(g) \in O[g_1] + T_a(O[g_1])$. Then $f := (T_\tau - id)(g)$ is such that $T_1(f) = f$ as T_1 commutes with $T_\tau - id$, and $(T_\tau - id)^n(f) \in O[g_1] + T_a(O[g_1])$. By the induction assumption, this implies $f \in O[g_1] + T_a(O[g_1])$. Then one has $T_(g) = g$ and $(T_\tau - id)(g) \in O[g_1] + T_a(O[g_1])$, which by (a) implies $g \in O[g_1] + T_a(O[g_1])$. This proves the statement at step $n + 1$.

(c) This follows from (b) and from the invariance of $g_1^n \cdot T_a(g_1)$ under T_1, which follows from that of g_1 and of $T_a(g_1)$, and from $(T_\tau - id)^{n+2}(g_1^n \cdot T_a(g_1)) = 0$. ■

Lemma 4.11 *One has* $O_S[g_1] = \sum_{s\in S} T_{\tilde{s}}(O[g_1])$ *(equality of subspaces of* $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$*).*

Proof For $s \in S$, one has $T_{\tilde{s}}(O) \subset O_S \subset O_S[g_1]$, where the first inclusion follows from Lemma 4.9(b). Moreover, $T_{\tilde{s}}(g_1) = g_1 + (T_{\tilde{s}} - id)(g_1) \in O_S[g_1]$ by Lemma 4.9(a). Then $T_{\tilde{s}}(O[g_1])$ is the subalgebra of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$ generated by $T_{\tilde{s}}(O)$ and $T_{\tilde{s}}(g_1)$, which are both contained in $O_S[g_1]$, which implies the inclusion $T_{\tilde{s}}(O[g_1]) \subset O_S[g_1]$ as $O_S[g_1]$ is an algebra. As $O_S[g_1]$ is stable under summation, this implies the inclusion $O_S[g_1] \supset \sum_{s\in S} T_{\tilde{s}}(O[g_1])$.

In order to show the opposite inclusion, let us first show that $\sum_{s\in S} T_{\tilde{s}}(O[g_1])$ is a subalgebra of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$. For each $s \in S$, the subspace $T_{\tilde{s}}(O[g_1])$ is a subalgebra of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$. So it is enough to prove that for any $s \neq t \in S$, one has $T_{\tilde{s}}(O[g_1]) \cdot T_{\tilde{t}}(O[g_1]) \subset T_{\tilde{s}}(O[g_1]) + T_{\tilde{t}}(O[g_1])$. By translation invariance, it suffices to prove this for $t = 0$, in which case $\tilde{t} = 0$, i.e., to prove

$$O[g_1] \cdot T_a(O[g_1]) \subset O[g_1] + T_a(O[g_1]) \tag{4.3.2}$$

for $a \in \mathbb{C}\backslash\Lambda$.

One has $O \cdot T_a(O) \subset O_{\{pr(0),pr(a)\}} = O + T_a(O) + \mathbb{C} \cdot (T_a - id)(g_1)$ where the first inclusion follows from the fact that both O and $T_a(O)$ are contained in $O_{\{pr(0),pr(a)\}}$ and that the latter set is an algebra and the second inclusion follows from Lemma 4.9(b) for $S = \{pr(0), pr(a)\}$. This implies $O \cdot T_a(O) \subset O[g_1] + T_a(O[g_1])$. Now $O[g_1] \cdot T_a(O[g_1])$ is the sub-$\mathbb{C}[g_1, T_a(g_1)]$-module of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$ generated by $O \cdot T_a(O)$, therefore in order to prove (4.3.2) it suffices to show that $O[g_1] + T_a(O[g_1])$ is a sub-$\mathbb{C}[g_1, T_a(g_1)]$-module of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$, i.e., is stable under multiplication by g_1 and $T_a(g_1)$. The inclusions $O[g_1] \cdot g_1 \subset O[g_1]$ and $T_a(O[g_1]) \cdot T_a(g_1) \subset T_a(O[g_1])$ are obvious, it therefore remains to prove

(4.3.3)
$$O[g_1] \cdot T_a(g_1) \subset O[g_1] + T_a(O[g_1]), \quad T_a(O[g_1]) \cdot g_1 \subset O[g_1] + T_a(O[g_1]).$$

For $f \in O$, one has $f \cdot T_a(g_1) = f(a)T_a(g_1) + (f - f(a))(T_a - id)(g_1) + (f - f(a))g_1$. Then $f(a)T_a(g_1) \in \mathbb{C}T_a(g_1)$, $(f - f(a))g_1 \in O[g_1]$ and $(f - f(a))(T_a - id)(g_1) \in O$ as $f - f(a)$ belongs to O and vanishes at a, while $(T_a - id)(g_1)$ belongs to $O_{\{0,a\}}$ and has only a simple pole at a. It follows that

$$O \cdot T_a(g_1) \subset O[g_1] + \mathbb{C}T_a(g_1). \tag{4.3.4}$$

Then one has $Og_1^n \cdot T_a(g_1) \subset O[g_1]g_1^n + \mathbb{C}T_a(g_1)g_1^n \subset O[g_1] + T_a(O[g_1])$ by virtue of (4.3.4) and (4.3.1). This proves the first inclusion in (4.3.3), the second inclusion is a consequence of it (applying T_{-a} and replacing a by $-a$).

This ends the proof of the fact that $\sum_{s\in S} T_{\tilde{s}}(O[g_1])$ is a subalgebra of $\mathcal{O}_{mer}(\mathbb{C}, pr^{-1}(S))$.

One has $O_S = \sum_{s\in S} T_s(O) + \sum_{s\in S\backslash\{pr(0)\}}(T_{\tilde{s}} - id)(g_1) \subset \sum_{s\in S} T_{\tilde{s}}(O[g_1])$, where the equality follows from Lemma 4.9(b) and the inclusion follows from $(T_{\tilde{s}} - id)(g_1) \in O[g_1] + T_{\tilde{s}}(O[g_1])$ for any $s \in S\backslash\{pr(0)\}$. One also has $g_1 \in T_0(O[g_1]) \subset \sum_{s\in S} T_{\tilde{s}}(O[g_1])$. These two statements, together with the fact that $\sum_{s\in S} T_{\tilde{s}}(O[g_1])$ is an algebra, implies the inclusion $O_S[g_1] \subset \sum_{s\in S} T_{\tilde{s}}(O[g_1])$. ∎

Lemma 4.12 *The vector subspace $\sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}_{\mathbb{C}}\{g_n \,|\, n \geq 0\}) \subset O_S[g_1]$ is such that*

$$\sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}_{\mathbb{C}}\{g_n \,|\, n \geq 0\}) + \partial(O_S[g_1]) = O_S[g_1];$$

in other words, the cokernel of the endomorphism ∂ of $O_S[g_1]$ is linearly spanned by the image of the family $\{T_{\tilde{s}}(g_n) \,|\, (s,n) \in S \times \mathbb{Z}_{\geq 0}\}$.

Proof One has

$$\begin{aligned} O_S[g_1] &= \sum_{s\in S} T_{\tilde{s}}(O[g_1]) = \sum_{s\in S} T_{\tilde{s}}(\partial(O[g_1]) + \mathrm{Span}_{\mathbb{C}}\{g_n \,|\, n \geq 0\}) \\ &= \sum_{s\in S} T_{\tilde{s}}(\partial(O[g_1])) + \sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}_{\mathbb{C}}\{g_n \,|\, n \geq 0\}) \\ &= \partial(\sum_{s\in S} T_{\tilde{s}}(O[g_1])) + \sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}_{\mathbb{C}}\{g_n \,|\, n \geq 0\}) \\ &= \partial(O_S[g_1]) + \sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}_{\mathbb{C}}\{g_n \,|\, n \geq 0\}), \end{aligned}$$

where the first and last equalities follow from Lemma 4.11, the second equality follows from (4.2.2), the third equality follows from the linearity of $T_{\tilde{s}}$, and the fourth equality follows from the commutativity of ∂ and $T_{\tilde{s}}$. ∎

Lemma 4.13 *The sum map*

$$\mathbb{C}1 \oplus \Big(\bigoplus_{s\in S} T_{\tilde{s}}\mathrm{Span}\{g_n|n > 0\}\Big) \oplus \partial(O_S[g_1]) \to O_S[g_1]$$

is a vector space isomorphism.

Proof One has

$$\begin{aligned} O_S[g_1] &= \sum_{s\in S} T_{\tilde{s}}(O[g_1]) = \sum_{s\in S} T_{\tilde{s}}(\mathbb{C}1 + \mathrm{Span}\{g_n|n > 0\} + \partial(O[g_1])) \\ &= \mathbb{C}1 + \sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}\{g_n|n > 0\}) + \partial(\sum_{s\in S} T_{\tilde{s}}(O[g_1])) \\ &= \mathbb{C}1 + \sum_{s\in S} T_{\tilde{s}}(\mathrm{Span}\{g_n|n > 0\}) + \partial(O_S[g_1]) \end{aligned}$$

where the second equality follows from Lemma 4.8(c), the third equality follows from the commutativity of $T_{\tilde{s}}$ with ∂, the fourth equality follows from Lemma 4.11; this implies that the said sum map is surjective.

Let us prove its injectivity. Let $P \in O_S[g_1]$ and $S \ni s \mapsto t_s \in \mathrm{Span}\{g_n|n > 0\}$ and $\lambda \in \mathbb{C}$ be such that

$$\lambda + \partial(P) + \sum_{s\in S} T_{\tilde{s}}(t_s) = 0.$$

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By Lemma 4.11, there exists a map $S \ni s \mapsto P_s \in O[g_1]$ such that $P = \sum_{s\in S} T_{\tilde{s}}(P_s)$. Then

$$\lambda + \sum_{s\in S} T_{\tilde{s}}(\partial(P_s) + t_s) = 0$$

(equality in $O_S[g_1] \subset \mathcal{O}_{mer}(\mathbb{C}, p^{-1}(S))$).

For each $s \in \tilde{S}$, the equality $T_{\tilde{s}}(\partial(P_s) + t_s) = -\lambda - \sum_{s'\in S\setminus\{s\}} T_{\tilde{s}'}(\partial(P'_s) + t'_s)$ implies that the set of poles of $T_{\tilde{s}}(\partial(P_s) + t_s)$ is contained in $p^{-1}(s) \cap p^{-1}(S\setminus\{s\})$, which is empty; so $T_{\tilde{s}}(\partial(P_s) + t_s)$ is an element of $O[g_1]$ which is holomorphic on $\mathbb{C}$, therefore is constant by Lemma 4.5. Therefore there exists a map $S \ni s \mapsto \lambda_s \in \mathbb{C}$ such that $T_{\tilde{s}}(\partial(P_s) + t_s) = \lambda_s$ for any $s \in S$, and therefore

$$\lambda + \sum_{s\in S} \lambda_s = 0.$$

Then $\partial(P_s) = T_{\tilde{s}}^{-1}(\lambda_s) - t_s = \lambda_s - t_s$ for any $s \in S$. By Lemma 4.8(c), one has for any $s \in S$, $t_s - \lambda_s = 0$ and $\partial(P_s) = 0$. Since $t_s \in \mathrm{Span}\{g_n | n > 0\}$ and by Lemma 4.8(b), one derives $t_s = \lambda_s = 0$. Therefore $\partial(P) = \sum_s \partial(P_s) = 0$, $\lambda = \sum_s \lambda_s = 0$, which implies the injectivity. ∎

4.4 The equality $\mathcal{G} = A_{\mathcal{E}_S}$

Lemma 4.14 *$\mathcal{G}$ is equal to the $O_S[g_1]$-submodule of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ generated by the functions $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)$, where $r \geq 0$, $n_1, \ldots, n_r \geq 0$ and $a_1, \ldots, a_r \in \tilde{S}$, namely*

$$\mathcal{G} = \sum_{\substack{r\geq 0,\, n_1,\ldots,n_r\geq 0,\\ a_1,\ldots,a_r\in\tilde{S}}} O_S[g_1] \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right).$$

Proof This follows from the shuffle identity satisfied by the functions $\tilde{\Gamma}$, which follows from the algebra morphism status of $k_{\vec{0}}$. ∎

Set $\mathrm{int} := \mathrm{int}_{\omega_0}$, where $\omega_0 := dz$. Then int is the endomorphism of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ given by $f \mapsto [z \mapsto \int_{z_0}^{z} f\omega_0]$, where z_0 is fixed in $\tilde{\mathcal{E}}_S$.

Definition 4.15 For $r \geq 0$, let us set

$$F_r(\mathcal{G}) := \sum_{r'\leq r} \sum_{((n_1,a_1),\ldots,(n_{r'},a_{r'}))\in(\mathbb{Z}_{\geq 0}\times\tilde{S})^{r'}} O_S[g_1] \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r'} \\ a_1 & a_2 & \cdots & a_{r'} \end{smallmatrix}; -\right).$$

One checks that $F_\bullet(\mathcal{G})$ is an algebra filtration of $\mathcal{G}$.

Lemma 4.16 *For any $r, n, n_1, \ldots, n_r \geq 0$, $a_1, \ldots, a_r \in \tilde{S}$ and $s \in S$, one has*

$$\mathrm{int}(T_{\tilde{s}}(g_n) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)) \in F_{r+1}(\mathcal{G}).$$

Proof Both $\mathrm{int}(T_{\tilde{s}}(g_n) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right))$ and $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r & n \\ a_1 & a_2 & \cdots & a_r & \tilde{s} \end{smallmatrix}; -\right) - \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r & n \\ a_1 & a_2 & \cdots & a_r & \tilde{s} \end{smallmatrix}; z_0\right)$ are elements of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$. By (3.3.1) their images by $d : \mathcal{O}_{hol}(\tilde{\mathcal{E}}_S) \to \Omega_{hol}(\tilde{\mathcal{E}}_S)$ coincide, and they both vanish at z_0. It follows that

$$\mathrm{int}(T_{\tilde{s}}(g_n) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)) = \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r & n \\ a_1 & a_2 & \cdots & a_r & \tilde{s} \end{smallmatrix}; -\right) - \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r & n \\ a_1 & a_2 & \cdots & a_r & \tilde{s} \end{smallmatrix}; z_0\right). \tag{4.4.1}$$

This identity implies the statement. ∎

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Lemma 4.17 *For any $r, n_1, \dots, n_r \geq 0$ and $a_1, \dots, a_r \in \tilde{S}$, one has*

$$\mathrm{int}(\partial(O_S[g_1]) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)) \subset F_r(\mathcal{G}).$$

Proof By induction on r. For $r = 0$, and $f \in O_S[g_1]$, one has $\mathrm{int}(\partial(f)) = f - f(z_0) \in F_0(\mathcal{G})$. Assume the statement at step $r-1$. For $f \in O_S[g_1]$, $n_1, \dots, n_r \geq 0$ and $a_1, \dots, a_r \in \tilde{S}$, integration by parts together with (3.3.1) yields

$$\begin{aligned}\mathrm{int}(\partial(f) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)) &= f \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right) - f(z_0) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; z_0\right) \\ &\quad - \mathrm{int}(f \cdot T_{a_r}(g_{n_r}) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right)).\end{aligned}$$

Note that $f \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right) \in F_r(\mathcal{G})$ and $f(z_0) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; z_0\right) \in \mathbb{C} \subset F_0(\mathcal{G})$. Moreover, $f \in O_S[g_1]$ and $T_{a_r}(g_{n_r}) \in O_S[g_1]$ by Lemma 4.12, which implies that $f \cdot T_{a_r}(g_{n_r}) \in O_S[g_1]$. By Lemma 4.12, one may then decompose $f \cdot T_{a_r}(g_{n_r})$ as $\partial(g) + \sum_{(n,a) \in \mathbb{Z}_{\geq 0} \times \tilde{S}} \lambda_{n,a} T_a(g_n)$, with $g \in O_S[g_1]$ and $\lambda_{n,a} \in \mathbb{C}$. Then

$$\begin{aligned}&\mathrm{int}(f \cdot T_{a_r}(g_{n_r}) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right)) \\ &= \mathrm{int}(\partial(g) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right)) + \sum_{(n,a) \in \mathbb{Z}_{\geq 0} \times \tilde{S}} \lambda_{n,a} \mathrm{int}(T_a(g_n) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right)).\end{aligned}$$

By the induction assumption, $\mathrm{int}(\partial(g) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right))$ belongs to $F_{r-1}(\mathcal{G})$. Moreover, for each (n, a), $\mathrm{int}(T_a(g_n) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_{r-1} \\ a_1 & a_2 & \cdots & a_{r-1} \end{smallmatrix}; -\right)) \in F_r(\mathcal{G})$by Lemma 4.16. All this implies $\mathrm{int}(\partial(f) \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right)) \in F_r(\mathcal{G})$, which concludes the inductive argument. ∎

Proposition 4.18 *For any $z_0 \in \tilde{\mathcal{E}}_S$, the linear endomorphism* int *of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ maps $\mathcal{G}$ to itself.*

Proof For $r \geq 0$, set

$$\mathrm{gr}_r(\mathcal{G}) := \sum_{\substack{n_1, \dots, n_r \geq 0, \\ a_1, \dots, a_r \in \tilde{S}}} O_S[g_1] \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; -\right).$$

One then has $F_r(\mathcal{G}) = \sum_{r' \in [\![0,r]\!]} \mathrm{gr}_{r'}(\mathcal{G})$. We will prove the inclusion $\mathrm{int}(F_r(\mathcal{G})) \subset F_{r+1}(\mathcal{G})$ by induction on $r \geq 0$. Let us first prove that $\mathrm{int}(F_0(\mathcal{G})) \subset F_1(\mathcal{G})$.

For $f \in O_S[g_1]$, one has $\mathrm{int}(\partial(f)) = f - f(z_0) \in O_S[g_1]$, which implies that

$$\mathrm{int}(\partial(O_S[g_1])) \subset O_S[g_1] \in F_0(\mathcal{G}). \tag{4.4.2}$$

It follows from (4.4.1) with $r = 0$ that, for $s \in S$ and $n \geq 0$, one has $\mathrm{int}(T_{\tilde{s}}(g_n)) = \tilde{\Gamma}\left(\begin{smallmatrix} n \\ \tilde{s} \end{smallmatrix}; -\right) - \tilde{\Gamma}\left(\begin{smallmatrix} n \\ \tilde{s} \end{smallmatrix}; z_0\right)$, which implies that

$$\mathrm{int}(T_{\tilde{s}}(g_n)) \in F_1(\mathcal{G}). \tag{4.4.3}$$

Then

$$\begin{aligned}\mathrm{int}(F_0(\mathcal{G})) &= \mathrm{int}(O_S[g_1]) = \mathrm{int}(\partial(O_S[g_1]) + \sum_{s \in S} T_{\tilde{s}}(\mathrm{Span}_{\mathbb{C}}\{g_n \mid n \geq 0\})) \\ &= \mathrm{int}(\partial(O_S[g_1])) + \sum_{s \in S, n \geq 0} \mathbb{C} \cdot \mathrm{int}(T_{\tilde{s}}(g_n)) \subset F_0(\mathcal{G}) + \sum_{s \in S, n \geq 0} F_1(\mathcal{G}) = F_1(\mathcal{G}),\end{aligned}$$

where the first equality follows from $F_0(\mathcal{G}) = O_S[g_1]$, the second equality follows from Lemma 4.12 and the inclusion follows from (4.4.2) and (4.4.3). This proves the initial step of the induction.

Assume that $r \geq 1$ and that $\mathrm{int}(F_{r-1}(\mathcal{G})) \subset F_r(\mathcal{G})$. Let us prove that $\mathrm{int}(F_r(\mathcal{G})) \subset F_{r+1}(\mathcal{G})$. One has $F_r(\mathcal{G}) = F_{r-1}(\mathcal{G}) + \mathrm{gr}_r(\mathcal{G})$, so by the induction assumption it suffices to prove that $\mathrm{int}(\mathrm{gr}_r(\mathcal{G})) \subset F_{r+1}(\mathcal{G})$.

Combining Lemma 4.12 with the definition of $\mathrm{gr}_r(\mathcal{G})$, one finds

$$\begin{aligned}\mathrm{gr}_r(\mathcal{G}) = &\sum_{\substack{n_1,\ldots,n_r\geq 0,\\ a_1,\ldots,a_r\in\tilde{S}}} \partial(O_S[g_1])\cdot\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r\end{smallmatrix};-\right)\\ &+\sum_{n\geq 0, s\in S}\sum_{\substack{n_1,\ldots,n_r\geq 0,\\ a_1,\ldots,a_r\in\tilde{S}}} \mathbb{C}\cdot T_{\tilde{s}}(g_n)\cdot\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r\end{smallmatrix};-\right),\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{int}(\mathrm{gr}_r(\mathcal{G})) = &\sum_{\substack{n_1,\ldots,n_r\geq 0,\\ a_1,\ldots,a_r\in\tilde{S}}} \mathrm{int}(\partial(O_S[g_1])\cdot\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r\end{smallmatrix};-\right))\\ &+\sum_{n\geq 0, s\in S}\sum_{\substack{n_1,\ldots,n_r\geq 0,\\ a_1,\ldots,a_r\in\tilde{S}}} \mathbb{C}\cdot \mathrm{int}(T_{\tilde{s}}(g_n)\cdot\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r\end{smallmatrix};-\right)),\end{aligned}$$

It follows from Lemma 4.17 (resp. Lemma 4.16) that the first (resp. second) summand of the right-hand side of this equality is contained in $F_r(\mathcal{G})$ (resp. $F_{r+1}(\mathcal{G})$), which implies that $\mathrm{int}(\mathrm{gr}_r(\mathcal{G})) \subset F_{r+1}(\mathcal{G})$. ■

Proposition 4.19 *The subspace $\mathcal{G}$ of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ is stable under the endomorphism* int_ω *for any $\omega \in \Omega(\mathcal{E}_S)$.*

Proof Since $\omega_0 = dz$ is a nowhere vanishing holomorphic differential, there exists $f_\omega \in O_S$ such that $\omega = f_\omega \cdot \omega_0$. Then for $f \in \mathcal{G}$, one has $\mathrm{int}_\omega(f) = \mathrm{int}(f_\omega \cdot f) \in \mathcal{G}$, where the equality follows from $\omega = f_\omega \cdot \omega_0$ and the inclusion follows from $f_\omega \cdot f \in \mathcal{G}$, which stems from the fact that $\mathcal{G}$ is an O_S-module, and from Proposition 4.18. ■

Theorem 4.20 *One has the equality $\mathcal{G} = A_{\mathcal{E}_S}$.*

Proof Proposition 4.19 implies that $\mathcal{G}$ is a subalgebra with unit of $\mathcal{O}_{hol}(\tilde{\mathcal{E}}_S)$ which is stable under the endomorphism int_ω for any $\omega \in \Omega(\mathcal{E}_S)$. Together with Lemma-Definition 2.2, this implies that $\mathcal{G} \supset A_{\mathcal{E}_S}$. The statement follows from the combination of this inclusion with Corollary 4.4. ■

4.5 An $\mathcal{O}(\mathcal{E}_S)$-basis of $A_{\mathcal{E}_S}$ arising from the functions $\tilde{\Gamma}$

Since the ring O_S is an integral domain, it injects into its fraction field, which is equal to the field $\mathbb{C}(\mathcal{E})$ of rational functions on $\mathcal{E}$ and is therefore independent of S, and will be denoted K. There is a unique extension of the derivation ∂ of O_S to a derivation of K, which will also be denoted ∂.

Lemma 4.21 *One has $O_S \cap \partial(K) = \partial(O_S)$ (equality of subspaces of K).*

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Proof For $f \in K\backslash\{0\}$, let $\mathrm{Pole}(f) \subset \mathcal{E}$ be the set of its poles. If f is nonconstant, then both f and $\partial(f)$ are in $K\backslash\{0\}$, and one has $\mathrm{Pole}(\partial f) = \mathrm{Pole}(f)$. On the other hand, $O_S = \{f \in K\backslash\{0\} \,|\, \mathrm{Pole}(f) \subset S\} \cup \{0\}$. Let now $f \in K$ be such that $\partial(f) \in O_S$. If $\partial(f) = 0$, then f is constant and so $f \in O_S$. If $\partial(f) \neq 0$, then f is nonzero and $\mathrm{Pole}(f) = \mathrm{Pole}(\partial f) \subset S$, therefore $f \in O_S$. All this proves the implication $(f \in K$ and $\partial(f) \subset O_S) \implies (f \in O_S)$, which implies the inclusion $O_S \cap \partial(K) \subset \partial(O_S)$. The opposite inclusion is obvious. ■

Under the isomorphism $O_S[X] \simeq O_S[g_1]$, the derivation ∂ of $O_S[g_1]$ is intertwined with the derivation ∂ of $O_S[X]$, uniquely defined by the conditions that it extends the derivation ∂ of O_S and that $\partial(X) = -E_2 \subset O \subset O_S$. This derivation further extends to a derivation ∂ of $K[X]$, extending the derivation ∂ of K.

Lemma 4.22 *The natural map $O_S[X]/\partial(O_S[X]) \to K[X]/\partial(K[X])$ is injective.*

Proof We first prove inductively on $n \geq 0$ the statement

$$(S_n): \quad (P \in O_S[X]_{\leq n}, Q \in K[X]_{\leq n+1}, P = \partial(Q)) \implies (Q \in O_S[X]_{\leq n+1}).$$

Let us prove (S_0). Let P, Q be polynomials satisfying the premise. Then there exist $a_0 \in O_S$ and $b_0, b_1 \in K$ such that $P = a_0$ and $Q = b_0 + b_1 X$. The coefficients of X and of 1 in $P = \partial(Q)$ respectively yields the equations

$$\partial(b_1) = 0, \quad a_0 + b_1 \cdot E_2 = \partial(b_0).$$

The first equation implies $b_1 \in \mathbb{C} \subset O_S$. The relation $b_1 \in \mathbb{C}$ then implies that the left-hand side of the second equation belongs to O_S, while $b_0 \in K$. By Lemma 4.21, this implies $b_0 \in O_S$. This proves that $Q \in O_S[X]_{\leq 1}$, and therefore (S_0).

Assume now (S_{n-1}) to be true for fixed $n \geq 1$, and let us prove (S_n). Let P, Q be polynomials satisfying the premise. Let $a_0, \dots, a_n \in O_S$ and $b_0, \dots, b_{n+1} \in K$ be such that $P = a_0 + \cdots + a_n X^n$ and $Q = b_0 + \cdots + b_{n+1} X^{n+1}$. The coefficients of X^{n+1} and X^n in $P = \partial(Q)$ respectively yield

$$\partial(b_{n+1}) = 0, \quad a_n + (n+1) b_{n+1} \cdot E_2 = \partial(b_n).$$

The first equation implies $b_{n+1} \in \mathbb{C} \subset O_S$. The relation $b_{n+1} \in \mathbb{C}$ then implies that the left-hand side of the second equation belongs to O_S, while $b_n \in K$. By Lemma 4.21, this implies $b_n \in O_S$. Let us set $\tilde{Q} := Q - b_{n+1} X^{n+1} - b_n X^n$ and $\tilde{P} := \partial(\tilde{Q})$. Then $\tilde{Q} \in K[X]_{\leq n-1}$, which implies $\tilde{P} \in K[X]_{\leq n-1}$ (because $\tilde{P} = \partial(\tilde{Q})$ and ∂ maps $K[X]_{\leq n-1}$ to itself) and $\tilde{P} \in O_S[X]$ (because $\tilde{P} = P - \partial(b_{n+1} X^{n+1} + b_n X^n)$), and therefore $\tilde{P} \in O_S[X]_{\leq n-1}$. The relation $\tilde{Q} \in K[X]_{\leq n-1}$ also implies $\tilde{Q} \in K[X]_{\leq n}$. Hence $(\tilde{P}, \tilde{Q})$ satisfy the assumptions of (S_{n-1}). It follows that $\tilde{Q} \in O_S[X]_{\leq n-1}$. Since $b_n, b_{n+1} \in O_S$, then $Q = \tilde{Q} + b_{n+1} X^{n+1} + b_n X^n \in O_S[X]_{\leq n+1}$. Therefore (P, Q) satisfy the conclusion of (S_n), which proves by induction that (S_n) is true for any $n \geq 0$.

Assume now that $Q \in K[X]$ is such that $\partial(Q) \in O_S[X]$. Let $n \neq 0$ be such that $Q \in K[X]_{\leq n}$. Then $\partial(Q) \in K[X]_{\leq n}$ because ∂ takes $K[X]_{\leq n}$ to itself, and since $\partial(Q) \in O_S[X]$ one has $\partial(Q) \in O_S[X]_{\leq n}$. But one also has $Q \in K[X]_{\leq n+1}$, hence (P, Q) satisfy the assumptions of (S_n), and therefore $Q \in O_S[X]$. All this implies the inclusion $O_S[X] \cap \partial(K[X]) \subset \partial(O_S[X])$. Since the opposite inclusion is obvious, one gets $O_S[X] \cap \partial(K[X]) = \partial(O_S[X])$, which concludes the proof. ■

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Lemma 4.23 *$\mathcal{O}_S[X]$ is an integral domain, and $\mathcal{O}_S[X] \hookrightarrow K(X)$ can be identified with the injection of $\mathcal{O}_S[X]$ in its fraction field. There is a unique derivation ∂ of $K(X)$ extending the derivation ∂ of $\mathcal{O}_S[X]$.*

Proof The first statement follows from fact that $\mathcal{O}_S$ is an integral domain. The field $K(X)$ contains $\mathcal{O}_S[X]$ as a subalgebra, and coincides with the smallest of its subfields containing $\mathcal{O}_S[X]$, which implies the identification of $\mathrm{Frac}(\mathcal{O}_S[X])$ with $K(X)$. The last statement follows from the fact the a derivation of an integral domain admits a unique extension to its fraction field. ∎

Recall the family $(1, (T_{\tilde{s}} g_n)_{s\in S, n\geq 1})$ of $\mathcal{O}_S[X]$ (see Lemma 4.12).

Lemma 4.24 *The image of the family $(1, (T_{\tilde{s}} g_n)_{s\in S, n\geq 1})$ in $K(X)/\partial(K(X))$ is $\mathbb{C}$-linearly independent.*

Proof Expansion for X at infinity induces a field extension $K(X) \subset K((X^{-1}))$, where $K((X^{-1}))$ is the field of Laurent series in the formal variable X^{-1} with coefficients in K. On the other hand, there is a double ring inclusion $\mathcal{O}_S[X] \subset K[X] \subset K(X)$, thus yielding a sequence of ring inclusions

$$\mathcal{O}_S[X] \subset K[X] \subset K(X) \subset K((X^{-1})). \tag{4.5.1}$$

There is a unique derivation ∂ of $K[X]$, extending ∂ on K and such that $\partial(X) = -E_2$. Since the derivation ∂ of $K(X)$ shares these properties, both derivations are compatible. The endomorphism of $K((X^{-1}))$ given by $\sum_{i\in\mathbb{Z}} f_i X^i \mapsto \sum_{i\in\mathbb{Z}}(\partial(f_i) - E_2(i+1)f_{i+1})X^i$ is well-defined and is a derivation of $K((X^{-1}))$. The restriction of this derivation to $\mathcal{O}_S[X]$ coincides with the derivation ∂ of this algebra, therefore its restriction to $K(X)$ also coincides with the derivation ∂ of this field. Hence one obtains a family of compatible derivations on the rings in the sequence of morphisms (4.5.1). From this one derives a sequence of linear maps

$$\begin{aligned}\mathcal{O}_S[X]/\partial(\mathcal{O}_S[X]) &\overset{\alpha}{\to} K[X]/\partial(K[X]) \\ &\overset{\beta}{\to} K(X)/\partial(K(X)) \overset{\gamma}{\to} K((X^{-1}))/\partial(K((X^{-1}))).\end{aligned} \tag{4.5.2}$$

The derivation ∂ of $K((X^{-1}))$ is compatible with the direct sum decomposition $K((X^{-1})) = K[X] \oplus X^{-1}K[[X^{-1}]]$, because it is compatible with the derivation ∂ of $K[X]$ and because $(\forall i \geq 0, f_i = 0) \implies (\forall i \geq 0, \partial(f_i) - (i+1)E_2 f_{i+1} = 0)$. It follows that the map $\gamma \circ \beta$ is injective, and so that β is injective. By Lemma 4.22, the map α is injective. It follows that $\beta \circ \alpha$ is injective, and in particular that it takes $\mathbb{C}$-linearly independent families to $\mathbb{C}$-linearly independent families. Then Lemma 4.13, which says that the image of $(1, (T_{\tilde{s}} g_n)_{s\in S, n\geq 1})$ in $\mathcal{O}_S[X]/\partial(\mathcal{O}_S[X])$ is $\mathbb{C}$-linearly independent, implies the statement. ∎

The following statement is a direct consequence of the main result of [DDMS].

Theorem 4.25 (see [DDMS], Theorem 1) *Let $(\mathcal{A}, \boldsymbol{d})$ be a commutative associative differential algebra with unit over a field $\mathbf{k}$ of characteristic 0, and let $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e., $\boldsymbol{d}(\mathcal{C}) \subset \mathcal{C}$). Let X be a set, $x \mapsto u_x$ be a map $X \to \mathcal{C}$, and let $(r, (x_1, \dots, x_r)) \mapsto f_{x_1,\dots,x_r}$ be a map $\sqcup_{r\geq 0} X^r \to \mathcal{A}$, such that $f_\varnothing = 1$ and $\boldsymbol{d}(f_{x_1,\dots,x_r}) = u_{x_1} \cdot f_{x_2,\dots,x_r}$ for any $r \geq 1$ and $x_1, \dots, x_r \in X$.*

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If the image of the family $x \mapsto u_x$ in $\mathcal{C}/\mathbf{d}(\mathcal{C})$ is $\mathbf{k}$-linearly independent, then the family $(r, (x_1, \ldots, x_r)) \mapsto f_{x_1,\ldots,x_r}$ is $\mathcal{C}$-linearly independent.

Proof The hypothesis of the statement is (iii) of Theorem 1 in [DDMS], and its conclusion is (i) in the same theorem; the statement then follows from Theorem 1 in *loc. cit.* ∎

Proposition 4.26 *(a) $\mathcal{G}$ is a free $O_S[g_1]$-module with basis*

$$(\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix}; -\right))_{((n_1,a_1),\ldots,(n_r,a_r))\in\sqcup_{r\geq 0}\{(n,a)\in\mathbb{Z}_{\geq 0}\times\tilde{S} \mid a=0 \text{ if } n=0\}^r}. \tag{4.5.3}$$

(b) $\mathcal{G}$ is a free O_S-module with basis

$$(g_1^i \cdot \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix}; -\right))_{i\geq 0,((n_1,a_1),\ldots,(n_r,a_r))\in\sqcup_{r\geq 0}\{(n,a)\in\mathbb{Z}_{\geq 0}\times\tilde{S} \mid a=0 \text{ if } n=0\}^r}. \tag{4.5.4}$$

Proof (a) It follows from Lemma 4.14 that (4.5.3) is a generating family of $\mathcal{G}$ as an $O_S[g_1]$-module.

Recall from Lemma 4.23 that $O_S[g_1]$ is a integral domain; it therefore fits in an algebra inclusion

$$O_S[g_1] \subset \mathrm{Frac}(O_S[g_1]). \tag{4.5.5}$$

On the other hand, there is an algebra inclusion

$$\mathcal{G} \subset \mathcal{O}_{mer}(\tilde{\mathcal{E}}_S), \tag{4.5.6}$$

where $\mathcal{O}_{mer}(\tilde{\mathcal{E}}_S)$ is the algebra of all meromorphic functions on $\tilde{\mathcal{E}}_S$. Moreover, this inclusion is compatible with (4.5.5) and with the module structure of both sides of (4.5.6) over the corresponding algebras of (4.5.5).

Let us show that the image of (4.5.3) under (4.5.6) is $\mathrm{Frac}(O_S[g_1])$-linearly independent. Set $(\mathcal{A}, \mathbf{d}) := (\mathcal{O}_{mer}(\tilde{\mathcal{E}}_S), \partial)$, set $\mathbf{k} := \mathbb{C}$, set $\mathcal{C} := \mathrm{Frac}(O_S[g_1])$, set $X := \{(n, a) \in \mathbb{Z}_{\geq 0} \times \tilde{S} \mid a = 0 \text{ if } n = 0\}$, let $x \mapsto u_x$ be the map $X \to \mathcal{C}$ given by $(0, 0) \mapsto 1$ and $(n, a) \mapsto T_a g_n$ for $n > 0$, let $(r, (x_1, \ldots, x_r)) \mapsto f_{x_1,\ldots,x_r}$ be the map $\sqcup_{r\geq 0} X^r \to \mathcal{A}$ given by (4.5.3).

It follows from (3.3.1) that the identities $f_\varnothing = 1$ and $\mathbf{d}(f_{x_1,\ldots,x_r}) = u_{x_1} \cdot f_{x_2,\ldots,x_r}$ are satisfied. It follows from Lemma 4.24 and from the identification $\mathrm{Frac}(O_S[g_1]) \simeq K(X)$ (see Lemma 4.23) that the image of $x \mapsto u_x$ in $\mathcal{C}/\mathbf{d}(\mathcal{C})$ is $\mathbb{C}$-linearly independent. We can therefore apply Theorem 4.25, which implies that the image of (4.5.3) under (4.5.6) is $\mathrm{Frac}(O_S[g_1])$-linearly independent. By the injectivity of the maps (4.5.5) and (4.5.6), this implies the announced statement.

(b) Let $\mathcal{F} := \sqcup_{r\geq 0}\{(n, a) \in \mathbb{Z}_{\geq 0} \times \tilde{S} \mid a = 0 \text{ if } n = 0\}$. Let $\mathbb{C}^{(\mathcal{F})}$ be the set of finitely supported complex functions on $\mathcal{F}$. Then (a) says that the map $O_S[g_1] \otimes \mathbb{C}^{(\mathcal{F})} \to \mathcal{G}$ induced by the family from (a) is a linear isomorphism. Moreover, Lemma 4.6 implies that the map $O_S \otimes \mathbb{C}^{(\mathbb{Z}_{\geq 0}\times\mathcal{F})} \to O_S[g_1] \otimes \mathbb{C}^{(\mathcal{F})}$ given by $f \otimes \delta_{(i,x)} \mapsto f \cdot g_1^i \otimes \delta_x$ for any $x \in \mathcal{F}$, $i \in \mathbb{Z}_{\geq 0}$, $f \in O_S$, is a linear isomorphism. Composing these two linear isomorphisms, one obtains that the map $O_S \otimes \mathbb{C}^{(\mathbb{Z}_{\geq 0}\times\mathcal{F})} \to \mathcal{G}$ induced by the family from (b) is a linear isomorphism, which proves (b). ∎

Theorem 4.27 *Both families (2.5.2) and (4.5.4) are O_S-bases of the O_S-module $A_{\mathcal{E}_S}$.*

Proof The statement on the family (2.5.2) follows from Lemma 2.19. The statement on the family (4.5.4) follows from Proposition 4.26(b) and from Theorem 4.20. ∎

4.6 Examples of relations between elliptic HLs and the functions $\tilde{\Gamma}$

The fact that the iterated integration basepoint for the elliptic HLs $L_{\alpha_{i_1}\cdots\alpha_{i_n}}$ from Section 2.5 is a fixed point z_0 *different* from 0, whereas the iterated integration basepoint for the functions $\tilde{\Gamma}$ is precisely the point 0, makes it a bit cumbersome to pass from one basis to the other. We provide below a few examples of expressions which relate elements of the two basis in the simplest setting where $S = pr(0)$ consists only of one point. In this case, the basis of differential forms considered in Section 2.5 reduces to $\alpha := dz$ and $\beta := E_2\, dz$.

Let us introduce the lighter notation

$$\tilde{\Gamma}(n_1 \dots n_r; -) := \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ 0 & 0 & \cdots & 0\end{smallmatrix}; -\right).$$

Since $\Gamma(\underbrace{0,\dots,0}_{n}; z) = \frac{z^n}{n!}$ and $L_{\underbrace{\alpha\cdots\alpha}_{n}}(z) = \frac{(z-z_0)^n}{n!}$, one immediately finds the formula

$$L_{\underbrace{\alpha\cdots\alpha}_{n}} = \sum_{j=0}^{n} \frac{(-z_0)^{n-j}}{(n-j)!}\tilde{\Gamma}(\underbrace{0,\dots,0}_{j}; -).$$

Furthermore, since $E_2 = -g_1'$, it follows that

$$L_\beta = -g_1 + g_1(z_0). \tag{4.6.1}$$

Combining this relation with the fact that $\tilde{\Gamma}(1; z) = \tilde{\Gamma}(1; z_0) + \int_{z_0}^{z} g_1(u)du$, one gets

$$\tilde{\Gamma}(1; -) = -L_{\beta\alpha} + g_1(z_0)L_\alpha + \tilde{\Gamma}(1; z_0).$$

Moreover, from (4.6.1) and the shuffle product identity $L_\beta^k = k! L_{\underbrace{\beta\cdots\beta}_{k}}$, one gets for $n \geq 1$

$$L_{\underbrace{\beta\cdots\beta}_{n}} = \frac{(g_1(z_0) - g_1)^n}{n!},$$

$$g_1^n = \sum_{k=0}^{n} \frac{(-1)^k\, n!}{(n-k)!}\, g_1^{n-k}(z_0) L_{\underbrace{\beta\cdots\beta}_{k}}. \tag{4.6.2}$$

Finally, let us express $\tilde{\Gamma}(2; -)$ in terms of elliptic HLs. By Equation (2.4.5), one has $g_2 = (g_1^2 - E_2 + e_2)/2$. Moreover, Equation (4.6.2) for $n = 2$ gives $g_1^2 = 2L_{\beta\beta} - 2g_1(z_0)L_\beta + g_1^2(z_0)$. Combining these two observations, one gets

$$\tilde{\Gamma}(2; -) = L_{\beta\beta\alpha} - g_1(z_0)L_{\beta\alpha} + \frac{e_2 + g_1^2(z_0)}{2} L_\alpha - \frac{1}{2} L_\beta + \tilde{\Gamma}(2; z_0).$$

5 Relation of the functions $\tilde{\Gamma}$ with the functions E_3

The purpose of this section is to give an alternative proof, based on the results of §4, of the fact that an algebra $\mathcal{A}_3$ attached to an elliptic curve $\mathcal{E}_{alg}$ equipped with a degree 2 ramified covering $\mathcal{E}_{alg} \to \mathbb{P}^1$ which was constructed in [BDDT] is stable

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under integration. In §5.1, we relate the framework of the present paper with the one of [BDDT]. In Section 5.2, we attach to each finite subset $S_0 \subset \mathbb{P}^1_{\mathbb{C}}$ containing ∞ an algebra $\mathcal{A}_3(S_0)$, which we prove to be stable under integration using Theorem. 4.20; we derive the fact that the inductive limit of the algebras $\mathcal{A}_3(S_0)$ is stable under integration, and identify this inductive limit with the algebra $\mathcal{A}_3$ of [BDDT], thereby giving an alternative proof of the result of this paper (Proposition 5.6(c)).

5.1 Reminders on uniformisation of elliptic curves

Let us denote by $(z,\tau) \mapsto E_2(z|\tau)$ (resp. $\tau \mapsto e_2(\tau)$) the meromorphic (resp. holomorphic) function on $\mathbb{C} \times \mathfrak{H}$ (resp. $\mathfrak{H}$) such that for any $\tau \in \mathfrak{H}$, the map $z \mapsto E_2(z|\tau)$ (resp. the element $e_2(\tau) \in \mathbb{C}$) coincides with the function E_2 (resp. the number $e_2 \in \mathbb{C}$) from §2.4.2. For $\tau \in \mathfrak{H}$, we set $\wp(z|\tau) := E_2(z|\tau) - e_2(\tau)$.

Denote by $\mathbb{C}^3_*$ be the complement of the diagonals in $\mathbb{C}^3$.

Lemma 5.1 *Let $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \in \mathbb{C}^3_*$. There exists $\tau \in \mathfrak{H}$, $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$ such that*

$$(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = (a\wp(1/2|\tau) + b, a\wp(\tau/2|\tau) + b, a\wp((\tau+1)/2|\tau) + b).$$

Proof The set $\mathbb{C}^3_*$ is equipped with the commuting actions of the group $\mathbb{C}^\times \ltimes \mathbb{C}$ by $(a,b) \cdot (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) := (a\mathbf{a}_1 + b, a\mathbf{a}_2 + b, a\mathbf{a}_3 + b)$ and of the symmetric group $\mathfrak{S}_3$ by permutation. The set $\mathbb{C}\backslash\{0,1\}$ is equipped with an action of $\mathfrak{S}_3$, generated by the involutions $\lambda \mapsto 1-\lambda$ and $\lambda \mapsto 1/\lambda$.

The map $\mathbb{C}^3_* \to \mathbb{C}\backslash\{0,1\}$ given by $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \mapsto \mathbf{a}_{21}/\mathbf{a}_{31}$, where we set $\mathbf{a}_{ij} = \mathbf{a}_i - \mathbf{a}_j$, is $\mathfrak{S}_3$-equivariant; it can be identified with the projection $\mathbb{C}^3_* \to (\mathbb{C}^\times \ltimes \mathbb{C})\backslash\mathbb{C}^3_*$ of $\mathbb{C}^3_*$ on its set of orbits under the action of $\mathbb{C}^\times \ltimes \mathbb{C}$.

The map $\mathbb{C}\backslash\{0,1\} \to \mathbb{C}$ induced by $\lambda \mapsto 256(1-\lambda+\lambda^2)^3/(\lambda^2(1-\lambda)^2)$ is S_3-invariant, and can be identified with the projection $\mathbb{C}\backslash\{0,1\} \to \mathfrak{S}_3\backslash(\mathbb{C}\backslash\{0,1\})$ of $\mathbb{C}\backslash\{0,1\}$ on the set of its orbits under the action of $\mathfrak{S}_3$.

The map $j : \mathfrak{H} \to \mathbb{C}$ defines a bijection $j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \to \mathbb{C}$. On the other hand, the map $\lambda : \mathfrak{H} \to \mathbb{C}\backslash\{0,1\}$ defined by $\tau \mapsto (\wp(\tau/2|\tau) - \wp(1/2|\tau))/(\wp((1+\tau)/2|\tau) - \wp(1/2|\tau))$ is compatible with the group morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_2) \simeq \mathfrak{S}_3$; it therefore defines a map $\lambda : \Gamma(2)\backslash\mathfrak{H} \to \mathbb{C}\backslash\{0,1\}$, which is a bijection. The map λ is also the composition with the projection $\mathbb{C}^3_* \to \mathbb{C}\backslash\{0,1\}$ of the map $\mathfrak{H} \to \mathbb{C}^3_*$ given by $\tau \mapsto (\wp(1/2|\tau), \wp(\tau/2|\tau), \wp((\tau+1)/2|\tau))$.

The situation is summarized in the diagram

$$\begin{tikzcd}
\mathbb{C}^3_* \arrow[r, "\mathbb{C}^\times \ltimes \mathbb{C}"] & \mathbb{C}\backslash\{0,1\} \arrow[r, "\mathfrak{S}_3"] & \mathbb{C} \\
\mathfrak{H} \arrow[u] \arrow[r, "\Gamma(2)"] \arrow[rr, bend right, "\mathrm{SL}_2(\mathbb{Z})"'] & \Gamma(2)\backslash\mathfrak{H} \arrow[u, "\lambda", "\sim"'] \arrow[r, "\mathfrak{S}_3"] & \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \arrow[u, "\sim", "j"']
\end{tikzcd}$$

where the notation $X \overset{\Gamma}{\to} Y$ means that the map $X \to Y$ is Γ-invariant, and sets up a bijection $\Gamma\backslash X \overset{\sim}{\to} Y$.

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Let now $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbb{C}^3_*$. Its image by the composition $\mathbb{C}^3_* \to \mathbb{C}\backslash\{0,1\} \to \mathbb{C} \overset{j^{-1}}{\to} \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ is a well-defined $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathfrak{H}$. Let $\tau_0 \in \mathfrak{H}$ be an element of this orbit. By the properties of the above diagram, there exists $\sigma \in \mathfrak{S}_3$ such that $\lambda(\tau_0) \in \mathbb{C}\backslash\{0,1\}$ is related to $\mathbf{a}_{21}/\mathbf{a}_{31}$ by $\mathbf{a}_{21}/\mathbf{a}_{31} = \sigma \cdot \lambda(\tau_0)$. Let $\tilde{\sigma} \in \mathrm{SL}_2(\mathbb{Z})$ be a lift of σ, one then has $\mathbf{a}_{21}/\mathbf{a}_{31} = \lambda(\tau)$, where $\tau = \sigma \cdot \tau_0$. Then the elements $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $(\wp(1/2|\tau), \wp(\tau/2|\tau), \wp((\tau+1)/2|\tau))$ of $\mathbb{C}^3_*$ have the same image in $\mathbb{C}\backslash\{0,1\}$, which implies that they are related by the action of $\mathbb{C}^\times \ltimes \mathbb{C}$. ∎

In the rest of §5.1, we fix $\vec{\mathbf{a}} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbb{C}^3_*$.

Definition 5.2 $\mathcal{E}_{\mathrm{alg}}$ is the projective curve in $\mathbb{P}^2(\mathbb{C})$ defined by the equation $Y^2T = (X - \mathbf{a}_1T)(X - \mathbf{a}_2T)(X - \mathbf{a}_3T)$, where $[X : Y : T]$ is the canonical system of projective coordinates on $\mathbb{P}^2(\mathbb{C})$.

Lemma 5.3 *Let $(a, b, \tau) \in \mathbb{C}^\times \times \mathbb{C} \times \mathfrak{H}$ be as in Lemma 5.1, let $\Lambda := \mathbb{Z} + \mathbb{Z}\tau$ and $\mathcal{E} := \mathbb{C}/\Lambda$. Let $a^{3/2}$ be a square root of a^3, and denote by $(z,\tau) \mapsto \wp'(z|\tau)$ the partial derivative of $(z,\tau) \mapsto \wp'(z|\tau)$ with respect to z.*

(a) There is a unique isomorphism $iso : \mathcal{E} \to \mathcal{E}_{\mathrm{alg}}$, given by $pr(z) \mapsto [a\wp(z|\tau) + b : (1/2)a^{3/2}\wp'(z|\tau) : 1]$ for $z \notin \Lambda$ and $pr(0) \mapsto [0 : 1 : 0]$.

(b) The image by iso of $pr(1/2)$, $pr(\tau/2)$ and $pr((1+\tau)/2)$ are respectively $[\mathbf{a}_1 : 0 : 1]$, $[\mathbf{a}_2 : 0 : 1]$ and $[\mathbf{a}_3 : 0 : 1]$. The isomorphism iso intertwines the involution $pr(z) \mapsto pr(-z)$ of $\mathcal{E}$ with the involution $[X : Y : T] \mapsto [X : -Y : T]$ of $\mathcal{E}_{\mathrm{alg}}$.

Proof (a) follows from the fact that functions $(z,\tau) \mapsto \wp(z|\tau)$ and $\wp'(z|\tau)$ satisfy the identity $\wp'(z|\tau)^2 = 4(\wp(z|\tau) - \wp(1/2|\tau))(\wp(z|\tau) - \wp(\tau/2|\tau))(\wp(z|\tau) - \wp((1+\tau)/2|\tau))$ (see [Kn], Theorem 6.15) and from the relation from Lemma 5.1. The first part of (b) follows from Lemma 5.1, from the vanishing for fixed τ of the function $z \mapsto \wp'(z|\tau)$ at $1/2$, $\tau/2$ and $(1+\tau)/2$, which follows from its oddness and from its Λ-periodicity, and its second part follows from the evenness of the function $z \mapsto \wp(z|\tau)$ and oddness of the function $z \mapsto \wp'(z|\tau)$ for τ being fixed. ∎

5.2 An alternative proof of a result of [BDDT]

Let $\vec{\mathbf{a}} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbb{C}^3_*$, let $(a, b, \tau) \in \mathbb{C}^\times \times \mathbb{C} \times \mathfrak{H}$ be as in Lemma 5.1 and let $\mathcal{E}, \mathcal{E}_{\mathrm{alg}}$ be as in Lemma 5.3 and Definition 5.2.

Let $\pi : \mathcal{E}_{\mathrm{alg}} \to \mathbb{P}^1(\mathbb{C})$ be the morphism given by the composition $\mathcal{E}_{\mathrm{alg}} \subset \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, where the last morphism is $[X : Y : T] \mapsto [X : T]$. Let $S_0 \subset \mathbb{P}^1(\mathbb{C})$ be a finite subset containing ∞. Let $S_{\mathrm{alg}} := \pi^{-1}(S_0)$ and $S := iso^{-1}(S_{\mathrm{alg}})$.

Then S is a finite subset of $\mathcal{E}$, which is stable under the involution $pr(z) \mapsto pr(-z)$, and which contains $\{pr(0)\}$; moreover, there is an isomorphism

$$iso : \mathcal{E}\backslash S \to \mathcal{E}_{\mathrm{alg}}\backslash S_{\mathrm{alg}}.$$

In [BDDT], (3.16), one defines the family of multivalued holomorphic functions

$$x \mapsto \mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right)$$

on $\mathcal{E}_{\mathrm{alg}}\backslash S_{\mathrm{alg}}$ (denoted $x \mapsto \mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{\mathbf{a}}\right)$in *loc. cit.* to underscore the dependence in $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$), indexed by tuples $(n_1, c_1), \ldots, (n_k, c_k)$ in $\sqcup_{k \geq 0} I^k$, where $I := (\mathbb{Z} \times S_0^{unr}) \sqcup (\mathbb{Z}_{\geq 0} \times S_0^{ram})$ and $S_0^{ram} := S_0\backslash(S_0 \cap \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \infty\})$ and $S_0^{unr} := S_0\backslash S_0^{ram}$; one

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therefore has for each $i \in [\![1,k]\!]$ the relations $(n_i, c_i) \in \mathbb{Z} \times S_0$, with[6] $n_i \geq 0$ whenever $c_i \in \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \infty\}$.

Recall that $\left(z \mapsto \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ a_1 & \cdots & a_k \end{smallmatrix}; z\right)\right)$ is a family of multivalued holomorphic functions on $\mathcal{E}\backslash S$, indexed by tuples $(n_1, a_1), \ldots, (n_k, a_k)$ in $\sqcup_{k\geq 0}(\mathbb{Z}_{\geq 0} \times \tilde{S})^k$.

Proposition 5.4 (see also [BDDT], Section 5) *The map iso^* induces an isomorphism of vector spaces between the linear spans* $\mathrm{Span}_{\mathbb{C}}\{x \mapsto \mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) \mid k \geq 0, (n_i, c_i) \in (\mathbb{Z} \times S_0^{unr}) \sqcup (\mathbb{Z}_{\geq 0} \times S_0^{ram})\}$ *and* $\mathrm{Span}_{\mathbb{C}}\{z \mapsto \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ a_1 & \cdots & a_k \end{smallmatrix}; z\right) \mid k \geq 0, (n_i, a_i) \in \mathbb{Z}_{\geq 0} \times \tilde{S}\}$.

Proof The family of functions $x \mapsto \mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right)$ with variable $(k, (n_1, c_1), \ldots, (n_k, c_k))$ is defined in [BDDT] as follows. One introduces in *loc. cit.* a family of multivalued meromorphic differentials $\varphi_n(c,t)dt$ on $\mathcal{E}_{\mathrm{alg}}$, where $(n, c) \in (\mathbb{Z} \times S_0^{unr}) \sqcup (\mathbb{Z}_{\geq 0} \times S_0^{ram})$ and $S_0^{ram} := S_0 \backslash (S_0 \cap \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \infty\})$ with sets of poles contained in $pr_{\mathrm{alg}}^{-1}(S_{\mathrm{alg}})$, where $pr_{\mathrm{alg}} : \tilde{\mathcal{E}}_{\mathrm{alg}} \to \mathcal{E}_{\mathrm{alg}}$ is a universal cover. The functions $x \mapsto \mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right)$ are the iterated integrals of these differentials starting from the point $[0 : \sqrt{-\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3} : 1]$ in $\mathcal{E}_{\mathrm{alg}}$, where $\sqrt{-\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3}$ is a square root of $-\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$. The isomorphism $iso : \mathcal{E} \to \mathcal{E}_{\mathrm{alg}}$ induces an isomorphism $\widetilde{iso} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}_{\mathrm{alg}}$, and the image by $\widetilde{iso}^*$ of $\mathrm{Span}_{\mathbb{C}}(\varphi_n(c,t)dt \mid (n,c) \in (\mathbb{Z} \times S_0^{unr}) \sqcup (\mathbb{Z}_{\geq 0} \times S_0^{ram}))$ is a space of multivalued meromorphic differentials on $\mathcal{E}$, with sets of poles contained in $pr^{-1}(S)$, which is shown in [BDDT], Section 5 to coincide with $\mathrm{Span}_{\mathbb{C}}(dz, T_a g_n \cdot dz, a \in pr^{-1}(S))$. It follows from (2.4.6) that the latter space is equal to $\mathrm{Span}_{\mathbb{C}}(dz, T_a g_n \cdot dz, a \in \tilde{S})$. So the linear span of the functions $x \mapsto \mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right)$ corresponds via iso^* to the linear span of the iterated integrals starting at $iso(0)$ of the elements of $\mathrm{Span}_{\mathbb{C}}(dz, T_a g_n \cdot dz, a \in \tilde{S})$. By Lemma 3.16(b), this linear span coincides with that of the regularized iterated integrals starting at $pr^{-1}(0)$ of the elements of $\mathrm{Span}_{\mathbb{C}}(dz, T_a g_n \cdot dz, a \in \tilde{S})$, which is the linear span of $z \mapsto \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ a_1 & \cdots & a_k \end{smallmatrix}; z\right)$ for variable $(k, (n_1, a_1), \ldots, (n_k, a_k))$. ∎

The authors of [BDDT] define in Equation (3.23) a multivalued function Z_3 on $\mathcal{E}_{\mathrm{alg}} \backslash \{\infty\}$, closely related to a classical elliptic integral, and prove that $iso^*(\mathrm{Z}_3) = 4g_1$ (see [BDDT], Equation (5.11)).

Proposition 5.5 *The subalgebra*

$$\mathcal{A}_3(S_0) := \mathcal{O}(\mathcal{E}_{\mathrm{alg}} \backslash S_{\mathrm{alg}})[\mathrm{Z}_3][\mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; -\right) | k, (n_1, c_1), \ldots, (n_k, c_k)]$$

of the algebra of holomorphic multivalued functions on $\mathcal{E}_{\mathrm{alg}} \backslash S_{\mathrm{alg}}$ is stable in the sense of Definition 2.1.

Proof Since $iso^*(\mathrm{Z}_3) = 4g_1$, it follows that the image by iso^* of this algebra is equal to $\mathcal{G}$ (see Definition 4.1). The result then follows from Proposition 4.19. ∎

Let us emphasize the dependence of S_{alg} in the finite subset S_0 such that $\{\infty\} \subset S_0 \subset \mathbb{P}^1_{\mathbb{C}}$ by denoting it $S_{\mathrm{alg}}(S_0)$. Fix: (a) for each finite subset subset S_0 such that $\{\infty\} \subset S_0 \subset \mathbb{P}^1_{\mathbb{C}}$, a universal cover $\mathcal{E}_{\mathrm{alg}} \widetilde{\backslash S_{\mathrm{alg}}}(S_0) \to \mathcal{E}_{\mathrm{alg}} \backslash S_{\mathrm{alg}}(S_0)$; (b) for

[6] The range of values $\mathbb{Z} \times \mathbb{P}^1_{\mathbb{C}}$ for the pairs (n_i, c_i) announced in [BDDT], (3.16) should in fact be restricted by the condition that $n_i \geq 0$ whenever $c_i \in \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \infty\}$, as one can derive from (3.31) and the discussion following it.

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each pair of finite sets S_0, S_0' with $\{\infty\} \subset S_0 \subset S_0' \subset \mathbb{P}^1_{\mathbb{C}}$, a holomorphic map $p_{S_0,S_0'} : \mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0')} \to \mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0)}$ such that

$$\begin{tikzcd}
\mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0')} \arrow[r, "p_{S_0,S_0'}"] \arrow[d] & \mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0)} \arrow[d] \\
\mathcal{E}_{\text{alg}}\backslash S_{\text{alg}}(S_0') \arrow[r] & \mathcal{E}_{\text{alg}}\backslash S_{\text{alg}}(S_0)
\end{tikzcd}$$

commutes, such that for any triple of finite sets S_0, S_0', S_0'' with $\{\infty\} \subset S_0 \subset S_0' \subset S_0'' \subset \mathbb{P}^1_{\mathbb{C}}$, one has $p_{S_0,S_0''} = p_{S_0,S_0'} \circ p_{S_0',S_0''}$. One can then form the inductive limit algebra $\varinjlim_{S_0} \mathcal{O}_{hol}(\mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0)})$; it contains the algebra $\varinjlim_{S_0} \mathcal{O}(\mathcal{E}_{\text{alg}}\backslash S_{\text{alg}}(S_0))$, which is the field $\mathbb{C}(\mathcal{E}_{\text{alg}})$ of rational functions on $\mathcal{E}_{\text{alg}}$.

Recall that the space of regular differentials on $\mathcal{E}_{\text{alg}}$ is linearly spanned by $(TdX - XdT)/(YT)$.

Proposition 5.6 *(a) For each pair of finite sets S_0, S_0' with $\{\infty\} \subset S_0 \subset S_0' \subset \mathbb{P}^1_{\mathbb{C}}$, the algebra inclusion $p^*_{S_0,S_0'} : \mathcal{O}_{hol}(\mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0)}) \to \mathcal{O}_{hol}(\mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0')})$ is such that $p^*_{S_0,S_0'}(\mathcal{A}_3(S_0)) \subset \mathcal{A}_3(S_0')$. Set $\mathcal{A}_3 := \varinjlim_{S_0} \mathcal{A}_3(S_0)$.*

(b) Let denote the endomorphism $\text{int}_{(TdX-XdT)/(YT)}$ of $\mathcal{O}_{hol}(\mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0)})$ (see §2.1) as int_{S_0}. Then for each pair of finite sets S_0, S_0' with $\{\infty\} \subset S_0 \subset S_0' \subset \mathbb{P}^1_{\mathbb{C}}$, one has $\text{int}_{S_0} \circ p^*_{S_0,S_0'} = p^*_{S_0,S_0'} \circ \text{int}_{S_0'}$; let $\varinjlim_{S_0} \text{int}_{S_0}$ be the corresponding endomorphism of $\varinjlim_{S_0} \mathcal{O}_{hol}(\mathcal{E}_{\text{alg}}\widetilde{\backslash S_{\text{alg}}(S_0)})$.

(c) $\mathcal{A}_3$ is stable under $\varinjlim_{S_0} \text{int}_{S_0}$.

Proof (a) and (b) are straightforward, and (c) follows from Proposition 5.5. ■

Note that Proposition 5.6(c) recovers the result from [BDDT], stated in 3 and proved in §6.

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