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CAUCHY INTEGRAL OF CALDERÓN ON THE GRAPHS OF FUNCTIONS WITH BMO DERIVATIVES

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ABSTRACT. We first note that each graph $(x, A(x))$ of a function $A(x)$ with BMO derivative is a chord-arc curve. Using this, Muckenhoupt's A_p theory, and the theory of Calderón–Zygmund operators, we shall derive weighted norm inequalities for the Cauchy integral on such graphs from a recent theorem of G. David on the L^2-boundedness of Cauchy integral on almost-lipschitzian curves.

1. Recently Murai [9] has proved the following theorem, related to the Cauchy integral of Calderón.

THEOREM A. *Let $A(x)$ be a real valued function on the real line $\mathbb{R}$ with $A'(x) \in$ BMO $(\mathbb{R})$, and consider the singular integral*

$$Tf(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1 + iA'(y)}{x - y + i(A(x) - A(y))} f(y)\, dy.$$

Then for any $w \in A_p$ $(1 < p < \infty)$, there exists a constant $C = C(p, w)$ such that

$$\|Tf\|_{L^p(wdx)} \leq C\|f\|_{L^p(wdx)},$$

and

$$\|T_* f\|_{L^p(wdx)} \leq C\|f\|_{L^p(wdx)}.$$

In the above, BMO$(\mathbb{R})$ is the set of all functions f of bounded mean oscillation, i.e., $\|f\|_{\text{BMO}} = \sup |I|^{-1} \int_I |f(x) - m_I f|dx < \infty$, where the supremum is taken over all intervals I, $m_I f = |I|^{-1} \int_I f(x)dx$, and $|I|$ is the length of I. A_p is the Muckenhoupt weight class, i.e. for $1 < p < \infty$ $A_p = \{w \in L^1_{\text{loc}}(\mathbb{R}); w \geq 0, \sup_I m_I w[m_I(w^{-1/(p-1)})]^{p-1} < \infty$.

$$T_* f(x) = \sup_{0<\epsilon<\eta} \left| \int_{\epsilon<|x-y|<\eta} K(x,y) f(y) dy \right|,$$

where $K(x,y) = (1 + iA'(y))/[x - y + i(A(x) - A(y))]$. We note that $K(x,y)$ is the Cauchy integral kernel of the curve $\Gamma_0 = \{(x, A(x)); x \in \mathbb{R}\}$, parametrized by the real variable x.

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Now a rectifiable curve $\Gamma = \{z(s) \in \mathbb{C};\ s \in \mathbb{R}\}$ on the complex plane $\mathbb{C}$, parametrized by the arc-length, is said to be a regular curve in the sense of Ahlfors or an *almost-lipschitzian* curve, if there exists a constant $C > 0$ such that for all $r > 0$ and any disc D of radius r, the length of $\Gamma \cap D$ is smaller than Cr. And a rectifiable curve Γ is said to be a *chord-arc* curve or a Lavrentiev curve, if there exists a constant $C > 0$ such that $|s - t| \leq (1 + C)|z(s) - z(t)|$. The infimum of C is called the chord arc constant. A chord-arc curve is always almost-lipschitzian. Recently G. David showed the following [4].

THEOREM B. *Let* $1 < p < \infty$. *Let* $\Gamma = \{z(s);\ s \in \mathbb{R}\}$ *be an almost-lipschitzian curve. Then, for any* $f \in L^p(\mathbb{R})$,

$$Sf(t) = \lim_{\epsilon \to 0} \int_{|z(t)-z(s)|>\epsilon} [z(t) - z(s)]^{-1} f(s)ds$$

exists for almost every t and

$$\|Sf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}.$$

Note that the Cauchy integral kernel of the curve Γ is $z'(s)[z(t) - z(s)]^{-1}$.

Murai has proved Theorem A directly by obtaining a good λ inequality. Recently, B. C. Krickeles has also obtained Theorem A implicitly in [8], i.e. one can obtain Theorem A from Corollary 3 in [8]. His way is also to get a good λ inequality for the kernel $K(x, y)$. The purpose of this note is to deduce Theorem A from Theorem B, using A_p weight theory. This is suggested by Y. Meyer.

Finally we note that in the case $A' \in L^\infty$,

$$T_1 f(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x - y + i(A(x) - A(y))} f(y)dy$$

is a Calderón-Zygmund operator, and hence weighted norm inequalities hold [3]. Especially, T is bounded from $L^1(\mathbb{R})$ to weak-$L^1(\mathbb{R})$. However, for general $A' \in \text{BMO}$, this is not true. Take, for example, $A'(y) = \log|y|$ and $f(y) = \chi_{(0,1)}(y)(y \log^2 2/y)^{-1}$, where χ_E is the characteristic function of the set E.

2. **Some properties of BMO and A_p weights**. First we give a condition under which a curve is a chord-arc curve. This is perhaps known, but as far as we know, it has not appeared in literatures.

LEMMA 1. *Let* $\Gamma = \{x + iA(x) \in \mathbb{C};\ x \in \mathbb{R}\}$ *and* $A(x)$ *be a real valued function with* $A'(x) \in \text{BMO}$. *Then* Γ *is a chord-arc curve with chord-arc constant smaller than* $C\|A'\|_{\text{BMO}}$ *where C is an absolute constant.*

PROOF. For $0 < a \leq 1$ we have

$$\int_{x_1}^{x_2} (1 + (A'(x))^2)^{1/2} dx \leq a^{-1} \int_{x_1}^{x_2} (1 + a^2(A'(x))^2)^{1/2} dx.$$

Now, since $\log(1 + ix)$ is a Lipschitz function on $\mathbb{R}$, with Lipschitz constant 1, we get

$$\|\log(1 + iaA'(x))\|_{\mathrm{BMO}} \leq 2a\|A'\|_{\mathrm{BMO}}.$$

Hence, if $2a\|A'\|_{\mathrm{BMO}} \leq \epsilon_2 < 1$ (ϵ_2 is a sufficiently small constant), then by Proposition 13 in [2], the curve $\{x + iaA(x)\}$ is a chord-arc curve of chord-arc constant smaller than $C_2(2a\|A'\|_{\mathrm{BMO}})^2$. Hence

$$\int_{x_1}^{x_2}(1 + a^2(A'(x))^2)^{1/2}dx \leq (1 + 4C_2a^2\|A'\|^2_{\mathrm{BMO}})[(x_2 - x_1)^2 + a^2(A(x_2) - A(x_1))^2]^{1/2}.$$

Thus taking $a = 1$ if $\|A'\|_{\mathrm{BMO}} \leq \epsilon_2/2$ and $a = \epsilon_2/(2\|A'\|_{\mathrm{BMO}})$ if $\|A'\|_{\mathrm{BMO}} \geq \epsilon_2/2$, we have

$$\int_{x_1}^{x_2}(1 + (A'(x))^2)^{1/2}dx \leq (1 + 2\epsilon_2^{-1}(1 + C_2\epsilon_2^2)\|A'\|_{\mathrm{BMO}}) \times [(x_2 - x_1)^2 + (A(x_2) - A(x_1))^2]^{1/2}.$$

This completes the proof.

LEMMA 2. *Let $f(z)$ be a nonnegative function on the complex plane $\mathbb{C}$ satisfying $|f(z_1) - f(z_2)| \leq C|z_1 - z_2|$ (for all $z_1, z_2 \in \mathbb{C}$) and $f(z) \geq a$ for some $a > 0$. Then for any real number k and any complex valued function $G(x) \in \mathrm{BMO}(\mathbb{R})$ it holds*

$$[f(G(x))]^k \in \bigcap_{1<p} A_p.$$

PROOF. We may assume $1 < p \leq 2$, since $A_r \subset A_s$ $(1 \leq r < s)$. Put $m = 1/(p - 1)$, so that $m \geq 1$. Then for $b > 0$ one can easily show that $\log(b + f(z))^{mk}$ is a Lipschitz function on $\mathbb{C}$ with Lipschitz constant smaller than $Cm|k|/(b + a)$. Hence we get

$$\|\log(b + f(G(x))^{mk}\|_{\mathrm{BMO}} \leq \frac{2Cm|k|}{b + a}\|G\|_{\mathrm{BMO}}.$$

So, for sufficiently large b, by John-Nirenberg's lemma ([5], p. 417, 3′ or [1], p. 41) we obtain

$$(b + f(G(x)))^{mk} = \exp[\log(b + f(G(x)))^{mk}] \in A_2.$$

Hence one gets easily

$$(b + f(G(x)))^k \in A_{(m+1)/m} = A_p.$$

Since $a(2b + 2a)^{-1}(b + f(G(x))) < f(G(x)) < b + f(G(x))$, we obtain the desired conclusion. q.e.d.

LEMMA 3. *Let $\Phi(t) = \int_0^t \phi(s)ds$ be a homeomorphism of the real line with $0 \leq \phi(t) \in A_\infty = \bigcup_{p>1} A_p$. Then for any $w(x) \in A_\infty$, we have $w(\Phi(t))\Phi'(t) \in A_\infty$.*

PROOF. This can easily be derived from the following characterization of A_∞ by Coifman and Fefferman [1]: $v(t) \in A_\infty$ if and only if there exist $C > 0$ and $\delta > 0$ such that for any interval I and any measurable subset E of I

$$\frac{\int_E v(t)dt}{\int_I v(t)dt} \leq C\left(\frac{|E|}{|I|}\right)^\delta. \qquad \text{q.e.d.}$$

LEMMA 4. *Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $\Phi(t) = \int_0^t \phi(s)ds$ be an increasing homeomorphism of the real line satisfying $\Phi'(\Phi^{-1}(x)) \in A_q$. Then there is a $C > 0$ such that for any $f \in L^1_{\text{loc}}(\mathbb{R})$ and $x \in \mathbb{R}$*

$$M(f(\Phi^{-1}))(\Phi(x)) \leq CM_p(f)(x).$$

Here $M_p(f)(x) = \sup (|I|^{-1} \int_I |f(x)|^p dx)^{1/p}$ and $M(f) = M_1(f)$, where the supremum is taken over all intervals I containing x.

PROOF. Let I be an arbitrary interval with $\Phi(x) \in I$. Then after applying Hölder's inequality to the right-hand side of the following identity

$$|I|^{-1} \int_I |f(\Phi^{-1})(t)|dt = |I|^{-1} \int_{\Phi^{-1}(I)} |f(t)|\Phi'(t)dt,$$

use the assumption $\Phi'(\Phi^{-1}) \in A_q$, and we obtain the desired assertion. q.e.d.

Finally in this section we quote a theorem of P. Jones [6].

LEMMA 5. *Let $\Phi(t) = \int_0^t \phi(s)ds$ be an increasing homeomorphism of the real line with $\phi \in L^1_{\text{loc}}(\mathbb{R})$. Then the following are equivalent each other.*

(*i*) $\phi \in A_\infty$;

(*ii*) $f(\Phi^{-1}) \in \text{BMO}(\mathbb{R})$ *for all* $f \in \text{BMO}(\mathbb{R})$;

(*iii*) $f(\Phi) \in \text{BMO}(\mathbb{R})$ *for all* $f \in \text{BMO}(\mathbb{R})$;

(*iv*) $(\Phi^{-1})'(t) = 1/\Phi'(\Phi^{-1}(t)) \in A_\infty$.

3. **Proof of Theorem A**. Let $\Gamma = \{(x, A(x)); x \in \mathbb{R}\}$, $s(x) = \Phi(x) = \int_0^x (1 + (A'(y))^2)^{1/2} dy$, and $z(s) = x + iA(x)$. Then by Lemma 1 we get

$$\left|\frac{1}{z(t) - z(s)}\right| \leq \frac{C}{|t - s|} \tag{3.1}$$

$$\left|\frac{\partial}{\partial t}\frac{1}{z(t) - z(s)}\right| \leq \frac{C}{|t - s|^2}. \tag{3.2}$$

One also sees that, for $f(s) = z'(s)g(s)$ with $g \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} Uf(t) &= \lim_{\epsilon \to 0} \int_{|t-s|>\epsilon} \frac{f(s)}{z(t) - z(s)} ds \\ &= \lim_{\epsilon \to 0} \int_{|z(t)-z(s)|>\epsilon} \frac{f(s)}{z(t) - z(s)} ds. \end{aligned} \tag{3.3}$$

Hence (3.1), (3.2) and Theorem B imply that U is a Calderón-Zygmund operator (as for the definition and basic properties of Calderón–Zygmund operators see, for example, [7]). Hence for any $v \in A_\infty$, we have, as in Theorem 3 in Coifman–Fefferman [1], $\|U_* f\|_{L^p(vdx)} \leq C_p \|Mf\|_{L^p(vdx)}$. So, putting $Vf = U(z'f)$, we have for any $0 < p < \infty$

$$(3.4) \qquad \|V_* f\|_{L^p(vdx)} \leq C_p \|Mf\|_{L^p(vdx)},$$

where

$$V_* f(x) = \sup_{\epsilon > 0} \left| \int_{|t-s|>\epsilon} z'(s)[z(t) - z(s)]^{-1} f(s)ds \right|.$$

Now let $w \in A_p$ $(1 < p < \infty)$ and put $\Psi(t) = \Phi^{-1}(t)$. Then by Lemma 2, $\Phi' \in \bigcap_{r>1} A_r$ and hence by Lemma 5 $\Psi'(t) \in A_\infty$. Thus by Lemma 3 we get $u(t) = w(\Psi(t))\Psi'(t) \in A_\infty$. (Unfortunately we cannot, for the present, assert $u \in A_p$.) We obtain next the following identity.

$$(3.5) \quad \int_{|x-y|>\epsilon} K(x,y)f(y)dy = \int_{|\Phi(x)-s|>\Phi(x+\epsilon)-\Phi(x)} L(\Phi(x),s)f(\Psi(s))ds + \int_{\Phi(x-\epsilon)}^{2\Phi(x)-\Phi(x+\epsilon)} L(\Phi(x),s)f(\Psi(s))ds,$$

where $K(x,y) = (1 + iA'(y))/[(x - y) + i(A(x) - A(y))]$ and $L(t,s) = z'(s)/[z(t) - z(s)]$. By Lemma 1 and (3.1) we see easily that the second term in the right-hand side of (3.5) is dominated by $CM(f \circ \Psi)(\Phi(x))$. So

$$(3.6) \qquad T_* f(x) \leq 2V_*(f \circ \Psi)(\Phi(x)) + CM(f \circ \Psi)(\Phi(x)).$$

Since $w \in A_p$, there is $1 < p' < p$ such that $w \in A_{p/p'}$ by a theorem of Coifman and Fefferman [1]. Since $\Phi' \in A_\infty$, by Lemma 5 we get $A'(\Psi(t)) \in \text{BMO}$, and hence by Lemma 2 $\Phi'(\Psi) \in \bigcap_{r>1} A_r$. Thus by Lemma 4 we get $M(f \circ \Psi)(\Phi(x)) \leq C_{p'} M_{p'}(f)(x)$, and so

$$(3.7) \qquad \|M(f \circ \Psi)(s)\|_{L^p(uds)} = \|M(f \circ \Psi)(\Phi(x))\|_{L^p(wds)} \leq C\|M_{p'}(f)\|_{L^p(wdx)} \leq C'\|f\|_{L^p(wdx)}.$$

The last inequality follows from $w \in A_{p/p'}$, because weighted norm inequalities hold for the Hardy-Littlewood maximal function $M(g)$. By (3.4) we get

$$(3.8) \quad \|V_*(f \circ \Psi)(\Phi(x))\|_{L^p(wdx)} = \|V_*(f \circ \Psi)(s)\|_{L^p(uds)} \leq C\|M(f \circ \Psi)(s)\|_{L^p(uds)}.$$

From (3.6), (3.7) and (3.8) we have

$$(3.9) \qquad \|T_* f\|_{L^p(wdx)} \leq C\|f\|_{L^p(wdx)}.$$

It is easily seen that for any $f \in C_0^\infty(\mathbb{R})$, p.v. $\int K(x,y)f(y)dy$ exists a.e. and equals $U((z'f) \circ \Psi)(\Phi(x))$ a.e.. Since $C_0^\infty(\mathbb{R})$ is dense in $L^p(wdx)$, from (3.9) it follows, by a standard argument, that for any $f \in L^p(wdx)$, p.v. $\int K(x,y)f(y)dy$ exists a.e.. Hence we have

$$\|Tf\|_{L^p(wdx)} \leq C\|f\|_{L^p(wdx)}.$$

This completes the proof of Theorem A.

4. **Final remark**. Murai has shown more in [9]. We note that by his method one can prove, for example, the following: Let $A(x)$ be a real valued function on $\mathbb{R}$ with $A'(x) \in \mathrm{BMO}$ and for nonnegative integer k

$$T_k[A,f](x) = \text{p.v.} \int_{-\infty}^{\infty} (x-y)^{-1} \left[\frac{A(x)-A(y)}{x-y} - A'(y)\right]^k \exp\left[i\frac{A(x)-A(y)}{x-y}\right] f(y)dy.$$

Then for any $w \in A_p$ $(1 < p < \infty)$, there are $C = C(p,w)$ and a nonnegative number N such that

$$\|T[A,f]\|_{L^p(wdx)} \leq C^k\|A'\|_{\mathrm{BMO}}^k(1 + \|A'\|_{\mathrm{BMO}})^N\|f\|_{L^p(wdx)}. \tag{4.1}$$

From this (the case $k = 1$) one can naturally prove Theorem A. Unlike the case $A' \in L^\infty$, it seems that one can not prove (4.1) from Theorem A or its variants, (cf. [3]).

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