

DEFORMATION INVARIANCE OF PLURIGENERA

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Abstract. We prove the invariance of plurigenera under smooth projective deformations in full generality.

§1. Introduction

Let X be a smooth projective variety and let K_X be the canonical bundle of X . The canonical ring

$$R(X, K_X) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a basic birational invariant of X . For every positive integer m , the m -th plurigenus $P_m(X)$ is defined by

$$P_m(X) := \dim H^0(X, \mathcal{O}_X(mK_X)).$$

The plurigenera are fundamental birational invariants of compact complex manifolds. It has been believed that the plurigenera is invariant under smooth projective deformations (although the plurigenera are not deformation invariant under nonprojective deformations ([9])).

In this paper, we shall prove the invariance of plurigenera under smooth projective deformations in full generality.

THEOREM 1.1. *Let $\pi : X \rightarrow \Delta$ be a smooth projective family of smooth projective varieties over the unit open disk.*

Then for every positive integer m , the m -th plurigenus $P_m(X_t)$ ($X_t := \pi^{-1}(t)$) is independent of $t \in \Delta$.

In the course of the proof of Theorem 1.1, we also prove the following theorem.

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THEOREM 1.2. *Let $\pi : X \rightarrow \Delta$ be a smooth projective family. Suppose that K_{X_0} is pseudoeffective. Then K_{X_t} is pseudoeffective for every $t \in \Delta$.*

Let us recall the former results concerning Theorem 1.1. S. Iitaka proved the invariance of plurigenera for deformations of compact complex surfaces ([3]) by using the classification of compact complex surfaces. N. Nakayama ([10]) pointed out that the invariance of plurigenera for smooth projective deformations can be derived if the minimal model program were completed for families (it is worth noting that his results (including the lowersemicontinuity of plurigenera under semistable degenerations) immediately follow from the L^2 -extension theorem ([13, p.200, Theorem]) without using Kollár's vanishing theorem ([5])).

Recently Y.-T. Siu ([14]) proved that for every $m \geq 1$, $P_m(X)$ is invariant under smooth projective deformations, if all the fibers are of general type. The proof of [14] has been translated into more algebraic way and the result in [14] has been slightly generalized by [4, 11] (although such translation is completely unnecessary for such generalizations).

For the proof of Theorem 1.1, the central problem is the existence of singular hermitian metric h on K_X such that the curvature current Θ_h is semipositive and $h|_{X_t}$ is an AZD (cf. Definition 2.3) of K_{X_t} for every $t \in \Delta$. As soon as we construct such a metric h , the L^2 -extension theorem ([13]) implies the invariance of the plurigenera. Here the key point is that the L^2 -extension theorem requires the semipositivity of the curvature of the singular hermitian metric, but it does not require any strict positivity of the curvature. In this sense the L^2 -extension theorem is similar to Kollár's vanishing theorem ([5]) in algebraic context.

The construction of the above singular hermitian metric h consists of the inductive estimates of singular hermitian metrics using Bergman kernels. This is more straightforward than the inductive comparison of multiplier ideal sheaves as in [14, p. 670, Proposition 5]. In this sense the proof is **quantitative** and not qualitative. The essential idea is the **dynamical construction** of an AZD of the canonical line bundle of a smooth projective variety with pseudoeffective canonical line bundle. This construction works only for pseudoeffective canonical line bundles. This clarifies why the canonical line bundle is special. Here the key ingredient of the proof is the L^2 -extension theorem of holomorphic sections [13, 8, 12]. And we use the fact that the operator norm of the interpolation operator is bounded from

above by a **universal constant**.

Hence the proof here is analytic in nature. The advantage of the L^2 -estimates of $\bar{\partial}$ -operators is that it assures the existence of solutions of $\bar{\partial}$ -equations with **estimates**, while the algebraic methods only indicate the existence of solutions. This is the main reason why the L^2 -estimates changed the complex analysis completely.

Another key of the proof of Theorem 1.1 is the theory of AZD. The theory of AZD can be viewed as a **minimal model theory of pseudo-effective line bundles**. The theory of AZD is very simple (cf. Theorem 2.2) and fits to the L^2 -extension theorem and many other analytic tools. As in [14, 11, 4], we may replace AZD's with some infinite sequence of multiplier ideal sheaves, if the line bundle is big. But this kind of translation breaks down when the line bundle is not big. This is because in general a singular hermitian metric with semipositive curvature current cannot be approximated by a sequence of singular hermitian metrics with algebraic singularities and semipositive curvature, while singular hermitian metrics with strictly positive curvature current can be approximated by a sequence of singular hermitian metric with algebraic singularities and semipositive curvature current ([1]).

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§2. Preliminaries

2.1. Multiplier ideal sheaves

In this subsection L will denote a holomorphic line bundle on a complex manifold M .

DEFINITION 2.1. A singular hermitian metric h on L is given by

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ -hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is an arbitrary function on M . We call φ a weight function of h .

The curvature current Θ_h of the singular hermitian line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi,$$

where $\partial\bar{\partial}$ is taken in the sense of a current. The L^2 -sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle (L, h) is defined by

$$\mathcal{L}^2(L, h) := \{\sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma, \sigma) \in L^1_{loc}(U)\},$$

where U runs over the open subsets of M . In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of (L, h) . If we write h as

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. For $\varphi \in L^1_{loc}(M)$ we define the multiplier ideal sheaf of φ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}).$$

Similarly we define

$$\mathcal{I}_\infty(h) := \mathcal{L}^\infty(\mathcal{O}_M, e^{-\varphi})$$

and call it the L^∞ -**multiplier ideal sheaf** of (L, h) .

DEFINITION 2.2. L is said to be pseudoeffective, if there exists a singular hermitian metric h on L such that the curvature current Θ_h is a closed positive current.

Also a singular hermitian line bundle (L, h) is said to be pseudoeffective, if the curvature current Θ_h is a closed positive current.

2.2. Analytic Zariski decompositions

DEFINITION 2.3. Let M be a compact complex manifold and let L be a holomorphic line bundle on M . A singular hermitian metric h on L is said to be an analytic Zariski decomposition, if the followings hold.

1. Θ_h is a closed positive current,
2. for every $m \geq 0$, the natural inclusion

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(M, \mathcal{O}_M(mL))$$

is an isomorphism.

Remark 2.1. If an AZD exists on a line bundle L on a smooth projective variety M , L is pseudoeffective by the condition 1 above.

THEOREM 2.1. ([15, 16]) *Let L be a big line bundle on a smooth projective variety M . Then L has an AZD.*

As for the existence for general pseudoeffective line bundles, now we have the following theorem.

THEOREM 2.2. ([2, Theorem 1.5]) *Let X be a smooth projective variety and let L be a pseudoeffective line bundle on X . Then L has an AZD.*

Proof of Theorem 2.2. Although the proof is in [2], we shall give a proof here, because we shall use it afterwards.

Let h_0 be a fixed C^∞ -hermitian metric on L . Let E be the set of singular hermitian metric on L defined by

$$E = \{h; h : \text{lowersemicontinuous singular hermitian metric on } L, \Theta_h \text{ is positive, } \frac{h}{h_0} \geq 1\}.$$

Since L is pseudoeffective, E is nonempty. We set

$$h_L = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. The supremum of a family of plurisubharmonic functions uniformly bounded from above is known to be again plurisubharmonic, if we modify the supremum on a set of measure 0 (i.e., if we take the uppersemicontinuous envelope) by the following theorem of P. Lelong.

THEOREM 2.3. ([7, p.26, Theorem 5]) *Let $\{\varphi_t\}_{t \in T}$ be a family of plurisubharmonic functions on a domain Ω which is uniformly bounded from above on every compact subset of Ω . Then $\psi = \sup_{t \in T} \varphi_t$ has a minimum uppersemicontinuous majorant ψ^* which is plurisubharmonic. We call ψ^* the uppersemicontinuous envelope of ψ .*

Remark 2.2. In the above theorem the equality $\psi = \psi^*$ holds outside of a set of measure 0 (cf.[7, p.29]).

By Theorem 2.3, we see that h_L is also a singular hermitian metric on L with $\Theta_h \geq 0$. Suppose that there exists a nontrivial section $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ for some m (otherwise the second condition in Definition 2.3 is empty). We note that

$$\log |\sigma|^{\frac{2}{m}}$$

gives the weight of a singular hermitian metric on L with curvature $2\pi m^{-1}(\sigma)$, where (σ) is the current of integration along the zero set of σ . By the construction we see that there exists a positive constant c such that

$$\frac{h_0}{|\sigma|^{\frac{2}{m}}} \geq c \cdot h_L$$

holds. Hence

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m))$$

holds. Hence in particular

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_L^m))$$

holds. This means that h_L is an AZD of L . □

Remark 2.3. By the above proof we have that for the AZD h_L constructed as above

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m)) \simeq H^0(X, \mathcal{O}_X(mL))$$

holds for every m .

It is easy to see that the multiplier ideal sheaves of $h_L^m (m \geq 1)$ constructed in the proof of Theorem 2.2 are independent of the choice of the C^∞ -hermitian metric h_0 . We call the AZD constructed as in the proof of Theorem 2.2 a **canonical AZD** of L .

2.3. L^2 -extension theorem

The following theorem is crucial in our proof of Theorem 1.1.

THEOREM 2.4. ([13, p.200, Theorem]) *Let X be a Stein manifold of dimension n , ψ a plurisubharmonic function on X and s a holomorphic function on X such that $ds \neq 0$ on every branch of $s^{-1}(0)$. We put $Y :=$*

$s^{-1}(0)$ and $Y_0 := \{x \in Y; ds(x) \neq 0\}$. Let g be a holomorphic $(n - 1)$ -form on Y_0 with

$$c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g} < \infty,$$

where $c_k = (-1)^{\frac{k(k-1)}{2}} (\sqrt{-1})^k$. Then there exists a holomorphic n -form G on X such that

$$G(x) = g(x) \wedge ds(x)$$

on Y_0 and

$$c_n \int_X e^{-\psi} (1 + |s|^2)^{-2} G \wedge \bar{G} \leq 1620\pi c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g}.$$

§3. Dynamical construction of an AZD

Let X be a smooth projective variety and let K_X be the canonical line bundle of X . Let n denote the dimension of X .

Let A be a sufficiently ample line bundle on X such that for every pseudoeffective singular hermitian line bundle (L, h_L)

$$\mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L)$$

and

$$\mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L)$$

are globally generated. This is possible by [14, p. 667, Proposition 1]. Let h_A be a C^∞ hermitian metric on A with strictly positive curvature.

Hereafter we shall assume that $\kappa(X) \geq 0$, in particular K_X is pseudoeffective. For $m \geq 0$, let h_m be the singular hermitian metrics on $A + mK_X$ constructed as follows. Let h_0 be a C^∞ -hermitian metric h_A on A with strictly positive curvature. Suppose that h_{m-1} ($m \geq 1$) has been constructed. Let $\{\sigma_0^{(m)}, \dots, \sigma_{N(m)}^{(m)}\}$ be an orthonormal basis of $V_m := H^0(X, \mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(h^{m-1}))$ with respect to the inner product :

$$\begin{aligned} (\sigma, \sigma') &:= \int_X \sigma \cdot \bar{\sigma}' \cdot h_{m-1} \\ &= \int_X \sigma \cdot \bar{\sigma}' \cdot (h_{m-1} \otimes (dV)^{-1}) \cdot dV, \end{aligned}$$

where dV is an arbitrary nowhere degenerate C^∞ volume form on X . We set

$$K_m := \sum_{i=0}^{N(m)} |\sigma_i^{(m)}|^2,$$

where $|\sigma_i^{(m)}|^2$ denotes $\sigma_i^{(m)} \cdot \overline{\sigma_i^{(m)}}$. We call K_m the **Bergman kernel** of $A + mK_X$ with respect to $h_{m-1} \otimes (dV)^{-1}$ and dV . Clearly it is independent of the choice of the orthonormal basis. We define the integer ν by

$$\nu := \overline{\lim}_{m \rightarrow \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(h^m))}{\log m}.$$

And we define the singular hermitian metric h_m on $A + mK_X$ by

$$h_m := m^\nu \cdot K_m^{-1}.$$

It is clear that K_m has semipositive curvature in the sense of currents. We note that for every $x \in X$

$$K_m(x) = \sup \left\{ |\sigma|^2(x); \sigma \in V_m, \int_X h_{m-1} \cdot |\sigma|^2 = 1 \right\}$$

holds by definition (cf. [6, p.46, Proposition 1.4.16]).

Let dV be a C^∞ -volume form on X with respect to a Kähler form ω on X . For a singular hermitian line bundle (F, h_F) on X , let $A^2(M, F, h_F, dV)$ denote the Hilbert space of L^2 holomorphic sections of F with respect to h_F and dV . We may also assume that for any pseudoeffective singular hermitian line bundle (L, h_L) and for any point $x \in X$, there exists an interpolation operator

$$I_x : A^2(x, K_X \otimes A \otimes L, dV^{-1}h_A h_L, \delta_x) \longrightarrow A^2(X, K_X \otimes A \otimes L, dV^{-1}h_A h_L, dV)$$

such that the operator norm of I_x is bounded from above by a positive constant independent of $x \in X$ and (L, h_L) , where δ_x denotes the Dirac measure at x . This is certainly possible, if we take A to be sufficiently ample.

In fact let x be a point on X and let (U, z_1, \dots, z_n) be a local coordinate neighbourhood of x which is biholomorphic to Δ^n and $z_i(x) = 0 (1 \leq i \leq n)$. Then by the successive use of Theorem 2.4, we find an interpolation operator

$$I_x^U : A^2(x, K_X \otimes A \otimes L, dV^{-1}h_A h_L, \delta_x) \longrightarrow A^2(U, K_X \otimes A \otimes L, dV^{-1}h_A h_L, dV)$$

such that the operator norm of I_x^U is bounded from above by a positive constant C_U independent of (L, h_L) . Now we note that the curvature of $h_A h_L$ is bounded from below by the Kähler form Θ_A . Let ρ be a C^∞ -function on X such that $\text{Supp } \rho \subset \subset U$, $0 \leq \rho \leq 1$ and $\rho \equiv 1$ on a neighbourhood of x .

Let σ_x be an element of $A^2(x, K_X \otimes A \otimes L, dV^{-1}h_A h_L, \delta_x)$. Then replacing (A, h_A) by its sufficiently high positive multiple, we may assume that

$$\Theta_A + (n + 1)\sqrt{-1}\partial\bar{\partial}(\rho \cdot \log \sum_{i=1}^n |z_i|^2) \geq \omega$$

holds on X . We also note that there exists a positive constant C'_U independent of (L, h_L) and σ_x such that

$$\begin{aligned} \int_X \exp(-(n + 1)\rho \cdot \log \sum_{i=1}^n |z_i|^2) \cdot |\bar{\partial}(\rho \cdot I_x^U \sigma_x)|^2 dV \\ \leq C'_U \cdot (dV^{-1}h_A h_L)(\sigma_x, \sigma_x) \end{aligned}$$

holds, where $|\bar{\partial}(\rho \cdot I_x^U \sigma_x)|^2$ denotes the norm with respect to $h_A \cdot h_L$ and ω . In fact C'_U only depends on C_U and the supremum of the norm of $\bar{\partial}\rho$ with respect to ω . Then by the usual L^2 -estimate, we may assume that we can solve the $\bar{\partial}$ -equation

$$\bar{\partial}u = \bar{\partial}(\rho \cdot I_x^U \sigma_x)$$

with

$$u(x) = 0$$

so that

$$\int_X h_A h_L |u|^2 \leq C \cdot (dV^{-1}h_A h_L)(\sigma_x, \sigma_x)$$

holds for a positive constant C independent of (L, h_L) and σ_x . Then

$$\rho \cdot I_x^U \sigma_x - u \in H^0(X, \mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L))$$

is an extension of σ_x . Since X is compact, moving x and U , by the above estimates this implies the assertion.

LEMMA 3.1. *Let h be a canonical AZD of K_X constructed as in the proof of Theorem 2.2. Then the inclusion:*

$$\mathcal{I}(h^m) \subseteq \mathcal{I}(h_m)$$

holds for every $m \geq 0$.

Proof. We prove this lemma by induction on m . If $m = 0$, then both sides are \mathcal{O}_X .

Suppose that the inclusion

$$\mathcal{I}(h^{m-1}) \subseteq \mathcal{I}(h_{m-1})$$

has been settled for some $m \geq 1$. Then we have that by the property of A as above

$$\mathcal{O}_X(K_X + (A + (m - 1)K_X)) \otimes \mathcal{I}(h^{m-1})$$

is a globally generated subsheaf of

$$\mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(h_{m-1})$$

Hence by the definition of $\mathcal{I}(h_m)$ we see that

$$\mathcal{I}(h^{m-1}) \subseteq \mathcal{I}(h_m)$$

holds. In particular

$$\mathcal{I}(h^m) \subseteq \mathcal{I}(h_m)$$

holds. By the induction on m , this completes the proof of Lemma 3.1. \square

By the choice of A and Lemma 3.1, h_m is well defined for every $m \geq 0$. Now we shall make the above lemma quantitative.

LEMMA 3.2. *There exists a positive constant C such that*

$$h_m \leq C^m \cdot h_A \cdot h^m$$

holds for every $m \geq 0$.

Proof. First if $m = 0$, both sides are h_0 . Suppose that for some $m \geq 1$,

$$h_{m-1} \leq C_{(m-1)} \cdot h_A \cdot h^{m-1}$$

holds for some positive constant $C_{(m-1)}$. Let dV be a C^∞ volume form on X . Let $C(1)$ be a positive constant such that

$$h \geq C(1) \cdot (dV)^{-1}$$

holds on X . Let us denote the Bergman kernel of V_m with respect to a singular hermitian metric H on $A + mK_X$ and the volume form dV by $K(A +$

mK_X, H, dV). In this notation K_m is expressed as $K(A + mK_X, h_{m-1} \otimes (dV)^{-1}, dV)$.

Then we have that

$$\begin{aligned} & K(A + mK_X, h_{m-1} \otimes (dV)^{-1}, dV) \\ & \geq C_{(m-1)}^{-1} \cdot K(A + mK_X, h_A \cdot h^{m-1} \cdot (dV)^{-1}, dV) \\ & \geq C_{(m-1)}^{-1} \cdot C(2) \cdot (h_A \cdot h^{m-1} \cdot (dV)^{-1})^{-1} \cdot m^\nu \\ & \geq C_{(m-1)}^{-1} \cdot C(1) \cdot C(2) \cdot (h_A \cdot h^m)^{-1} \cdot m^\nu \end{aligned}$$

hold for some positive constant $C(2)$. The first inequality follow from the formula :

$$\begin{aligned} & K(A + mK_X, h_{m-1} \otimes (dV)^{-1}, dV)(x) \\ & = \sup\{|\sigma|^2(x); \sigma \in V_m; \int_X h_{m-1} |\sigma|^2 = 1\} (x \in X) \end{aligned}$$

and the similar formula for $K(A + mK_X, h_A \cdot h^{m-1} \cdot (dV)^{-1}, dV)$.

The 2-nd inequality follows from the L^2 -extension theorem (Theorem 2.4) applied to the extension from a point to X as above. We note that for every point $x \in X$ and every general ν -dimensional subvariety $Y \ni x$, $(K_x, h)|_Y$ is big, hence $h|_Y$ is dominated by a metric with strictly positive curvature. The factor m^ν comes from the fact that the norm of the local interpolation I_x^U as above is $O(m^{-\nu})$ (cf. [17, p.105, (1.11)]). Hence we may assume that $C(2)$ is independent of m . Now we can take C to be the constant $C(1)^{-1} \cdot C(2)^{-1}$. This completes the proof of Lemma 3.2. \square

LEMMA 3.3. *There exists a positive constant \tilde{C} such that for every $m \geq 1$,*

$$h_A \cdot K_m \leq \tilde{C}^m (dV)^m \cdot m^\nu$$

holds.

Proof. Let $p \in X$ be an arbitrary point. Let (U, z_1, \dots, z_n) be a local coordinate around x such that

1. $z_1(p) = \dots = z_n(p) = 0$,
2. U is biholomorphic to the open unit polydisk in \mathbf{C}^n with center $O \in \mathbf{C}^n$ by the coordinate,
3. z_1, \dots, z_n are holomorphic on a neighbourhood of the closure of U ,

4. there exists a holomorphic frame \mathbf{e} of A on the closure of U .

We set

$$\Omega := (-1)^{\frac{n(n-1)}{2}}(\sqrt{-1})^n dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

For every $m \geq 0$, we set

$$C_m := \sup_{x \in U} \frac{h_A \cdot K_m}{\Omega^m}.$$

We note that for any $x \in X$

$$K_m(x) = \sup \left\{ |\phi|^2(x); \phi \in V_m, \int_X h_{m-1} |\phi|^2 = 1 \right\}$$

holds. Let ϕ_0 be the element of $\Gamma(X, \mathcal{O}_X(A + mK_X))$ such that

$$K_m(x) = |\phi_0|^2(x)$$

and

$$\int_X h_{m-1} |\phi_0|^2 = 1.$$

Then there exists a holomorphic function f on U such that

$$\phi_0|_U = f \cdot (dz_1 \wedge \cdots \wedge dz_n)^m \cdot \mathbf{e}$$

holds. Then

$$\int_U h_A |\phi_0|^2 \Omega^{-(m-1)} = \int_U |f|^2 h_A(\mathbf{e}, \mathbf{e}) \Omega$$

holds. On the other hand by the definition of C_{m-1} we see that

$$\int_U h_A |\phi_0|^2 \Omega^{-(m-1)} \leq C_{m-1} \int_U h_{m-1} |\phi_0|^2 \leq C_{m-1}$$

hold. Combining above inequalities we have that

$$\int_U |f|^2 h_A(\mathbf{e}, \mathbf{e}) \Omega \leq C_{m-1}$$

holds. Let $0 < \delta \ll 1$ be a sufficiently small number. Let U_δ be the inverse image of

$$\{(y_1, \dots, y_n) \in \mathbf{C}^n \mid |y_i| < 1 - \delta\}$$

by the coordinate (z_1, \dots, z_n) .

Then by the subharmonicity of $|f|^2$, there exists a positive constant C_δ independent of m such that

$$|f(x)|^2 \leq C_\delta \cdot C_{m-1}$$

holds for every $x \in U_\delta$. Then we have that

$$K_m(x) \leq C_\delta \cdot C_{m-1} \cdot |\mathbf{e}|^2 \otimes \Omega^m(x)$$

holds for every $x \in U_\delta$. Summing up the estimates for orthonormal basis, moving p , by the compactness of X we see that there exists a positive constant \tilde{C} such that

$$h_A \cdot K_m \leq \tilde{C}^m \cdot (dV)^m \cdot m^\nu$$

holds on X . This completes the proof of Lemma 3.3. □

By Lemma 3.2 and Lemma 3.3

$$K_\infty := \text{the uppersemicontinuous envelope of } \limsup_{m \rightarrow \infty} \sqrt[m]{K_m}$$

is a well defined volume form on X which does not vanish outside of a set of measure 0. We set

$$h_\infty := \frac{1}{K_\infty}.$$

Then by Lemma 3.2, we see that

$$h_\infty \leq C \cdot h$$

holds. By the definition of h_∞ , it is clear that the curvature Θ_{h_∞} is semi-positive in the sense of current. Hence by the construction of h (see the proof of Theorem 2.2), we see that there exists a positive constant C' such that the opposite estimate:

$$h_\infty \geq C' \cdot h$$

holds.

Hence we have the following theorem.

THEOREM 3.1. *Let h_∞ be the above singular hermitian metric on K_X . Then h_∞ is an AZD of K_X .*

§4. Proof of Theorem 1.1

Now we shall prove Theorem 1.1. Let $\pi : X \rightarrow \Delta$ be a smooth projective family of projective varieties as in Theorem 1.1. We set $X_t := \pi^{-1}(t)$. If for every $t \in \Delta$, K_{X_t} is not pseudoeffective, then $P_m(X_t) = 0$ holds for every $t \in \Delta$ and every $m \geq 1$. Hence in this case there is nothing to prove.

Now we shall assume that for some $t_0 \in \Delta$, say $t_0 = 0$, K_{X_0} is pseudoeffective. Shrinking Δ , if necessary, we may assume that there exists an ample line bundle A on X such that for any pseudoeffective singular hermitian line bundle (L, h_L)

$$\mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L)$$

and

$$\mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L)$$

are globally generated and for every $t \in \Delta$ and for any pseudoeffective singular hermitian line bundle (L_t, h_{L_t}) on X_t ,

$$\mathcal{O}_{X_t}(A|_{X_t} + L_t) \otimes \mathcal{I}(h_{L_t})$$

and

$$\mathcal{O}_{X_t}(K_{X_t} + A|_{X_t} + L_t) \otimes \mathcal{I}(h_{L_t})$$

are globally generated. Let h_A be a C^∞ -hermitian metric on A such that Θ_{h_A} is a Kähler form on X . We set

$$h_0 := h_A$$

and

$$h_{0,t} := h_A|_{X_t} \quad (t \in \Delta).$$

Let us fix an arbitrary $t \in \Delta$. As in Section 3, inductively we shall define the sequences of singular hermitian metrics $\{h_m\}$ on X and $\{h_{m,t}\}$ on $X_t (t \in \Delta)$. In this case X is noncompact, but using the sections which restricts to $V_{m,t}$ above and the normalization constants $m^{\nu+1} (m \geq 1)$ (here we use the Cauchy estimates on the disk of radius $O(1/\sqrt{m})$), the construction works as in Section 3. However we should note that we do not know the pseudoeffectivity of K_X or $K_{X_t} (t \in \Delta^*)$ apriori. Hence at this stage h_m and $h_{m,t} (t \in \Delta^*)$ are **not well defined** for $m \geq 2$.

But by the L^2 -extension theorem (Theorem 2.4 or [8]) (as in the proof of Lemma 4.1), we have the following lemma.

LEMMA 4.1. *Let t be a point on Δ . Suppose that $h_{m-1}, h_{m-1,t}$ have been defined and*

$$\mathcal{I}(h_{m-1,t}) \subseteq \mathcal{I}(h_{m-1} |_{X_t})$$

holds on X_t . Then every element of

$$H^0(X_t, \mathcal{O}_{X_t}(K_{X_t} + A |_{X_t} + (m - 1)K_{X_t}) \otimes \mathcal{I}(h_{m-1,t}))$$

extends to an element of

$$H^0(X, \mathcal{O}_X(K_X + A + (m - 1)K_X) \otimes \mathcal{I}(h_{m-1})).$$

Proof. We note that since Theorem 2.4 is stated for Stein manifolds, we cannot apply it directly. Let U be a Zariski open Stein subset of X such that $X_t \cap U$ is nonempty and $K_X |_U$ is holomorphically trivial. Then for every element σ_t of

$$H^0(X_t, \mathcal{O}_{X_t}(K_{X_t} + A |_{X_t} + (m - 1)K_{X_t}) \otimes \mathcal{I}(h_{m-1,t})),$$

$\sigma_t |_{U \cap X_t}$ extends to an element σ of

$$H^0(U, \mathcal{O}_X(K_X + A + (m - 1)K_X) \otimes \mathcal{I}(h_{m-1})).$$

with the L^2 -condition

$$\int_U |h_{m-1} | \sigma|^2 < \infty.$$

But this L^2 -condition implies that σ extends to a section of

$$H^0(X, \mathcal{O}_X(K_X + A + (m - 1)K_X) \otimes \mathcal{I}(h_{m-1})).$$

□

Since K_{X_0} is pseudoeffective, using Lemma 4.1, we have that h_m is well defined for every $m \geq 0$ and the inclusion $\mathcal{I}(h_{m,0}) \subset \mathcal{I}(h_m |_{X_0})$ holds for every $m \geq 0$ inductively. Hence for every $m \geq 1$ we have a proper Zariski closed subset S_m in Δ such that for every $t \in \Delta - S_m$, $h_{m,t}$ is well defined. In particular K_{X_t} is pseudoeffective for every $t \in \Delta - \cup_{m \geq 1} S_m$. This implies, by Grauert’s coherency theorem that K_{X_t} is pseudoeffective for every $t \in \Delta$. Then using Lemma 4.1 and Lemma 3.1, by induction on m , we have the following lemma.

LEMMA 4.2. $h_m, h_{m,t} (t \in \Delta)$ are well defined for every $m \geq 0$ and $t \in \Delta$. And

$$\mathcal{I}(h_{m,t}) \subseteq \mathcal{I}(h_m |_{X_t})$$

holds for every $m \geq 0$ and $t \in \Delta$.

We define

$$h_\infty := \text{the lowersemicontinuous envelope of } \liminf_{m \rightarrow \infty} \sqrt[m]{h_m}$$

and

$$h_{\infty,t} := \text{the lowersemicontinuous envelope of } \liminf_{m \rightarrow \infty} \sqrt[m]{h_{m,t}}.$$

Although X is noncompact, the argument in Section 3 is still valid. In fact since X admits a continuous plurisubharmonic exhaustion function, Lemma 3.2 holds in this case, if we restrict the family to a relatively compact subset of Δ . Also Lemma 3.3 is valid on every relatively compact subset of X , since the proof is local. Hence $h_\infty, h_{\infty,t}$ are well defined AZD's on K_X and K_{X_t} respectively.

Again by the L^2 -extension theorem (Theorem 2.4), as Lemma 3.2, we have the following lemma.

LEMMA 4.3. For every t , there exists a positive constant C such that

$$h_m |_{X_t} \leq C^m \cdot h_{m,t}$$

holds for every $m \geq 0$. In particular

$$h_\infty |_{X_t} \leq C \cdot h_{\infty,t}$$

holds.

Here we have used the fact that for every

$$\sigma_t \in A^2(X_t, K_{X_t} \otimes A \otimes K_X^{\otimes(m-1)}, h_{m-1} |_{X_t}),$$

there exists an element of

$$\sigma \in A^2(X, A \otimes K_X^{\otimes m}, h_{m-1})$$

such that $\sigma |_{X_t} = \sigma_t$ and

$$\| \sigma \| \leq C_t \| \sigma_t \|,$$

where C_t is a positive constant depending only on t (if $t = 0$, we may take C_0 to be $2\sqrt{1620\pi}$ by Theorem 2.4). Here we note that the L^2 -spaces above are determined without specifying volume forms.

Now by Lemma 4.3 and Theorem 3.1, we see that $h_\infty|_{X_t}$ is an AZD of K_{X_t} . Then by the L^2 -extension theorem (Theorem 2.4, see also the proof of Lemma 4.1), we see that for every $m \geq 1$, every element of

$$H^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}) \otimes \mathcal{I}(h_\infty^{m-1}|_{X_t})) \simeq H^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$$

extends to an element of

$$H^0(X, \mathcal{O}_X(K_X + (m - 1)K_X) \otimes \mathcal{I}(h_\infty^{m-1})).$$

Hence we see that $P_m(X_t)$ is lowersemicontinuous. By the upper semicontinuity theorem for cohomologies, we see that $P_m(X_t)$ is independent of $t \in \Delta$. This completes the proof of Theorem 1.1.

Theorem 1.2 has already been proved in the course of the above proof.

§5. Further generalization

The proof of the following generalization of Theorem 1.1 is completely parallel to that of Theorem 1.1 (see the assumption on Y in Theorem 2.4).

THEOREM 5.1. *Let $\pi : X \rightarrow \Delta$ be a flat projective family. Suppose that π is semistable and $X_0 = \sum_{i \in I} \Gamma_i$ is the only singular fiber. Then*

$$\sum_{i \in I} P_m(\Gamma_i) \leq P_m(X_t)$$

holds for every $m \geq 1$ and $t \in \Delta^$.*

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Note added in proofs

For a generalization of Theorem 1.1, see the author's preprint: "Subadjunction theorem for pluricanonical divisors", math.AG/0111311.

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