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CARTESIAN NETS AND GROUPOIDS

BY
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Introduction. Aczel has conjectured, [1, p. 448], the possibility of developing a net theory for structures more general than quasigroups. Steps in this direction have been taken by Havel who considers nets associated with multigroupoids [2]. The work presented here introduces a generalization of 3-nets and their algebraization which is wide enough to encompass most algebraic structures based on a single binary operation.

The main theorems are concerned primarily with groupoids and the nets corresponding to them when under the constraints of the Thomsen and Reidemeister closure conditions. A net is used as a "geometric model" of a class of isotopic binary systems and the closure conditions may be formulated either "geometrically", i.e., in terms of the net, or algebraically. This makes it possible to develop theorems in the intuitive "geometric model" and transform them into purely algebraic terms.

Notation and Definitions.

DEFINITION. A *net* is a set S which is partitioned into four subsets X, Y, Z (called line sets) and P (called the point set) together with a binary relation I. The members of the line sets are called lines and the members of the point set are called points.

The conditions N1–N6 are placed on I and S;

N1: If lIp then one and only one of l and p is a point.

N2: If l is a line then there exists a point p such that lIp.

N3: If p is a point then there exist lines $x \in X$, $y \in Y$, $z \in Z$ such that xIp, yIp, zIp.

A useful notation is the following:

$|xyz| \Leftrightarrow$ lines x, y, z are from different line sets and there exists a point p such that xIp, yIp, zIp. (Read $|xyz|$ as "the lines x, y and z are concurrent".)

N4: If l_1 and l_2 are members of the same line set and l_1Ip, l_2Ip, then $l_1=l_2$.

N5: If l_1, l_2 belong to the same line set, and $|l_1bc| \Leftrightarrow |l_2bc|$ for all b, c concurrent with l_1 or l_2, implies $l_1=l_2$.

N6: If p_1, p_2 are points and x, y, z are from different line sets, xIp_1, yIp_1, zIp_1, xIp_2, yIp_2, $zIp_2 \Rightarrow p_1=p_2$.

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Nets will be denoted by quintuplets of the form (X, Y, Z, I, P) or triples (X, Y, Z). Where a net is not specifically designated by a quintuplet or a triple it can be assumed to be of the form (X, Y, Z, I, P).

J will be used to denote an indexing set.

DEFINITION. A *net isomorphism* or simply an *isomorphism*, H, from

$$N_1 = (X_1, Y_1, Z_1, I_1, P_1) \quad \text{to} \quad N_2 = (X_2, Y_2, Z_2, I_2, P_2)$$

is a bijection

$$H: X_1 \cup Y_1 \cup Z_1 \cup P_1 \to X_2 \cup Y_2 \cup Z_2 \cup P_2$$

such that

$$H(S_1) = S_2$$

where S is replaceable by X, Y, Z, or P. Also,

$$lI_1p \Rightarrow H(l)I_2H(p),$$

and

$mI_2q \Rightarrow$ there exist a line l from N_1 and a point p from P_1 such that,

$$m = H(l), \qquad q = H(p) \quad \text{and} \quad lI_1p.$$

REMARK. Isomorphy between nets is an equivalence relation.

DEFINITION. A *binary system*,

$$\cdot : S_1 \times S_2 \to B(S_3),$$

is a binary operation on part of the cartesian product of two sets S_1, S_2, into a set of subsets, $B(S_3)$, of a third set S_3.

The operation on the ordered pair (s_1, s_2) giving the set $\{s_j\}$, $s_1 \in S_1$, $s_2 \in S_2$, $s_j \in S_3, j \in J$, is denoted by

$$s_1 \cdot s_2 = \{s_j\}.$$

A method of associating a net with a binary system is given below. This method requires that additional technical restrictions be included in the definition of binary system. These are that the domain of the operation is always taken as being nonvoid and the binary system has the properties:

1(a) For all $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 \cdot s_2$ is defined.

1(b) If $t_1 \in S_1$, $s_2 \in S_2$ are such that whenever $t_1 \cdot s_2$, $s_1 \cdot s_2$ are defined $t_1 \cdot s_2 = s_1 \cdot s_2$, $s_2 \in S_2$, then $t_1 = s_1$.

2(a) For all $s_2 \in S_2$ there exists $s_1 \in S_1$ such that $s_1 \cdot s_2$ is defined.

2(b) If $t_2 \in S_2$, $s_2 \in S_2$ are such that whenever $s_1 \cdot t_2$, $s_1 \cdot s_2$ are defined, $s_1 \cdot t_2 = s_1 \cdot s_2$, $s_1 \in S_1$, then $t_2 = s_2$.

3(a) For all $s_3 \in S_3$ there exists $s_1 \in S_1$, $s_2 \in S_2$ such that $s_3 \in s_1 \cdot s_2$.

3(b) If $t_3 \in S_3$, $s_3 \in S_3$ are such that whenever $s_3 \in s_1 \cdot s_2$, also $t_3 \in s_1 \cdot s_2$, and whenever $t_3 \in t_1 \cdot t_2$ also $s_3 \in t_1 \cdot t_2$, then $s_3 = t_3$.

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The set $\{s_j\}$ is called a product of s_1 and s_2. If all the defined products are singleton sets, then the set of an element is identified with the element. Such a binary system is called a halfgroupoid.

A halfgroupoid is called a groupoid if $S_1=S_2=S_3=S$, and if the domain of the binary operation is $S\times S$.

Binary systems will be denoted by quadruplets $(S_1, S_2, S_3, \cdot)$. Groupoids will be denoted by pairs $(S, \circ)$.

DEFINITION. A binary system $(S_1^1, S_2^1, S_3^1, \circ)$ is said to be *isotopic* with a binary system $(S_1^2, S_2^2, S_3^2, \cdot)$ if there exist bijections

$$f:S_1^1 \to S_1^2, \qquad g:S_2^1 \to S_2^2, \qquad h:S_3^1 \to S_3^2$$

such that

$$h(\{s_1 \circ s_2\}) = f(s_1)\cdot g(s_2), \qquad s_1 \in S_1^1, \qquad s_2 \in S_2^1, \qquad s_1 \circ s_2 \subset S_3^1.$$

Isomorphic nets and isotopic binary systems. We define a binary system on the net $N=(X, Y, Z, I, P)$ of the form

$$\cdot : X\times Y \to B(Z)$$

by

$$x\cdot y = \{z_j\} \Leftrightarrow xIp_j, \quad yIp_j, \quad z_jIp_j, \qquad j\in J.$$

We denote such a binary system by $S(N)$.

Conversely, we can derive a net from a given binary system. Suppose $(S_1, S_2, S_3, \cdot)$ is the given binary system. We choose mutually disjoint sets X, Y, Z with the same cardinalities as S_1, S_2, S_3 respectively.

Let $f:S_1\to X$, $g:S_2\to Y$, $h:S_3\to Z$ be bijections. Consider now all distinct statements of the form

$$s_j \in s_1\cdot s_2, \qquad s_1\in S_1, \qquad s_2 \in S_2, \qquad s_j \neq \phi \quad j\in J$$

which are obtainable from the binary system, and let P be an index for the set of all such statements.

We then define the net (X, Y, Z, I, P) by,

$$\begin{aligned} s_1\cdot s_2 = \{s_j\} &\Rightarrow f(s_1)Ip_j \\ &\Rightarrow g(s_2)Ip_j \\ &\Rightarrow h(s_j)Ip_j \qquad j\in J, \end{aligned}$$

where $p_j\in P$ is the index of the statement $s_j \in s_1\cdot s_2$.

Clearly, we do not obtain a unique net this way, but a family of isomorphic nets. Any net obtained this way from a binary system B will be written $N(B)$.

Classical, and Cartesian nets and their algebraic counterparts. Two types of nets to which we give specific consideration are the Classical and Cartesian nets.

DEFINITION. A *Classical net* is a net in which any two members of different line sets are related to a unique point; i.e. if l_1, l_2 are from different line sets then there exists a unique $p \in P$ such that l_1Ip, l_2Ip.

If N is a Classical net then $S(N)$ is isotopic with a quasigroup [1].

DEFINITION. A *Cartesian net* is a net with the property that if $x \in X$, $y \in Y$ then there exists a unique $p \in P$ such that xIp, yIp.

It is easily shown that if G is a groupoid then $N(G)$ is a Cartesian net. However it is not necessarily true that if N is a Cartesian net, $S(N)$ is isotopic to a groupoid, e.g. the net illustrated in Diagram 0. The binary system associated with this net has the multiplication table

	y_1	y_2	y_3	y_4
x_1	z_1	z_2	z_3	z_4
x_2	z_2	z_3	z_4	z_4
x_3	z_3	z_4	z_4	z_5

A finite groupoid must have a square multiplication table.

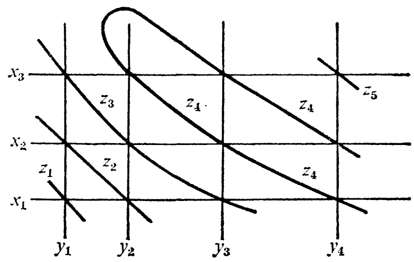

DIAGRAM 0

Closure conditions.

DEFINITION. In a net (X, Y, Z) an array of the form

$$\begin{array}{cc} |x_1y_2z_1| & |x_2y_1z_1| \\ |x_3y_1z_2| & |x_1y_3z_2| \\ \multicolumn{2}{c}{|x_3y_2z_3|} \end{array} \qquad \begin{array}{l} x_i \in X, \quad y_i \in Y \\ z_i \in Z \\ i = 1, 2, 3. \end{array}$$

is called a *T-configuration*. If such an array implies that $|x_2y_3z_3|$ then we say that the *Thomsen condition*, or T, holds in the net.

DEFINITION. The *R-configuration* is

$$\begin{array}{ccc} |x_1y_2z_1| & & |x_1y_4z_3| \\ |x_2y_1z_1| & |x_3y_2z_2| & |x_2y_3z_3| \\ |x_4y_1z_2| & & |x_4y_3z_4| \end{array} \qquad \begin{array}{l} x_i \in X, \quad y_i \in Y \\ z_i \in Z \\ i = 1, 2, 3, 4. \end{array}$$

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This configuration is said to close if $|x_3y_4z_4|$. If all R-configurations close in a net then we say that the *Reidemeister condition*, or R, holds in that net.

DEFINITION. The B_1-*configuration* is

$$\begin{array}{lll} |x_1y_2z_1| & & |x_1y_3z_3| \\ |x_2y_1z_1| & |x_3y_2z_2| & |x_2y_2z_3| \\ |x_4y_1z_2| & & |x_4y_2z_4| \end{array} \qquad \begin{array}{l} x_i \in X, \quad y_i \in Y, \quad z_i \in Z. \\ i = 1, 2, 3, 4. \end{array}$$

This configuration is said to close if $|x_3y_3z_4|$. The B_1 *condition*, or B_1, is said to hold in a net if all B_1-configurations close.

DEFINITION. The B_2-*configuration* is

$$\begin{array}{lll} |x_1y_2z_1| & & |x_1y_4z_3| \\ |x_2y_1z_1| & |x_2y_2z_2| & |x_2y_3z_3| \\ |x_3y_1z_2| & & |x_3y_3z_4|. \end{array}$$

This configuration is said to close if $|x_2y_4z_4|$ and the B_2 *condition*, or B_2, is said to hold if all such configurations close.

We conclude this section with the introduction of two notations.

$\|l_1l_2\| \Leftrightarrow l_1, l_2$ are from different line sets and there exists a point p such that l_1Ip, l_2Ip, (read $\|l_1l_2\|$ as "the lines l_1 and l_2 intersect).

$L(l_1, l_2)$ is the set of all lines l such that $|l\, l_1\, l_2|$.

Theorems. Theorem 1 is a summary of several theorems, the proofs of which may be found in [1]. Theorem 2 is the summary of several theorems developed in [4].

THEOREM 1. *If T, R, B_1, or B_2 (respectively) hold in a Classical net N, then $S(N)$ is isotopic with an abelian group, group, right Bol loop or left Bol loop (respectively).*

The Bol identities are,

$$\text{right Bol}: [(xy)z]y = x[(yz)y]$$

$$\text{left Bol}: y[z(xy)] = [y(zy)]x$$

DEFINITION. A loop which is both left and right Bol is called a *Moufang* loop.

THEOREM 2. *Let $N=(X, Y, Z)$ be a Cartesian net in which T, R, B_1 or B_2 hold. If there exists $x_0 \in X$ such that $\{L(x_0, y) \mid y \in Y\}=Z$ and if there exists $z_0 \in Z$ such that $\|yz_0\|$, for all $y \in Y$, then N is a Classical net.*

If we combine Theorems 1 and 2, we get,

THEOREM 3. *Let N be a Cartesian net under the conditions of Theorem 2, then $S(N)$ is isotopic to an abelian group, group, right Bol loop or left Bol loop (respectively).*

Theorem 3 is a mixture of "algebraic" and "geometric" forms. It is possible to give a version of Theorem 3 which is completely algebraic. This is done by noting that when dealing with groupoids the closure conditions may be defined without recourse to nets.

The Closure conditions in a groupoid $(G, \cdot)$

DEFINITION. The *Thomsen condition*, or T, holds in $(G, \cdot)$ if for all x_1, x_2, x_3, y_1 $y_2, y_3 \in G$

$$x_1 \cdot y_2 = x_2 \cdot y_1, \qquad x_1 \cdot y_3 = x_3 \cdot y_1$$

implies that

$$x_2 \cdot y_3 = x_3 \cdot y_2.$$

DEFINITION. The *Reidemeister condition*, or R holds in $(G, \cdot)$ if for all $x_1, x_2, x_3,$ $x_4, y_1, y_2, y_3, y_4 \in G$

$$x_1 \cdot y_2 = x_2 \cdot y_1, \qquad x_1 \cdot y_4 = x_2 \cdot y_3, \qquad x_3 \cdot y_2 = x_4 \cdot y_1$$

implies that

$$x_3 \cdot y_4 = x_4 \cdot y_3.$$

DEFINITION. The *B_1 condition*, or B_1, holds in $(G, \cdot)$ if for all $x_1, x_2, x_3, x_4, y_1, y_2,$ $y_3 \in G$

$$x_1 \cdot y_2 = x_2 \cdot y_1, \qquad x_2 \cdot y_2 = x_1 \cdot y_3, \qquad x_4 \cdot y_1 = x_3 \cdot y_2$$

implies that

$$x_4 \cdot y_2 = x_3 \cdot y_3.$$

DEFINITION. The *B_2 condition*, or B_2, holds in $(G, \cdot)$ if for all $x_1, x_2, x_3, y_1, y_2,$ $y_3, y_4 \in G$

$$x_1 \cdot y_2 = x_2 \cdot y_1, \qquad x_2 \cdot y_2 = x_3 \cdot y_1, \qquad x_1 \cdot y_4 = x_2 \cdot y_3$$

implies that

$$x_2 \cdot y_4 = x_3 \cdot y_3.$$

It is easily checked that these definitions are equivalent to the original definitions provided we restrict ourselves to nets which give rise to isotopes of groupoids or arise from isotopes of groupoids.

We now have,

THEOREM 4. *Let $(G, \cdot)$ be a groupoid in which T, R, B_1 or B_2 hold (respectively). If there exist $g_1, g_2 \in G$ such that $g_1 \cdot G = G$ and the equation $x \cdot y = g_2$ has a solution in x for all $y \in G$ then $(G, \cdot)$ is isotopic to an abelian group, group, right Bol loop or left Bol loop (respectively).*

This is the purely algebraic form of Theorem 3.

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We shall now proceed to develop several theorems about Cartesian nets and the algebraic structures which arise from them, and use these results to give theorems of a purely algebraic nature.

PROPOSITION 1. *Let N be a Cartesian net in which there exist $x_0 \in X$, $y_0 \in Y$ such that for every given $z \in Z$ there are unique $x \in X$, $y \in Y$ which satisfy $L(x_0, z)=y$, $L(y_0, z)=x$, then $S(N)$ is isotopic with a groupoid which possesses a unit element.*

Proof. We construct a groupoid of the required form on the Z line set. Define mappings,

$$f: Z \to X, \qquad g: Z \to Y$$

by

$$f(z) = L(y_0, z) = x \text{ (say)} \qquad z \in Z,$$
$$g(z) = L(x_0, z) = y \text{ (say)} \qquad x \in X, \quad y \in Y.$$

Clearly f and g are bijections.

We define a groupoid $\circ: Z \times Z \to Z$, by

$$z_1 \circ z_2 = f(z_1)g(z_2), \qquad z_1, z_2 \in Z.$$

where $f(z_1)g(z_2)$ is a product in $S(N)$.

This groupoid, $(Z, \circ)$, is an isotope of $S(N)$, and its operation may be described in net terms by (Diagram 1)

$$z_1 \circ z_2 = f(z_1)g(z_2) = L(L(y_0, z_1), L(x_0, z_2)) = L(f(z_1), g(z_2)).$$

If we write $z_0 = L(x_0, y_0)$ then, for all $z_1 \in Z$,

$$z_1 \circ z_0 = L(L(y_0, z_1), L(x_0, z_0)) = L(L(y_0, z_1), y_0) = z_1$$
$$z_0 \circ z_1 = L(L(y_0, z_0), L(x_0, z_1)) = L(x_0, L(x_0, z_1)) = z_1$$

i.e. z_0 is the unit element for $(Z, \circ)$.

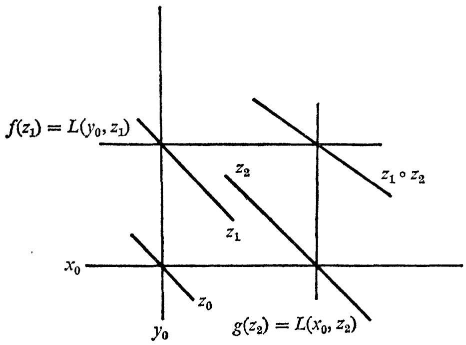

DIAGRAM 1

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THEOREM 5. *Let $N=(X, Y, Z)$ be a Cartesian net with $x_0 \in X$, $y_0 \in Y$ such that $\{L(x_0, y) \mid y \in Y\}=\{L(x, y_0) \mid x \in X\}=Z$. If T holds in N then $S(N)$ is isotopic with a commutative semigroup S. Moreover S is embeddable in a group.*

In proving Theorem 5 we require a result which was obtained in [4], which we quote here without proof.

LEMMA. 1. *Let $N=(X, Y, Z)$ be a Cartesian net in which there exist $x_0 \in X$, $y_0 \in Y$ such that $\{L(x_0, y) \mid y \in Y\}=\{L(x, y_0) \mid x \in X\}=Z$, and in which T holds. If $L(y_0, x_1)=L(y_0, x_2)$ $x_1, x_2 \in X$ then $x_1=x_2$ and if $L(x_0, y_1)=L(x_0, y_2)$, $y_1, y_2 \in Y$, then $y_1=y_2$.*

Proof of Theorem 5. Lemma 1 shows that x_0, y_0 fulfill the conditions of the Proposition 1, so we can construct $(Z, \circ)$ in the manner of Proposition 1. For the remainder of this proof $(Z, \circ)$ will refer to the groupoid specifically constructed for this proof.

First we show that $(Z, \circ)$ is commutative. Let $z_1, z_2 \in Z$.

$$z_1 \circ z_2 = f(z_1)g(z_2)$$
$$z_2 \circ z_1 = f(z_2)g(z_1).$$

From the definition of f and g,

$$f(z_1) = L(y_0, z_1) = x_1 \quad \text{(say)}$$
$$f(z_2) = L(y_0, z_2) = x_2 \quad \text{(say)}$$
$$g(z_1) = L(x_0, z_1) = y_1 \quad \text{(say)}$$
$$g(z_2) = L(x_0, z_2) = y_2 \quad \text{(say)}$$

i.e.,

$$L(y_0, x_1) = L(x_0, y_1)$$
$$L(y_0, x_2) = L(x_0, y_2)$$

This represents a T-configuration which closes, (Diagram 2), and

$$f(z_1)g(z_2) = L(L(y_0, z_1), L(x_0, z_2)) = L(x_1, y_2)$$
$$f(z_2)g(z_1) = L(L(y_0, z_2), L(x_0, z_1)) = L(x_2, y_1).$$

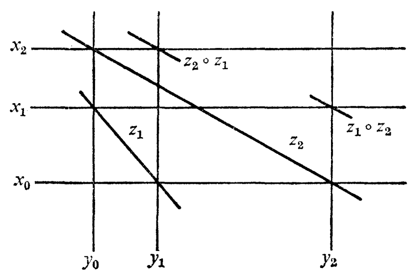

DIAGRAM 2

Thus we conclude that $(Z, \circ)$ is commutative.

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We shall now show that $(Z, \circ)$ is associative. Consider

$$\begin{aligned} z_1 \circ z_2 &= L(y_0, f(z_1 \circ z_2)) \\ z_2 \circ z_1 &= L(f(z_2), g(z_1)) \\ z_3 \circ z_2 &= L(y_0, f(z_3 \circ z_2)) \\ z_2 \circ z_3 &= L(f(z_2), g(z_3)). \end{aligned} \qquad z_1, z_2, z_3 \in Z$$

$(Z, \circ)$ is commutative so

$$L(y_0, f(z_1 \circ z_2)) = L(f(z_2), g(z_1))$$
$$L(y_0, f(z_3 \circ z_2)) = L(f(z_2), g(z_3)).$$

This represents a T-configuration which closes (Diagram 3) to give

$$L(f(z_1 \circ z_2), g(z_3)) = L(f(z_3 \circ z_2), g(z_1))$$

i.e.

$$(z_1 \circ z_2) \circ z_3 = (z_3 \circ z_2) \circ z_1.$$

Employing the commutativity of $(Z, \circ)$ we find

$$(z_1 \circ z_2) \circ z_3 = z_1 \circ (z_2 \circ z_3).$$

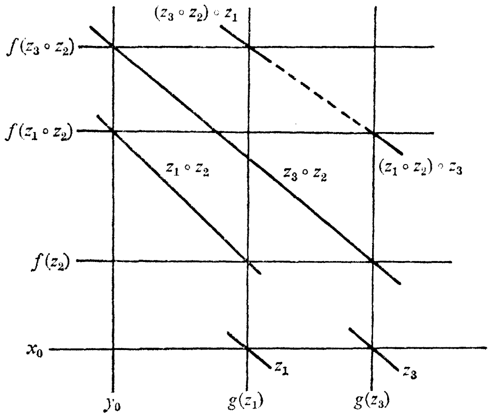

DIAGRAM 3

In order to complete the proof we have to embed $(Z, \circ)$ in a group. This is always possible for a commutative semigroup provided it is cancellative; we will show that $(Z, \circ)$ is cancellative. Suppose,

$$z_1 \circ z_2 = z_1 \circ z_3 \qquad z_1, z_2, z_3 \in Z$$

In net terms this becomes

$$L(f(z_1), g(z_2)) = L(f(z_1), g(z_2)) = z_1 \circ z_2$$

and also

$$L(x_0, g(z_1 \circ z_2)) = z_1 \circ z_2.$$

The equations

$$L(f(z_1), g(z_2)) = L(x_0, g(z_1 \circ z_2))$$

$$L(f(z_1), g(z_3)) = L(x_0, g(z_1 \circ z_2))$$

represent a T-configuration in which x_0 takes the part of both "x_2" and "x_3" in the usual configuration. This configuration closes with,

$$L(x_0, g(z_2)) = L(x_0, g(z_3)) \quad \text{(Diagram 4)}$$

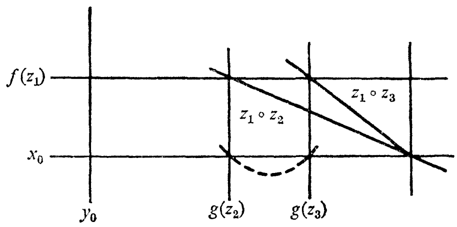

DIAGRAM 4

Hence, $g(z_2)=g(z_3)$, as a consequence of Lemma 1, and in particular, as g is bijective,

$$z_2 = z_3.$$

$(Z, \circ)$ is, therefore, left cancellative. The symmetry of the Thomsen condition with respect to the X and Y line sets decrees that the groupoid is also right cancellative.

Thus $(Z, \circ)$ is a cancellative commutative semigroup and as such may be embedded in a group.

THEOREM. 6. *Let N be a Cartesian net with $x_0 \in X$, $y_0 \in Y$ as in Theorem 5. If R holds in N then $S(N)$ is isotopic with a semigroup S. Moreover S can be represented in the form of two non-intersecting subsemigroups S_1, S_2 such that*

$$S = S_1 \cup S_2$$

where S_1 is a group and S_2 is a two sided ideal.

The following pair of lemmas which were proved in [4] will be used in the proof of Theorem 6.

LEMMA 2. *If B_1 holds in a net (X, Y, Z) then $|xy_1z|$, $|xy_2z|$ implies $y_1=y_2$, for all $y_1, y_2 \in Y$, $x \in X$, $z \in Z$.*

LEMMA 3. *If B_2 holds in a net (X, Y, Z) then $|x_1yz|$, $|x_2yz|$ implies $x_1=x_2$, for all $x_1, x_2 \in X$, $y \in Y$, $z \in Z$.*

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Proof of Theorem 6. If the condition R holds in a net, then B_1 and B_2 also hold in that net. Consequently, by Lemmas 2 and 3, $S(N)$ is cancellative, and the isotope $(Z, \circ)$ of $S(N)$ is also cancellative. A cancellative semigroup with a unit element can be represented as the union of two subsemigroups with the properties given in the statement of the theorem, so it is sufficient to show that $(Z, \circ)$ is a semigroup [3, p. 261].

Consider

$$z_2 = L(f(z_2), y_0) = L(x_0, g(z_2)) \qquad z_1, z_2, z_3 \in Z$$
$$z_1 \circ z_2 = L(f(z_1 \circ z_2), y_0) = L(f(z_1), g(z_2))$$
$$z_2 \circ z_3 = L(x_0, g(z_2 \circ z_3)) = L(f(z_2), g(z_3)).$$

This represents an R-configuration (Diagram 5) with closure

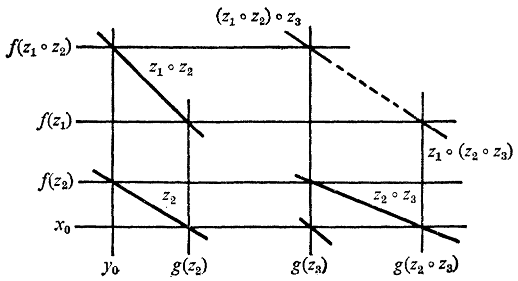

DIAGRAM 5

$$L(f(z_1 \circ z_2), g(z_3)) = L(f(z_1), g(z_2 \circ z_3))$$

i.e.

$$(z_1 \circ z_2) \circ z_3 = z_1 \circ (z_2 \circ z_3).$$

The associativity identity, $x(yz)=(xy)z$, implies a set of identities known as the alternativity identities.

These are,

$$\text{right alternative:} \quad x(yy) = (xy)y$$
$$\text{left alternative:} \quad x(xy) = (xx)y.$$

The right (left) alternative identity is a special case of the right (left) Bol identity.

DEFINITION. A *subset* $N_1=(X_1, Y_1, Z_1, I_1, P_1)$ of a net $N=(X, Y, Z, I, P)$ is any net for which

$$X_1 \subset X, \qquad Y_1 \subset Y, \qquad Z_1 \subset Z, \qquad P_1 \subset P,$$

and I_1 is the restriction of I to the union of X_1, Y_1, Z_1 and P_1.

THEOREM 7. *Let N be a Cartesian net in which there exist $x_0 \in X$, $y_0 \in Y$ such that $\{L(x_0, y) \mid y \in Y\}=\{L(x, y_0) \mid x \in X\}=Z$.*

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If B_1 and B_2 hold in N then $S(N)$ is isotopic to a groupoid G_1, with a unit element, in which the left and right alternative identities hold. Moreover,

$$G_1 = M \cup G_2 \qquad M \cap G_2 = \phi$$

where M is a Moufang loop G_2 is a two sided ideal.

Proof. We consider the groupoid $(Z, \circ)$ set up in the usual way, and we show that $(Z_1, \circ)$, $(Z_2, \circ)$ where

$$Z_1 = \{z \mid |zxy|, x \in X_1, y \in Y_1\}$$
$$X_1 = \{x \mid L(x, y) = z_0, y \in Y\}, \qquad z_0 = L(x_0, y_0)$$
$$Y_1 = \{y \mid L(x, y) = z_0, x \in X\},$$

and

$$Z_2 = Z - Z_1,$$

have the properties required by the theorem.

First we show that the subnet $N_1 = (X_1, Y_1, Z_1)$ of N is a Classical net.

Let $x_1 \in X$, $y_1 \in Y$ be such that $L(x_1, y_1) = z_0$, $x_1 \neq x_0$. (If such x_1 does not exist then the groupoid $(Z_1, \circ)$ becomes a one element groupoid which is trivially a Moufang loop). Now there exists $z_1 \in Z$ such that $z_1 = L(y_0, x_1)$ and there exists $y_2 = L(x_0, z_1)$. If $x_2 \in X$ then there exists $x_3 \in X$ such that

$$L(y_2, x_2) = L(y_0, x_3) \quad \text{(Diagram 6)}.$$

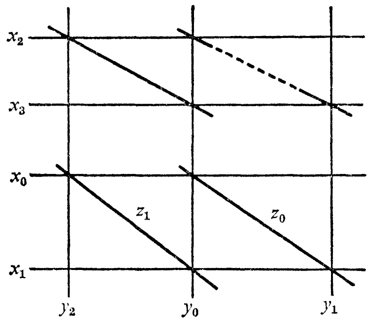

DIAGRAM 6

B_1 holds in the net so

$$L(x_2, y_0) = L(x_3, y_1).$$

This holds for all $x_2 \in X$, hence

$$\{L(x, y_0) \mid x \in X\} \subset \{L(x, y_1) \mid x \in X\}.$$

As the left hand side of the above expression is equal to Z,

$$\{L(x, y_1) \mid x \in X\} = Z.$$

The roles of X and Y may be interchanged, as B_2 also holds, to give

$$\{L(x_1, y) \mid y \in Y\} = \{L(x, y_1) \mid x \in X\} = Z.$$

We now show that if $z_1 \in Z$ and $x_1 \in X$ then $y_1 = L(x_1, z_1)$ implies that $y_1 \in Y_1$. The definition of Z_1 ensures that if $z_1 \in Z_1$ then

$$z_1 = L(x_0', y_0') \qquad x_0' \in X_1, \quad y_0' \in Y_1.$$

x_0' and y_0' respectively have all the properties of x_0 and y_0 respectively so by the first part of the proof, if $y_1 = L(x_1, z_1)$ then

$$\{L(y_1, x) \mid x \in X\} = Z,$$

and in particular there exists $x_2 \in X$ such that

$$L(y_1, x_2) = z_0.$$

Hence

$$y_1 \in Y_1.$$

In a similar manner we can show that

$$x_1 = L(y_1, z_1), \qquad y_1 \in Y_1, \quad z_1 \in Z_1$$

implies $x_1 \in X_1$.

We have proved that for $x \in X_1$, $y \in Y_1$, $z \in Z_1$ $\|x_1y_1\|$, $\|y_1z_1\|$ and $\|z_1x_1\|$. The Cartesian property of the net N and the fact that Lemmas 2 and 3 are applicable in this case ensure that N_1 is a Classical net. Consequently $(Z_1, \circ)$ is a loop which is left and right Bol, i.e., a Moufang loop.

In order to show that $(Z_2, \circ)$ is a two sided ideal we have to prove that

$$z \circ Z \subset Z_2 \qquad z \in Z_2$$
$$Z \circ z \subset Z_2$$

This is equivalent to

$$\{L(L(z, y_0), y) \mid y \in Y\} \subset Z_2 \qquad z \in Z_2$$
$$\{L(L(z, x_0), x) \mid x \in X\} \subset Z_2.$$

Suppose

$$L(L(z, y_0), y_1) = z_1 \in Z_1 \qquad z \in Z$$

then as $x_0 \in X_1$ and as there exists $y_2 \in Y$ such that,

$$L(x_0, y_2) = z_1$$

it follows that

$$y_2 \in Y_1.$$

However x_0, y_2, z_1 fulfill the conditions of the first part of the proof, hence, because $\|L(y_0, z)z_1\|$,

$$L(y_0, z) \in X_1.$$

Consequently, as $y_0 \in Y_1$,

$$L(L(y_0, z), y_0) \in Z_1$$

i.e.

$$z \in Z_1.$$

From this we deduce that

$$\{L(L(z, y_0), y) \mid y \in Y\} \subset Z_2 \qquad z \in Z_2.$$

Similarly it may be shown that

$$\{L(L(z, x_0), x) \mid x \in X\} \subset Z_2.$$

It only remains to confirm that the left and right alternative identities hold in (Z, ∘). This follows immediately from Lemmas 4 and 5 below.

LEMMA 4. *Let N be a Cartesian net in which there exist $x_0 \in X$, $y_0 \in Y$ such that given $z \in Z$ there are unique $x \in X$, $y \in Y$ which satisfy $L(x_0, z)=y$, $L(y_0, z)=x$. If B_1 holds in N then $S(N)$ is isotopic to a groupoid with a unit element in which the right alternative identity holds.*

LEMMA 5. *Let N be a Cartesian net in which there exist $x_0 \in X$, $y_0 \in Y$ as in Lemma 4. If B_2 holds in N then $S(N)$ is isotopic to a groupoid with a unit element in which the left alternative identity holds.*

The proofs of these two lemmas are similar, so we give only the proof of Lemma 4.

Proof of Lemma 4. Construct $(Z, \circ)$ in the usual manner. Let $z_1, z_2 \in Z$. There exist $x_1, x_2, x_3 \in X$, $y_1, y_2 \in Y$ such that

$$L(y_0, x_1) = L(x_0, y_1) = z_2$$
$$L(x_1, y_1) = L(x_0, y_2) = z_2 \circ z_2$$
$$L(x_2, y_0) = z_1$$
$$L(x_2, y_1) = L(x_3, y_0) = z_1 \circ z_2.$$

These represent a B_1 configuration (Diagram 7), which closes to give

$$L(x_3, y_1) = L(x_2, y_2).$$

However

$$L(x_3, y_1) = (z_1 \circ z_2) \circ z_2$$

and

$$L(x_2, y_2) = z_1 \circ (z_2 \circ z_2).$$

Hence left alternatively holds in $(Z, \circ)$.

DEFINITION. The *union*, $\bigcup_j N_j, j \in J$, of a set of subnets $N_j=(X_j, Y_j, Z_j, I_j, P_j)$ of

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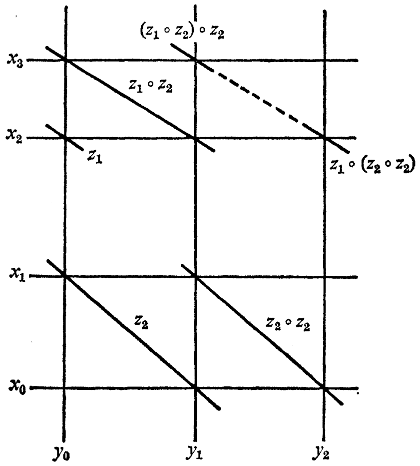

DIAGRAM 7

a net $N=(X, Y, Z, I, P)$ is that net

$$\bigcup_j N_j = \left(\bigcup_j X_j, \bigcup_j Y_j, \bigcup_j Z_j, I', \bigcup_j P_j\right)$$

where I' is the restriction of I to the union of all lines and points in $\bigcup_j N_j$.

THEOREM 8. *Let N be a Cartesian net in which there exists $x_0 \in X$, $y_0 \in Y$ such that $\{L(x_0, y) \mid y \in Y\}=\{L(y_0, x) \mid x \in X\}=Z$, and in which $L(x_0, y_0)=L(x, y_0) \Rightarrow x_0=x$.*

If B_1 holds in N then N is the union of subnets $N_j=(X_j, Y, Z)$ $j \in J$, and the $S(N_j)$ $j \in J$ are isotopic to groupoids which possess a unit element and in which the right alternative identity holds.

THEOREM 9. *Let N be a Cartesian net in which there exists $x_0 \in X$, $y_0 \in Y$ such that $\{L(x_0, y) \mid y \in Y\}=\{L(y_0, x) \mid x \in X\}=Z$, and in which $L(x_0, y_0)=L(x_0, y) \Rightarrow y=y_0$.*

If B_2 holds in N then N is the union of subnets $N_j=(X, Y_j, Z)j \in J$, and the $S(N_j)j \in J$ are isotopic to groupoids which possess a unit element and in which the left alternative identity holds.

Again we prove only one of the theorems.

Proof of Theorem 8. Lemma 2 ensures that given $z \in Z$ there is a unique $y \in Y$ such that $L(x_0, y)=z$.

Consider the subsets $P(z)$ of X given by

$$P(z) = L(z, y_0), \qquad z \in Z.$$

Define $\{X_j\}_{j\in J}$ to be the set of sets of X-lines such that each set X_j contains one and only one member of $P(z)$, $z\in Z$.

The nets (X_j, Y, Z) are Cartesian nets, each of which fulfills the conditions of Lemma 4.

The union of the nets is clearly equal to N.

We present now purely algebraic forms of Theorem 5, 6 and 7.

In the following theorems, G is a groupoid $(G, \cdot)$ and there exist elements g_1, $g_2\in G$ such that $g_1\cdot G=G\cdot g_2=G$.

THEOREM 10. *If T holds in G then G is isotopic to a commutative semigroup which is embeddable in a group.*

THEOREM 11. *If R holds in G then G is isotopic to a semigroup S which can be represented in the form of two-intersecting subsemigroups S_1, S_2 such that $S=S_1\cup S_2$, where S_1 is a group and S_2 is a two sided ideal.*

THEOREM 12. *If B_1 and B_2 hold in G then G is isotopic to a groupoid G_1, with unit, in which the left and right alternative identities hold and also, $G_1=M\cup G_2$, $M\cap G_2=\varnothing$ where M is a Moufang loop and G_2 a two sided ideal.*

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