Proceedings of the Edinburgh Mathematical Society (1994) 37, 423–444 ©

PROPERLY DISCONTINUOUS ACTIONS ON Λ-TREES

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(Received 21st September 1992)

The main result is a theorem giving several possibilities for the action of a 2-generator group acting on a Λ-tree, generalising the result that, if the action is free then the group is either free or free abelian. This involves investigation of several cases in which the action is shown to be properly discontinuous. This leads to a generalisation of results of Culler and Morgan, characterising abelian, dihedral and irreducible actions on $\mathbb{R}$-trees, to arbitrary Λ-trees.

1991 *Mathematics subject classification*: 20E08.

1.

Harrison's Theorem for Λ-trees (Theorem 1 in [3]) states that if a group G acts freely on a Λ-tree, then every 2-generator subgroup is either free of rank 2 or abelian. In [5] there is a more general result, describing various possibilities for an action of a free group of rank 2 on a Λ-tree, but a complete proof is not given. In this paper we give a proof of a version of Steiner's theorem, dropping the assumption that the group acting on the tree is free (which makes little difference). Before stating the result, we need the following definition, which appears in [5].

Definition. Let G be a group acting on a Λ-tree (X,d) (so d is the metric on X). The action is called *free and properly discontinuous* if for all $x\in X$, the set $\{d(x,gx)\,|\,g\in G, g\neq 1\}$ has a positive lower bound in Λ.

Essentially, the main result is as follows.

Main Theorem. *Let $G=\langle s,t\rangle$ act on a Λ-tree X with hyperbolic length function ℓ. Then one of the following possibilities occurs*

(i) *G acts freely and properly discontinuously on X, and s, t freely generate G.*

(ii) *By finite succession of elementary Nielsen transformations, s and t can be transformed into new generators u and v such that at least one of $\ell(u)$, $\ell(v)$ is zero.*

(iii) *$\ell(s)>0$, $\ell(t)>0$ but $sts^{-1}t^{-1}$ has a fixed point.*

(iv) *By a finite succession of elementary Nielsen transformations, s and t can be transformed into new generators u and v such that $\ell(u)>0$, $\ell(v)>0$ but $uvu^{-1}v$ has a fixed point.*

In fact, more detail can be given in Case (iv), and Harrison's Theorem can then be easily deduced. Our method of proof is similar to the proof of Theorem 1 in [3], with some modifications in one case which are suggested by [5]. It is based on a series of six propositions, corresponding to Lemmas 2–6 in [3]. In Section 2 we prove the first two of these propositions, which we regard as the good cases. The remaining four, which we view as pathological cases arising when Λ is non-archimedean, are proved in Section 3. The proof of the main result is in Section 4, and an application of Proposition 1 to the classification of actions on trees is given. The proofs in Section 2 and Section 3 are regrettably rather technical, in that they involve consideration of many cases, although they are based only on the simple idea of considering the effect of successively applying powers of s and t to points of the tree, where $G=\langle s,t\rangle$ is a group acting on some Λ-tree. To understand the proofs, it is necessary also to understand the proofs of Lemmas 2–6 in [3], as well as some of the basic theory of Λ-trees (see [1]). If s and t are hyperbolic isometries which meet coherently, we shall use L and R to denote the left and right-hand endpoints of $A_s \cap A_t$, when they exist (see the introduction in [3]). As in [1] and [3], $\Delta(s,t)$ will denote the diameter of $A_s \cap A_t$ when this is non-empty, and in the case that $A_s \cap A_t=[L,R]$, we may take $\Delta(s,t)$ to be the length of the segment $[L,R]$.

We shall use the Bridge Proposition ((2.17) in [1]) and the idea of the bridge between two closed subtrees. If A is a closed subtree of a Λ-tree X and $x\in X$, the bridge between $\{x\}$ and A has the form $[x,p]$ for some $p\in A$. We call p the projection of x onto A. It is consistent with the notation of [1] to denote p by $pr_A(x)$, but we shall not use this notation.

Before proceeding we prove two lemmas which are used in the proofs of all the six propositions which follow.

Lemma 1. *Let G be a group acting on a Λ-tree X with metric d, and let X' be a (non-empty) G-invariant subtree. Let $g\in G$. If $x\in X$, then there exists $v\in X'$ such that $d(x,gx)\geqq d(v,gv)$.*

Proof. If g is not an inversion, this follows from 6.6 in [1]. Suppose g is an inversion, so that A_{g^2} is the subtree of fixed points of g^2. Again by 6.6 of [1], we can choose $z\in A_{g^2}\cap X'$. Since $z\in A_{g^2}$, we have $gz\in A_{g^2}$, hence $[z,gz]\subseteq A_{g^2}\cap X'$. Let v be the projection of x onto $[z,gz]$. Since $g[z,gz]=[z,gz]$, gv is the projection of gx onto $[z,gz]$. Also g has no fixed points in X, so $gv\neq v$, hence $[x,gx]=[x,v,gv,gx]$ by the Piecewise Geodesic Proportion [1; 2.14], therefore $d(x,gx)\geqq d(v,gv)$. □

The second lemma is a trivial observation but will be used repeatedly.

Lemma 2. *Let G be a group acting on a Λ-tree X with metric d. Suppose Y is a subset of X with $GY=X$ and suppose that there exists $k\in\Lambda$ such that for all $y\in Y$ and $1\neq g\in G$, $d(y,gy)\geqq k$. Then $d(x,gx)\geqq k$ for all $x\in X$ and $1\neq g\in G$. If $d(y,gy)>k$ for all $y\in Y$ and $1\neq g\in G$, then $d(x,gx)>k$ for all $x\in X$ and $1\neq g\in G$.*

Proof. If $x\in X$ and $1\neq g\in G$, there exists $h\in G$ and $y\in Y$ such that $x=hy$. Then $d(x,gx)=d(hy,ghy)=d(y,(h^{-1}gh)y)\geqq k$, with strict inequality in the last part of the lemma. □

2.

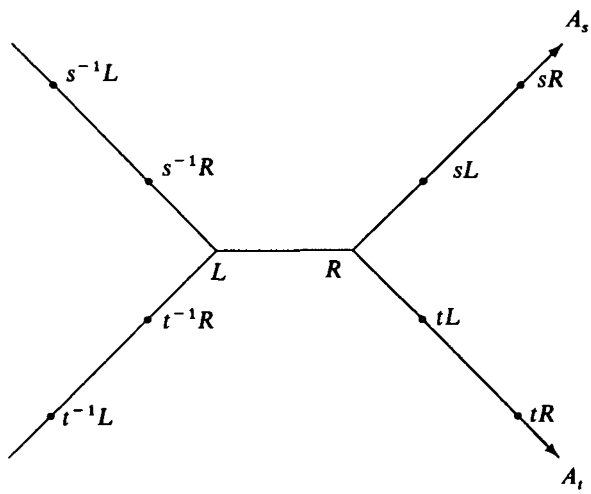
The first proposition is concerned with the situation in Lemma 2 of [3], which is described in the hypotheses of the following lemma. When considering an action on a tree, we shall use ℓ to denote the corresponding hyperbolic length function.

Lemma 3. *Let $G=\langle s,t\rangle$ act on a Λ-tree (X,d), suppose that $A_s \cap A_t \neq \varnothing$ and s, t meet coherently. Let $\Delta=\Delta(s,t)$ and assume $\ell(s)>\Delta$ and $\ell(t)>\Delta$. Put $k=\min\{d(R,sL),d(R,tL)\}=\min\{\ell(s)-\Delta,\ell(t)-\Delta\}$. Let $g\in G$, and write $g=u_1\ldots,u_n$ where $u_1,u_2,\ldots,u_n$ is a freely reduced word in $\{s^{\pm 1},t^{\pm 1}\}$ and $n\geqq 1$. Then*

(a) $d([L,R],g[L,R])\geqq nk$,

(b) *if $n>1$, then $d([R,sL],g[R,sL])\geqq k$ and $d([R,tL],g[R,tL])\geqq k$.*

Proof. If $n=1$ it is easy to see that (a) is true, using the fact that A_s and A_t are linear subtrees, and the situation is illustrated by the following diagram, which is part of the diagram for Case 2 on p. 362 of [1].



We therefore assume $n>1$. By Lemma 2 of [3],

$$[R,gR]=[R,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gR]$$

where Q_i is either L or R for $1\leqq i\leqq n-1$. Again, it is easy to see that either $L\in[R,u_1Q_1]$ or $R\in[L,u_1Q_1]$, and either $L\in[u_n^{-1}Q_{n-1},R]$ or $R\in[u_n^{-1}Q_{n-1},L]$. Thus there are four possibilities:

(i) $[R,gL]=[R,L,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gR,gL]$,

(ii) $[L,gL]=[L,R,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gR,gL]$,

(iii) $[R,gR]=[R,L,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gL,gR]$,

https://doi.org/10.1017/S0013091500018885 Published online by Cambridge University Press

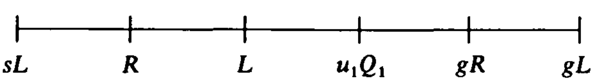
(iv) $[L,gR]=[L,R,u_1Q_1,u_1u_2Q_2,\dots,u_1\dots u_{n-1}Q_{n-1},gL,gR]$.

It is easily checked that the distance between two successive points on the right-hand side in (i)–(iv), other than the first two points or the last two points, is at least k. Therefore $d([L,R],g[L,R])$ is the distance between the second and penultimate points on the right-hand side, and is $\geqq nk$, as required.

Since $[L,sL]=[L,R,sL]$, so that $g[L,sL]=g[L,R,sL]$, it follows that $d([R,sL],g[R,sL])\geqq(n-2)k$ by the triangle inequality, and similarly it follows that $d([R,tL],g[R,tL])\geqq(n-2)k$, so (b) follows if $n\geqq3$, and it remains to prove (b) when $n=2$.

Put $d=d([R,sL],g[R,sL])$. One of the four cases (i)–(iv) above occurs and we consider them in turn.

Case (i). The situation is illustrated by the following diagram.



We leave it to readers to check that either $gsL\in[u_1Q_1,gR]$ or $[gsL,gR]\cap[gR,u_1Q_1]=\{gR\}$, hence $d\geqq d(L,u_1Q_1)\geqq k$.

Case (ii). In this case the diagram is

Now $[R,sL]\cap[R,u_1Q_1]=\{R\}$ unless $u_1=s$, in which case $sL\in[R,u_1Q_1]$, and $g[R,sL]\cap[gR,u_1Q_1]=\{gR\}$ unless $u_2=s^{-1}$, in which case $gsL\in[u_1Q_1,gR]$. Since $d(R,u_1Q_1)\geqq k$ and $d(gR,u_1Q_1)\geqq k$, and both $u_1=s$ and $u_2=s^{-1}$ are impossible (since u_1,u_2 is a reduced word), we see that, in all cases, $d\geqq k$.

In Case (iii) we see in a similar way that $d\geqq d(L,u_1Q_1)+d(u_1Q_1,gL)\geqq2k$, and Case (iv) follows by applying Case (i) with g^{-1} in place of g. The other assertion in (b), that $d([R,tL],g[R,tL])\geqq k$, is proved by interchanging the roles of s and t in this argument. □

Lemma 4. *In the situation of Lemma 3, suppose $1\neq g\in G$ and $x\in X$. Then $d(x,gx)\geqq k$.*

Proof. Let $X'=\bigcup_{g\in G}[R,gR]$. It follows from (2.11)(a)(ii) in [1] that X' is a G-invariant subtree of X. By Lemma 1, we may assume that $x\in X'$. Let $Y=[s^{-1}L,sR]\cup[t^{-1}L,tR]$. Then Y is a subtree of X' (see the diagram on the previous page); note that $[s^{-1}L,sR]\subseteq[s^{-2}R,sR]\subseteq X'$, since $\ell(s)>\Delta$, and similarly with t in place of s, so $Y\subseteq X'$.) By Lemma 2 in [3], if $x\in X'$, $x\in[hQ,huQ']$ for some $h\in G$, where $u\in\{s^{\pm1},t^{\pm1}\}$ and $Q,Q'\in\{L,R\}$. It is easily seen that $[Q,uQ']\subseteq Y$, hence $X'=GY$. Now let $Z=[L,sL]\cup[L,tL]$, a subtree of Y. Again it is easy to see that $Y\subseteq Z\cup\bigcup_{u\in\{s^{\pm1},t^{\pm1}\}}uZ$, hence $X'=GZ$.

Thus by Lemma 2 we may assume that $x\in Z$. Note that

$Z=[L,R]\cup[R,sL]\cup[R,tL]$. If $x\in[L,R]$, then $d(x,gx)\geqq k$ by Lemma 3(a); also, if $n>1$ and $x\in[R,sL]\cup[R,tL]$, then $d(x,gx)\geqq k$ by Lemma 3(b). Suppose $n=1$ and $x\in[R,sL]\cup[R,tL]$.

If $x\in[R,sL]$, there are the following possibilities.

(i) $g=t$. Then $gx\in t[R,sL]$ and $d([R,sL],t[R,sL])=d(R,tR)\geqq d(R,tL)\geqq k$.

(ii) $g=t^{-1}$. Then $d([R,sL],t^{-1}[R,sL])=d(R,t^{-1}R)=d(R,tR)\geqq k$ as in (i).

(iii) $g=s^{\pm 1}$. Then $d(x,gx)=\ell(s)=d(R,sR)\geqq d(R,sL)\geqq k$, since $[R,sL]\subseteq A_s$.

Thus in all cases $d(x,gx)\geqq k$. Similarly, interchanging the roles of s and t, we find that if $n=1$ and $x\in[R,tL]$, then $d(x,gx)\geqq k$, and the lemma is proved. □

Proposition 1. *In the circumstances of Lemma 3, G acts freely and properly discontinuously on X, and s, t freely generate G.*

Proof. This follows at once from Lemmas 3 and 4 (the fact that s and t are free generators also follows from Lemma 2 in [2]). □

The second proposition deals with the situation of Lemma 3 in [3]. The proof is quite similar to that of Proposition 1, with the bridge $[S,T]$ between A_s to A_t playing a role analogous to that of $[L,R]$ in Proposition 1.

Lemma 5. *Suppose that $G=<s,t>$ acts on a Λ-tree (X,d), with $\ell(s)$, $\ell(t)>0$, and $A_s\cap A_t=\varnothing$. Let $[S,T]$ be the bridge between A_s and A_t, where $S\in A_s$ and $T\in A_t$. Put $k=\min\{\ell(s),\ell(t)\}$. Let $g\in G$, and write $g=u_1\ldots u_n$, where $u_1,\ldots,u_n$ is a freely reduced word in $\{s^{\pm 1},t^{\pm 1}\}$. Then*

(a) $d([T,S],g[T,S])\geqq nk$,

(b) *if* $n>1$, *then* $d([S,sS],g[S,sS])\geqq k$ *and* $d([T,tT],g[T,tT])\geqq k$.

Proof. If $n=1$ it is easy to see that (a) is true (see the diagram for the case $\ell(s)>0$, $\ell(t)>0$ on p. 355 of [1]). We may therefore assume $n>1$. By Lemma 3 of [3],

$$[S,gS]=[S,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gS]$$

where Q_i is either S or T for $1\leqq i\leqq n-1$. Again, it is easy to see that either $T\in[S,u_1Q_1]$ or $S\in[T,u_1Q_1]$, and either $T\in[u_n^{-1}Q_{n-1},S]$ or $S\in[u_n^{-1}Q_{n-1},T]$. Thus there are four possibilities:

(i) $[S,gT]=[S,T,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gS,gT]$,

(ii) $[T,gT]=[T,S,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gS,gT]$,

(iii) $[S,gS]=[S,T,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gT,gS]$,

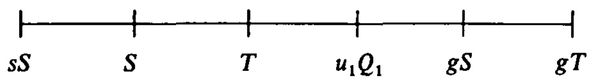
(iv) $[T,gS]=[T,S,u_1Q_1,u_1u_2Q_2,\ldots,u_1\ldots u_{n-1}Q_{n-1},gT,gS]$.

It is easily checked that the distance between two successive points on the right-hand side in (i)–(iv), other than the first two points or the last two points, is at least k. Therefore $d([T,S],g[T,S])$ is the distance between the second and penultimate points on the right-hand side, and is $\geqq nk$, as required.

Since $[T,sS]=[T,S,sS]$, so that $g[T,sS]=g[T,S,sS]$, it follows from the triangle inequality that $d([S,sS],g[S,sS])\geqq(n-2)k$, and similarly it follows that $d([T,tT],g[T,tT])\geqq(n-2)k$, so (b) follows if $n\geqq 3$, and it remains to prove (b) when $n=2$.

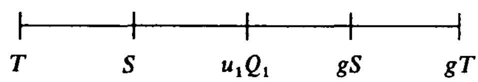
Put $d=d([S,sS],g[S,sS])$. One of the four cases (i)–(iv) above occurs and we consider them in turn.

Case (i). The situation is illustrated by the following diagram.



Once again an easy check shows that either $gsS\in[u_1Q_1,gS]$ or $[gsS,gS]\cap[gS,u_1Q_1]=\{gS\}$, hence $d\geqq d(T,u_1Q_1)\geqq k$.

Case (ii). In this case the diagram is



Now $[S,sS]\cap[S,u_1Q_1]=\{S\}$ unless $u_1=s$, in which case $sS\in[S,u_1Q_1]$, and $g[S,sS]\cap[gS,u_1Q_1]=\{gS\}$ unless $u_2=s^{-1}$, in which case $gsS\in[u_1Q_1,gS]$. Since $d(S,u_1Q_1)\geqq k$ and $d(gS,u_1Q_1)\geqq k$, and both $u_1=s$ and $u_2=s^{-1}$ is impossible (since u_1,u_2 is a reduced word), we see that, in all cases, $d\geqq k$.

In Case (iii) we have $d\geqq d(T,u_1Q_1)+d(u_1Q_1,gT)\geqq 2k$, and (iv) follows from (i) applied with g^{-1} in place of g. The other assertion in (b), that $d([T,tT],g[T,tT])\geqq k$, is proved by interchanging the roles of s and t in this argument, which has the effect of interchanging S and T. □

Lemma 6. *In the situation of Lemma 5, suppose $1\neq g\in G$ and $x\in X$. Then $d(x,gx)\geqq k$.*

Proof. Let $X'=\bigcup_{g\in G}[S,gS]$, a G-invariant subtree. By Lemma 1, we may assume that $x\in X'$. Let $Y=[s^{-1}T,t^{-1}S]\cup[sT,tS]$. Then Y is a subtree of X' (again see the diagram on p. 355 of [1]; note that $[s^{-1}T,t^{-1}S]\subseteq[S,t^{-1}S]\cup[S,s^{-1}t^{-1}S]\subseteq X'$, and $[sT,tS]\subseteq[S,stS]\cup[S,tS]\subseteq X'$, so $Y\subseteq X'$.) By Lemma 3 in [3], if $x\in X', x\in[hQ,huQ']$ for some $h\in G$, where $u\in\{s^{\pm1},t^{\pm1}\}$ and $Q,Q'\in\{S,T\}$. It is easily seen that $[Q,uQ']\subseteq Y$, hence $X'=GY$. Now let $Z=[sS,tT]=[sS,S,T,tT]$, a subtree of Y. Again it is easy to see that $Y\subseteq Z\cup\bigcup_{u\in\{s^{\pm1},t^{\pm1}\}}uZ$, hence $X'=GZ$. By Lemma 2, we may assume that $x\in Z$. Note that $Z=[S,sS]\cup[S,T]\cup[T,tT]$.

If $x\in[S,T]$, then $d(x,gx)\geqq k$ by Lemma 5(a); also, if $n>1$ and $x\in[S,sS]\cup[T,tT]$, then $d(x,gx)\geqq k$ by Lemma 5(b). Suppose $n=1$ and $x\in[S,sS]\cup[T,tT]$. If $x\in[S,sS]$, there are the following possibilities.

(i) $g=t$. Then $gx \in t[S,sS]$ and $d([S,sS],t[S,sS])=d(S,tS)\geqq d(T,tT)=l(t)\geqq k$, since $[S,tS]=[S,T,tT,tS]$.

(ii) $g=t^{-1}$. Then $d([S,sS],t^{-1}[S,sS])=d(S,t^{-1}S)=d(S,tS)\geqq k$ as in (i).

(iii) $g=s^{\pm 1}$. Then since $[S,sS]\subseteq A_s$, we have $d(x,gx)=\ell(s)\geqq k$.

Thus in all cases $d(x,gx)\geqq k$. Similarly, interchanging the roles of s and t, we find that if $n=1$ and $x\in[T,tT]$, then $d(x,gx)\geqq k$, and the lemma is proved.

Proposition 2. *In the circumstances of Lemma 5, G acts freely and properly discontinuously on X, and s, t freely generate G.*

Proof. This follows at once from Lemmas 5 and 6 (we can also use Lemma 3 in [3] to see s,t freely generate G). □

3.

We now consider in turn the situations in Lemmas 4, 5 and 6 of [3]. Lemma 5 deals with two cases and it is necessary here to treat them separately. It will be left to readers to draw their own pictures to illustrate the proofs. We begin with the situation of Lemma 4 in [3]. First, we need a lemma which is also used in the next case.

Lemma 7. *Let g be a hyperbolic isometry of a Λ-tree (X,d) and let $a,b\in X$. If $[a,b]\cap[ga,gb]\neq\varnothing$, then at least one endpoint of $[a,b]\cap[ga,gb]$ is in A_g. If $[a,b]\cap[ga,gb]=\varnothing$ then either the bridge joining $[a,b]$ and $[ga,gb]$ is contained in A_g, or this bridge is of the form $[w,gw]$ for some $w\in[a,b]$.*

Proof. If $[a,b]\cap A_g=\varnothing$ then let $[w,x]$ be the bridge between $[a,b]$ and A_g with $w\in[a,b]$ and $x\in A_g$. Then $[a,b]\cap[ga,gb]=\varnothing$ and the bridge joining them is $[w,x,gx,gw]$. Otherwise, $[a,b]\cap A_g=[x,y]$ for some x,y with $x\leqq_g y$. If $gx<_g y$ then $[a,b]\cap[ga,gb]=[gx,y]$ has both endpoints in A_g. If $gx=y$ then $[a,b]\cap[ga,gb]=[y,w]$ for some w, and the endpoint $y\in A_g$. Finally if $y<_g gx$, then $[a,b]\cap[ga,gb]=\varnothing$ and the bridge joining $[a,b]$ and $[ga,gb]$ is $[y,gx]$ which is contained in A_g. □

Before stating the next lemma we need to introduce the simple notion of signed distance. Let A be a linear subtree of a Λ-tree (X,d), so there is an isometry $\alpha: A\to\Lambda$. If $x,y\in A$, the signed distance from x to y (relative to α) is $\alpha(y)-\alpha(x)$. Thus $d(x,y)=|\alpha(y)-\alpha(x)|$. If $A=A_g$ is the axis of a hyperbolic isometry g then we choose α so that $\alpha(gx)=\alpha(x)+\ell(g)$ for $x\in A$, that is, so that the signed distance from x and y is positive if and only if $x<_g y$. In Lemma 8 we shall use signed distance for points which are in the intersection of the axes of two hyperbolic isometries s,t which meet coherently, so it does not matter whether we use A_s or A_t to measure signed distance. We denote the signed distance from x to y by $\vec{d}(x,y)$.

This next lemma is an elaboration of Lemma 4 in [3], and is concerned with the behaviour of a point $Q\in[L,R]=A_s\cap A_t$ when powers of s and t are successively applied. Roughly, if $u=s^{m_1}t^{n_1}\dots s^{m_k}t^{n_k}$ where only n_k can be zero, then $uQ\in[L,R]$ only if the points $t^{n_1}Q, s^{m_1}t^{n_1}Q, t^{n_2}s^{m_1}t^{n_1}Q, \ldots$ all lie in $[L,R]$, and are translated back and forth along $[L,R]$ by successive application of $t^{n_1}, s^{m_1}, t^{n_2}$, etc. Also, if one of these points leaves $[L,R]$, then the projection of uQ onto A_s behaves in the same way as it does in the case $Q=R$, which is treated in Lemma 4 of [3].

Lemma 8. *Let $G=\langle s,t\rangle$ be a group acting on a Λ-tree (X,d) with $\ell(s)>0$ and $\ell(t)>0$. Assume s,t meet coherently, and that $A_s\cap A_t$ has both a left and right-hand endpoint (denoted by L,R respectively). Let $\Delta=\Delta(s,t)$ and suppose that $\ell(t)<\ell(s)<\Delta<\ell(s)+\ell(t)$, that $\Delta\notin\langle\ell(s),\ell(t)\rangle$ and that $\ell(s),\ell(t)$ are $\mathbb{Z}$-linearly independent. Let Q be any point of $[L,R]$ and let $u=s^{m_1}t^{n_1}\ldots s^{m_k}t^{n_k}$, where the m_i and n_i are integers, $k\geqq 1$ and only n_k is allowed to be zero. Let w be the projection of uQ onto A_s. Then*

(1) *If $uQ\notin[L,R]$, then either $sL\leqq_s w$ or $w\leqq_s s^{-1}R$. Moreover:*
(a) *If $w\in[sL,R]$, then either $d(w,R)=\alpha\ell(s)+\beta\ell(t)$ for some integers α, β with $\alpha<0$, $\beta\geqq 0$, or $d(w,L)=\alpha\ell(s)+\beta\ell(t)$ for some integers α,β with $\alpha>0$, $\beta\leqq 0$. In either case, $w\neq R$.*
(b) *If $w\in[L,s^{-1}R]$, then either $d(w,L)=\alpha\ell(s)+\beta\ell(t)$ for some integers α,β with $\alpha<0$, $\beta\geqq 0$, or $d(w,R)=\alpha\ell(s)+\beta\ell(t)$ for some integers α,β with $\alpha>0$, $\beta\leqq 0$. In either case, $w\neq L$.*
(2) *If $uQ\in[L,R]$ then $s^{m_i}t^{n_i}\ldots s^{m_k}t^{n_k}Q\in[L,R]$ and $t^{n_i}\ldots s^{m_k}t^{n_k}Q\in[L,R]$ for $1\leqq i\leqq k$. Further, either all $m_i=1$ and all $n_i\leqq 0$, or all $m_i=-1$ and all $n_i\geqq 0$. In either case, the signed distance $\vec{d}(Q,uQ)=p\ell(s)+q\ell(t)$ where $p=\sum_{i=1}^k m_i=\pm k$ and $q=\sum_{i=1}^k n_i$.*

Proof. The proof is by induction on k. Suppose $k=1$. If $n_1=0$ then $w=uQ=s^{m_1}Q$ can be in $[L,R]$ only when $m_1=\pm 1$, because $\Delta<2\ell(s)$, and the lemma is clearly true in this case. Suppose $n_1\neq 0$. If $t^{n_1}Q\notin[L,R]$ then w is either $s^{m_1}L$ or $s^{m_1}R$, depending on the sign of n_1, and $uQ\notin[L,R]$. It is easily checked that the conditions on w are satisfied. Suppose $t^{n_1}Q\in[L,R]$. Then $w=uQ$, and because $\Delta<\ell(s)+\ell(t)$, uQ can be in $[L,R]$ only when $m_1=\pm 1$. For the same reason, if $n_1<0$, $m_1=1$ and if $n_1>0$, $m_1=-1$. It is now clear that the lemma is true in this case.

Now assume the result is true for $k-1$, and put $u'=s^{m_2}t^{n_2}\ldots s^{m_k}t^{n_k}$. If $u'Q\notin[L,R]$, then an analysis like that given in the proof of Lemma 4 in [3] shows that the conditions on w are satisfied and that $uQ\notin[L,R]$. Suppose that $u'Q\in[L,R]$. By induction there are two possibilities:

Case 1. $m_i=1$ and $n_i\leqq 0$ for $2\leqq i\leqq k$.

Case 2. $m_i=-1$ and $n_i\geqq 0$ for $2\leqq i\leqq k$.

In Case 1, if $t^{n_1}u'Q\notin[L,R]$, then $w=s^{m_1}R$ or $w=s^{m_1}L$ and $uQ\notin[L,R]$. Suppose $t^{n_1}u'Q\in[L,R]$. Since inductively $V:=t^{n_2}\ldots s^{m_k}t^{n_k}Q\in[L,R]$ and $m_2=1$, $t^{n_1}u'Q=t^{n_1}sV\notin[L,R]$ if $n_1>0$, again because $\Delta<\ell(s)+\ell(t)$, so $n_1<0$. Then $w=uQ=s^{m_1}t^{n_1}u'Q$, which can be in $[L,R]$ only when $m_1=1$ for similar reasons, and either $w\leqq_s s^{-1}R$ or $sL\leqq_s w$. Further, when $uQ\in[L,R]$, $\vec{d}(Q,uQ)=\vec{d}(Q,u'Q)+\ell(s)+n_1\ell(t)$, and the result follows by induction in this case. Case 2 is dealt with in a similar manner. □

Lemma 9. *Under the hypotheses of Lemma 8, for all $g\in G\backslash\{1\}$ and all $Q\in[L,R]$, $d(Q,gQ)>0$.*

Proof. Let $u=s^{m_1}t^{n_1}\dots s^{m_k}t^{n_k}$, where the m_i and n_i are integers, $k\geqq 1$ and only n_k is allowed to be zero, as in Lemma 8. If $g=u$, then the result follows easily from Lemma 8 because if $uQ\in[L,R]$, the coefficient p in $\vec{d}(Q,uQ)$ is non-zero, and $\ell(s)$, $\ell(t)$ are linearly independent, so $d(Q,uQ)=|\vec{d}(Q,uQ)|\neq 0$.

Suppose $g=t^n u$, where $n\neq 0$. If $uQ\in[L,R]$, then again by considering the signed distance from Q to gQ along the axis A_t, we see that $d(Q,gQ)>0$. Assume $uQ\notin[L,R]$.

Let w be the projection of uQ onto A_s, and let w' be the projection of gQ onto A_s. By Lemma 8, there are four possibilities.

Case 1. $R<_s w$. Then it is easily checked that

$$w'=\begin{cases} R & \text{if } n>0\\ t^nR & \text{if } n<0 \text{ and } L<_t t^nR\\ L & \text{if } n<0 \text{ and } t^nR<_t L.\end{cases}$$

Note that $t^nR\neq L$ since otherwise $\Delta=|n|\ell(t)$, so $\Delta\in\langle\ell(s),\ell(t)\rangle$, contrary to hypothesis.

Also, in all cases, $d(gQ,A_s)\geqq d(R,uQ)>0$, so $d(gQ,Q)>0$.

Case 2. $w\in[sL,R]$. By Lemma 8, $t^nw\neq L,R$, and it is easily checked that

$$w'=\begin{cases} t^nw & \text{if } t^nw\in[L,R]\\ R & \text{if } R<_t t^nw\\ L & \text{if } t^nw<_t L.\end{cases}$$

In all cases it is easy to see that $gQ\notin[L,R]$, in fact $d(gQ,A_t)=d(uQ,w)>0$.

Case 3. $w<_s L$. Similar argument to Case 1.

Case 4. $w\in[L,s^{-1}R]$. Similar argument to Case 2.

The only other possibility is that $g=t^n$ where $n\neq 0$. But $\ell(t^n)=|n|\ell(t)\neq 0$, so $d(gQ,Q)>0$ for all $Q\in X$.

We cannot yet prove that the action is properly discontinuous, but it does now follow that the action is free.

Lemma 10. *Under the hypotheses of Lemma 8, the action of G on X is free.*

Proof. We first show that for all $1\neq g\in G$ and all $x\in X$, we have $d(x,gx)>0$. As in Lemma 4, $Y=\bigcup_{g\in G}[R,gR]$ is a G-invariant subtree of X and by Lemma 1 we may assume $x\in Y$. Write $g=u_1\dots u_n$ where $u_i\in\{s^{\pm1},t^{\pm1}\}$. By (2.14) in [1],

$$[R,gR]\subseteq[R,u_1R]\cup[u_1R,u_1u_2R]\cup\dots\cup[u_1\dots u_{n-1}R,gR]$$

and each of the intervals in this union is in the orbit of either $[s^{-1}R,R]$ or $[t^{-1}R,R]$. But $[t^{-1}R,R]\subseteq[s^{-1}R,R]$, so $Y=G[s^{-1}R,R]$. Since $[s^{-1}R,R]\subseteq[L,R]$, it now follows from Lemma 9 and Lemma 2 that $d(x,gx)>0$. Thus no element of $G\backslash\{1\}$ has a fixed point. Now it follows from Lemma 4 in [3] that G is freely generated by s and t, so G is torsion-free. Hence, if $1 \neq g \in G$, $g^2 \neq 1$, so g^2 has no fixed points, hence g is not an inversion, so g must be a hyperbolic isometry. □

Lemma 11. *Under the hypotheses of Lemma 8, let* $A = \langle \ell(s), \ell(t), \Delta \rangle$. *For* $g \in G$, *let* w_g *be the projection of* gR *onto* A_s. *Then*:

(1) *For all* $g \in G \backslash \{1\}$, $d(R, gR) \geqq a > 0$ *for some* $a \in A$.

(2) *For all* $g \in G \backslash \{1\}$, $d(L, gL) \geqq a > 0$ *for some* $a \in A$.

(3) *For all* $g \in G$, w_g *is in the orbit of either* R *or* L.

(4) *For all* $g \in G \backslash \{1\}$ *and* $h \in G$, $d(gw_h, w_h) \geqq a > 0$ *for some* $a \in A$.

Proof. (2) follows from (1) on replacing s, t by s^{-1}, t^{-1} respectively, and (4) follows from (1)–(3), so we have to prove (1) and (3). Let $u = s^{m_1} t^{n_1} \dots s^{m_k} t^{n_k}$, where the m_i and n_i are integers and only n_k is allowed to be zero. An inspection of the proof of Lemma 4 in [**3**] shows that the conclusions of that lemma are valid even when $n_k = 0$. If $g = u$, further inspection of the proof of Lemma 4 of [**3**] shows that (3) holds, and that $d(R, w_u) \geq a > 0$ for some $a \in A$, hence $d(R, uR) \geqq a$. Thus (1) holds in this case.

If $g = t^n u$ where $n > 0$, then a similar inspection of the proof of Lemma 9 above shows that (1) and (3) hold in this case. For example, if $w_u \in [sL, R]$, $uQ \notin [L, R]$ and $R <_t t^n w_u$, then $R = w_g$ and $d(R, gR) \geqq d(R, t^n w_u) = n\ell(t) - d(R, w_u) > 0$, and $d(R, w_u) \in A$ by Lemma 8, so (1) and (3) hold. Finally, the result is clear if $g = t^n$. □

Lemma 12. *Under the hypotheses of Lemmas 8 and 11, suppose that there exists* $0 < c \in \Lambda$ *such that* $c \leqq a$ *for all* $0 < a \in A$. *Then for all* $1 \neq g \in G$ *and all* $Q \in [s^{-1}R, R]$, $d(Q, gQ) \geqq c$.

Proof. Take $g \neq 1$; note that, by Lemma 10, g is a hyperbolic isometry. Let w be the projection of gR onto A_s, and let w' be the projection of $gs^{-1}R$ onto A_s. If either $w <_s s^{-1}R$ and $R <_s w'$, or $w' <_s s^{-1}R$ and $R <_s w$, then by (2.14) in [**1**],

$$\begin{aligned} d(s^{-1}R, R) &= d(gs^{-1}R, gR) \\ &= d(gs^{-1}R, w') + d(w', w) + d(w, gR) \\ &\geqq d(w', w) > d(s^{-1}R, R) \end{aligned}$$

a contradiction. We therefore need to consider the following three cases.

Case 1. Either $w \in [s^{-1}R, R]$ or $w' \in [s^{-1}R, R]$ and $w \neq w'$. Then $[s^{-1}R, R] \cap [gs^{-1}R, gR]$ is one of

$$[w, w'], [w, R], [w, s^{-1}R], [w', R], [w', s^{-1}R]$$

and by Lemma 7, one of the endpoints of this intersection, say z, is in A_g. By Lemma 11, $d(z, gz) \geqq c$. Hence $\ell(g) \geqq c$, so $d(x, gx) \geqq c$ for all $x \in X$, in particular for $x \in [s^{-1}R, R]$.

Case 2. Suppose $w = w' \in [s^{-1}R, R]$. If $w \in [gs^{-1}R, gR]$, then we are finished as in Case 1. Otherwise, $[s^{-1}R, R] \cap [gs^{-1}R, gR] = \varnothing$ and w is an endpoint of the bridge between $[s^{-1}R, R]$ and $[gs^{-1}R, gR]$. By Lemma 7, either $w \in A_g$ and we are finished as in Case 1, or this bridge is $[w, gw]$, and the result follows using Lemma 11 and the Bridge Proposition (2.17(c) in [**1**]).

Case 3. Suppose $w, w' <_s s^{-1}R$ or $R <_s w, w'$. Then either R or $s^{-1}R$ is an endpoint of the bridge joining $[s^{-1}R, R]$ and $[gs^{-1}R, gR]$, and again the result follows using Lemmas 7 and 11. □

We still cannot prove that the action is properly discontinuous under the hypotheses of Lemmas 8, but we can prove this if we assume in addition the existence of c as in Lemma 12. This weaker result will eventually be sufficient to prove the main theorem.

Proposition 3. *Under the hypotheses of Lemmas 8 and 12, for all $1 \neq g \in G$ and all $x \in X$, $d(x, gx) \geqq c$, and s, t freely generate G.*

Proof. The first assertion is proved by an argument like that used in Lemma 10, using Lemma 12 in place of Lemma 9. Now s, t generate infinite cyclic groups since they are hyperbolic, and by Lemma 4 in [**3**], G is their free product. □

We now consider the situation of (1) in Lemma 5 of [**3**]. The strategy will be similar to that used to prove Proposition 3.

Lemma 13. *Let $G = <s, t>$ be a group acting on a Λ-tree (X, d) with $\ell(s) > 0$ and $\ell(t) > 0$ and assume s, t meet coherently, and that $A_s \cap A_t$ has both a left and right-hand endpoint (denoted by L, R respectively). Let $\Delta = \Delta(s, t)$ and suppose that $\ell(t^r) < \ell(s)$ for all integers r, and $\ell(s) < \Delta < \ell(s) + \ell(t)$. Let Q be any point of $[L, R]$ and let $u = s^{m_1}t^{n_1} \dots s^{m_k}t^{n_k}$, where the m_i and n_i are integers, $k \geqq 1$ and only n_k is allowed to be zero. Let w be the projection of uQ onto A_s. Then*:

(1) *Suppose $uQ \notin [L, R]$; if $w \in [L, R]$ then either $w = s^{-1}R$ or $w = sL$.*

(2) *If $uQ \in [L, R]$, then $k = 1$, $t^{n_1}Q \in [L, R]$ and either $m_1 = 1$ and $n_1 \leqq 0$, or $m_1 = -1$ and $n_1 \geqq 0$.*

Proof. The proof is by induction on k. Suppose $k = 1$. If $n_1 = 0$ then $w = uQ$, which can be in $[L, R]$ only when $m = \pm 1$. If $n_1 \neq 0$ then an argument like that in Lemma 8 shows that w is either $uQ, s^{m_1}L$ or $s^{m_1}R$, and $uQ \in [L, R]$ only when $t^{n_1}Q \in [L, R]$ and either $n_1 < 0$ and $m_1 = 1$, *or* $n_1 > 0$ and $m_1 = -1$.

Now assume $k > 1$ and the result is true for $k - 1$. Put $u' = s^{m_2}t^{n_2} \dots s^{m_k}t^{n_k}$. If $u'Q \notin [L, R]$, then an analysis like that given in the proof of Lemma 5 in [**3**] shows that the conditions on w are satisfied and that $uQ \notin [L, R]$. Suppose that $u'Q \in [L, R]$. By induction $k = 2$ and there are two possibilities:

(1) $m_2 = 1$ and $n_2 \leqq 0$.

(2) $m_2 = -1$ and $n_2 \geqq 0$.

In Case 1, if $t^{n_1}u'Q\notin[L,R]$, then $w=s^{m_1}R$ or $w=s^{m_1}L$ and $uQ\notin[L,R]$, and $w\in[L,R]$ only when $w=s^{-1}R$ or $w=sL$.

Suppose $t^{n_1}u'Q\in[L,R]$. Since $t^{n_2}Q\in[L,R]$ and $m_2=1$, $t^{n_1}st^{n_2}Q\notin[L,R]$ if $n_1>0$, because $\Delta<\ell(s)+\ell(t)$, so $n_1<0$. Then $w=uQ=s^{m_1}t^{n_1}u'Q$, which can be in $[L,R]$ only when $m_1=1$ for similar reasons. But then it is easy to see that $Q<_s uQ$ and $d(Q,uQ)=2\ell(s)-(n_1+n_2)\ell(t)\geqq\ell(s)+\ell(t)>\Delta$, so $uQ\notin[L,R]$. Case 2 is dealt with in a similar manner. □

Lemma 14. *Under the hypotheses of Lemma 13, for all $g\in G\backslash\{1\}$ and all $Q\in[L,R]$, $d(Q,gQ)>0$.*

Proof. Let $u=s^{m_1}t^{n_1}\dots s^{m_k}t^{n_k}$, where the m_i and n_i are integers, $k\geqq1$ and only n_k is allowed to be zero, as in Lemma 8. If $g=u$, then the result follows easily from Lemma 8 because if $uQ\in[L,R], d(Q,uQ)=\ell(s)-|n_1|\ell(t)>0$.

Suppose $g=t^n u$, where $n\neq0$. If $uQ\in[L,R]$, then $d(Q,gQ)=\ell(s)-|n_1|\ell(t)\pm n\ell(t)>0$. Assume $uQ\notin[L,R]$.

Let w be the projection of uQ onto A_s, and let w' be the projection of gQ onto A_s. By Lemma 13, there are four possibilities.

Case 1. $R<_s w$. Then it is easily checked that $w'=R$ if $n>0$ and $w'=t^nR$ if $n<0$, and that $d(gQ,A_s)\geqq d(R,w)>0$, so $d(gQ,Q)>0$.

Case 2. $w=sL$. Then it is easily checked that $w'=t^nsL$ if $n<0$ and $w'=R$ if $n>0$ and that $d(gQ,A_s)\geqq d(uQ,A_s)>0$, *so* $d(gQ,Q)>0$.

Case 3. $w<_s L$. Similar argument to Case 1.

Case 4. $w=s^{-1}R$. Similar argument to Case 2.

The only other possibility is that $g=t^n$ where $n\neq0$. But $\ell(t^n)=|n|\ell(t)\neq0$, so $d(gQ,Q)>0$ for all $Q\in X$. □

Lemma 15. *Under the hypotheses of Lemma 13, the action of G on X is free.*

Proof. This is proved in the same way as Lemma 10, observing that by Lemma 5(1) in [3], s and t freely generate G, so G is torsion-free, and using Lemma 14. □

Lemma 16. *Under the hypotheses of Lemma 14, let $c=\min\{\ell(s)+\ell(t)-\Delta,\Delta-\ell(s)\}$. Then:*

(1) *For all $g\in G\backslash\{1\}$, $d(R,gR)\geqq c$.*

(2) *For all $g\in G\backslash\{1\}$, $d(L,gL)\geqq c$.*

(3) *If w_g is the projection of gR onto A_s, then for all $g\in G$, w_g is in the orbit of either R or L.*

(4) *For all $g\in G\backslash\{1\}$ and $h\in G$, $d(gw_h,w_h)\geqq c$.*

Proof. As in Lemma 11, (2) follows from (1) on replacing s,t by s^{-1}, t^{-1} respectively, and (4) follows from (1)–(3), so we have to prove (1) and (3). Let $u=s^{m_1}t^{n_1}\dots s^{m_k}t^{n_k}$, where the m_i and n_i are integers and only n_k is allowed to be zero. An inspection of the proof of Lemma 5(1) in [3] shows that the conclusions of that lemma are valid even when $n_k=0$. If $g=u$, further inspection of the proof of Lemma 5 of [3] shows that (3) holds. One of the conclusions of that lemma is that either $stL \leqq_s w_u$ or $w_u \leqq_s sL$, so either $d(w_u, R) \geqq d(R, stL) = \ell(s)+\ell(t)-\Delta$ or $d(w_u, R) \geqq d(R, sL) = \Delta - \ell(s)$. Thus in either case $d(R, uR) \geqq d(R, w_u) \geqq c$, and the lemma is true when $g=u$.

If $g=t^n$ where $n \neq 0$, then w_g is either R or t^nR depending on the sign of n, and $d(R,gR) = |n|\ell(t) \geqq \ell(t) \geqq c$ because of the hypothesis $\ell(s) < \Delta < \ell(s)+\ell(t)$. Thus we need only verify (1) and (3) when $g=t^nu$, where $n \neq 0$. We consider several cases.

Case 1. Suppose $R <_s w_u$. Then if $n>0$, $w_g = R$ and $d(R,gR) = d(R,t^nR) + d(R,uR) \geqq d(R,t^nR) \geqq c$, as already noted. If $n<0$, then $w_g = t^nR \in [L,R]$ and again $d(R,gR) = d(R,t^nR) + d(R,uR) \geqq d(R,t^nR) \geqq c$.

Case 2. Suppose $w_u <_s L$. If $n<0$ then $w_g = L$ and again $d(R,gR) = d(R,uR) + |n|\ell(t) \geqq \ell(t) \geqq c$. If $n>0$, then $w_g = t^nL$ and $d(R,gR) \geqq d(t^nL, R) = \Delta - n\ell(t) > \ell(t) \geqq c$.

Case 3. Suppose $w_u = s^{-1}R$. If $n>0$, then $w_g = t^ns^{-1}R \in [L,R]$ and $d(R,gR) \geqq d(R,w_g) = \ell(s) - n\ell(t) > \ell(t) \geqq c$. If $n<0$, then $w_g = L$ and $d(R,gR) \geqq d(L,R) = \Delta \geqq \Delta - \ell(s) \geqq c$.

Case 4. Suppose $w_u = sL$. If $n>0$ then $w_g = R$ and $d(R,gR) \geqq d(R, t^nsL) \geqq d(R, tsL) = \ell(s) + \ell(t) - \Delta \geqq c$. If $n<0$, then $w_g = t^nsL \in [L,R]$ and $d(R,gR) \geqq d(R,w_g) = \Delta - \ell(s) + |n|\ell(t) > \Delta - \ell(s) \geqq c$.

By Lemma 5 in [3], these are the only possibilities for w_u, so this completes the proof. □

Lemma 17. *Under the hypotheses of Lemmas 13 and 16, for all* $1 \neq g \in G$ *and all* $Q \in [s^{-1}R, R]$, $d(Q,gQ) \geqq c$.

Proof. This proceeds in the same way as the proof of Lemma 12, using Lemmas 14 and 16 together with Lemma 7, and details are omitted. □

Proposition 4. *Under the hypotheses of Lemmas 13 and 16, for all* $1 \neq g \in G$ *and all* $x \in X$, $d(x,gx) \geqq c$. *Hence the action of G on X is properly discontinuous. Further, s and t freely generate G.*

Proof. Again this is proved by an argument like that used in Lemma 10, and Proposition 3, using Lemma 17 in place of Lemma 9, and using Lemma 5(1) in [3] to see s and t are free generators. □

We now consider the situation of Lemma 5(2) in [3]. This involves a somewhat different approach from that used in Propositions 3 and 4, and is simpler.

Lemma 18. *Let* $G = \langle s,t \rangle$ *be a group acting on a* Λ*-tree* (X,d) *with* $\ell(t)>0$ *and assume s, t meet coherently, and that* $A_s \cap A_t$ *has both a left and right-hand endpoint (denoted by L, R respectively). Let* $\Delta = \Delta(s,t)$ *and suppose that* $\ell(t^r) < \Delta$ *for all integers r and* $\Delta < \ell(s)$. *Let* $u = s^{m_1}t^{n_1}\ldots s^{m_k}t^{n_k}$, *where the* m_i *and* n_i *are integers,* $k \geqq 1$ *and only* n_k *is allowed to be zero. Then* $[L,R] \cap u[L,R] = \varnothing$, *and the bridge between* $[L,R]$ *and* $u[L,R]$ *contains one of the following points:*

$$s^{m_1}R, s^{m_1}t^{n_1}R (\text{with } n_1<0), s^{m_1}L, s^{m_1}t^{n_1}L \ (\text{with } n_1>0)$$

and all the listed points are at distance at least $\ell(s)-\Delta$ from $[L,R]$.

Proof. First note that all the listed points lie on A_s and if w is one of them, either $w<_s s^{-1}R<_s L$ or $R<_s sL<_s w$, and since $d(s^{-1}R,L)=d(R,sL)=\ell(s)-\Delta$, the last part of the lemma follows. The rest of the lemma is proved by induction on k. Suppose that $k=1$. If $n_1\leqq 0$, then $t^{n_1}R\in[L,R]$, $t^{n_1}L\leqq_t L$, and the bridge joining $[L,R]$ to $u[L,R]$ is $[R,s^{m_1}L]$ if $m_1>0$, and is $[s^{m_1}t^{n_1}R,L]$ if $m_1<0$.

If $n_1>0$, then $t^{n_1}L\in[L,R]$ and $R<_t t^{n_1}R$, and the bridge joining $[L,R]$ to $u[L,R]$ is $[R,s^{m_1}t^{n_1}L]$ if $m_1>0$, and is $[s^{m_1}R,L]$ if $m_1<0$. Thus the lemma is true when $k=1$.

Now assume the result is true for $k-1$. Put $u'=s^{m_2}t^{n_2}\dots s^{m_k}t^{n_k}$. Then the bridge joining $[L,R]$ to $u'[L,R]$ contains one of the listed points with m_2, n_2 in place of m_1,n_1. Denote such a point by w', so as noted at the beginning either $R<_s w'$ or $w'<_s L$. Suppose that $R<_s w'$, so that the bridge joining $[L,R]$ to $u'[L,R]$ is of the form $[R,w',p]$ for some $p\in u'[L,R]$. *If* $n_1>0$ then the bridge joining $[L,R]$ to $u[L,R]$ has the form $[R,s^{m_1}R,s^{m_1}t^{n_1}R,s^{m_1}t^{n_1}w',s^{m_1}t^{n_1}p]$ if $m_1>0$, and the form $[L,s^{m_1}R,s^{m_1}t^{n_1}R,s^{m_1}t^{n_1}w',s^{m_1}t^{n_1}p]$ if $m_1<0$, so contains $s^{m_1}R$. If $n_1<0$, the bridge joining $[L,R]$ to $u[L,R]$ has the form $[R,s^{m_1}t^{n_1}R,s^{m_1}t^{n_1}w',s^{m_1}t^{n_1}p]$ if $m_1>0$, and if $m_1<0$ it has the form $[L,s^{m_1}t^{n_1}R,s^{m_1}t^{n_1}w',s^{m_1}t^{n_1}p]$, so it contains $s^{m_1}t^{n_1}R$. The case $w'<_s L$ is dealt with similarly, by replacing s,t with their inverses, interchanging the roles of L and R. □

Lemma 19. *Under the hypotheses of Lemma 18, the bridge joining $[R,sL]$ to $u[R,sL]$ contains one of the points listed there and $d([R,sL],u[R,sL])\geqq\ell(t)$.*

Proof. The proof is similar to the proof of Lemma 18 and the details are left to the reader. There is one point to note. All the listed points are at distance at least $\ell(t)$ from $[R,sL]$, except, of course, sL. However, this only arises in the case $k>1$, the bridge joining $u'[R,sL]$ to $[R,sL]$ contains a point w' on the list with $w'<_s s^{-1}R<_s L$, and $n_1<0$, $m_1=1$. But then the bridge joining $u[R,sL]$ to $[R,sL]$ contains the segment $[sL,st^{n_1}L,st^{n_1}w']$ of length at least $d(sL,st^{n_1}L)=|n_1|\ell(t)\geqq\ell(t)$. □

Lemma 20. *Under the hypotheses of Lemma 18, suppose $1\neq g\in G$ and $Q\in[L,sL]$. Let $c=\min\{\ell(s)-\Delta,\ell(t)\}$. Then $d(Q,gQ)\geqq c$.*

Proof. Note that $[L,sL]=[L,R]\cup[R,sL]$. If $g=u$, where u is as described in Lemma 18, the result therefore follows by Lemmas 18 and 19. Suppose $g=t^n u$, when $n\neq 0$. If $Q\in[L,R]$, take a point w in the list in Lemma 18 which belongs to the bridge between $[L,R]$ and $u[L,R]$. If $sL\leqq_s w$, then the bridge between $[L,R]$ and $g[L,R]$ contains the segment $[t^nR,t^nw]$, regardless of the sign of n, and this segment has length $d(R,w)\geqq\ell(s)-\Delta$. The other possibility is that $w\leqq_s s^{-1}R$. In this case we similarly find that the bridge joining $[L,R]$ to $g[L,R]$ contains the segment $[t^nL,t^nw]$, of length $d(L,w)\geqq\ell(s)-\Delta$. The argument in case $Q\in[R,sL]$ is similar.

The only other possibility is that $g=t^n$ with $n\neq 0$, in which case $d(Q,gQ)\geqq|n|\ell(t)\geqq\ell(t)\geqq c$.

Proposition 5. *Under the hypotheses of Lemma 18, and with c defined as in Lemma 20, so that* $c>0$, *we have* $d(x,gx)\geqq c$ *for all* $x\in X$ *and* $1\neq g\in G$. *Hence the action of G on X is properly discontinuous. Also, s and t freely generate G.*

Proof. Let $Y=\bigcup_{g\in G}[L,gL]$, a G-invariant subtree of X. By Lemma 1 we may assume $x\in Y$, and arguing as in Lemma 10 with L in place of R, every point of Y is in the orbit of a point in $[L,sL]\cup[L,tL]=[L,sL]$. By Lemma 2, we may assume that $x\in[L,sL]$, and the action is properly discontinuous by Lemma 20. It follows from Lemma 5(2) in [3] that s and t are free generators of G. □

The final case is that of Lemma 6 in [3]. Although we do not have the complication in the previous case that $sL\notin[L,R]$, the fact that $sL=R$ gives new complications, and a somewhat longer analysis than that in Lemma 18 is necessary.

Lemma 21. *Let* $G=<s,t>$ *be a group acting on a Λ-tree* (X,d) *with* $\ell(t)>0$, *assume s, t meet coherently, and that* $A_s\cap A_t$ *has both a left and right-hand endpoint (denoted by L, R respectively). Let* $\Delta=\Delta(s,t)$ *and suppose that* $\ell(t^r)<\ell(s)$ *for all integers r, and that* $\ell(s)=\Delta$. *Assume that* $sts^{-1}t$ *has no fixed point in X. Let P be the projection of* $st^{-1}L$ *onto* A_t. *Let* $u=s^{m_1}t^{n_1}\dots s^{m_k}t^{n_k}$, *where the* m_i *and* n_i *are integers, only* n_k *is allowed to be zero, and* $k\geqq 1$. *Put* $c=d(P,tR)$ *and* $I=[L,R]$. *Then if* $k\geqq 2$, $I\cap uI=\varnothing$, $d(I,uI)\geqq c>0$ *and the bridge joining I to uI contains one of the following points:*

$$sts^{-1}P,\ s^{-1}t^{-1}P,\ st^{-1}L,\ s^{-1}tR$$

Proof. The proof is by induction on k. As noted at the beginning of the proof of Lemma 6 in [3], $c>0$, and since $d(R,st^{-1}L)=\ell(t)=d(R,tR)$, $c=d(P,st^{-1}L)$ and $c<\ell(t)$. It will be helpful to refer to Figure 2 in [3]. It suffices to show that the bridge joining I and uI contains one of the four listed points, for they are all at distance at least c from I. Suppose that $k=2$. We shall denote the bridge between closed subtrees Y, Z of X by $B(Y,Z)$.

Assume that $n_2\geqq 0$. We consider three possibilities for m_2.

Case 1. Suppose $m_2>0$. Then $s^{m_2}t^{n_2}I\cap I=\varnothing$ and the bridge joining these segments is $[R,s^{m_2}t^{n_2}L]$, which is a subset of A_s, except in the case $n_2=0$, $m_2=1$, when $sI\cap I=\{R\}$.

If $n_1>0$, then $B(I,t^{n_1}s^{m_2}t^{n_2}I)$ is $[R,t^{n_1}R,t^{n_1}s^{m_2}t^{n_2}L]$, and $B(I,uI)$ is thus $[R,s^{m_1}R,s^{m_1}t^{n_1}R,uL]$ if $m_1>0$, and is $[L,s^{m_1}R,s^{m_1}t^{n_1}R,uL]$ if $m_1<0$. Further, if $m_1=-1$, $B(I,uI)$ contains the point $s^{-1}tR$, while if $m_1>0$, $sts^{-1}P\in B(I,uI)$, and if $m_1<-1$ then $s^{-1}t^{-1}P\in B(I,uI)$.

If $n_1<0$, then $B(I,t^{n_1}s^{m_2}t^{n_2}I)=[t^{n_1}R,t^{n_1}s^{m_2}t^{n_2}L]$, except in the case $n_2=0$, $m_2=1$, when $I\cap t^{n_1}s^{m_2}t^{n_2}I=\{t^{n_1}R\}$. Thus if $m_1<0$, we have $B(I,uI)=[L,s^{m_1}t^{n_1}R,uL]$, so $s^{-1}t^{-1}P\in B(I,uI)$ since $s^{m_1}t^{n_1}R\leqq_s s^{-1}t^{-1}R<_s s^{-1}t^{-1}P$. If $m_1>0$, then we have $B(I,uI)=[R,sts^{-1}P,st^{n_1}R,s^{m_1}t^{n_1}R,uL]$ since $d(L,ts^{-1}P)=c<\ell(t)$ and $\Delta>(n_1+1)\ell(t)$, so $ts^{-1}P<_s t^{n_1}R$ in I.

Case 2. Suppose $m_2=-1$. Then $I\cap s^{-1}t^{n_2}I=\{L\}$, and if $n_2\neq 0$, $s^{-1}t^{n_2}I=[s^{-1}t^{n_2}L,L,s^{-1}P,s^{-1}t^{n_2}R]$. Suppose $n_1>0$. Then

$$I \cap t^{n_1}s^{-1}t^{n_2}I = \begin{cases} [t^{n_1}s^{-1}P, t^{n_1}L] & \text{if } n_2 \neq 0 \\ \{t^{n_1}L\} & \text{if } n_2 \neq 0 \end{cases}$$

hence

$$B(I, uI) = \begin{cases} [R, s^{m_1}t^{n_1}s^{-1}P] & \text{if } m_1 > 0 \text{ and } n_2 \neq 0 \\ [R, s^{m_1}t^{n_1}L] & \text{if } m_1 > 0 \text{ and } n_2 = 0 \\ [L, s^{m_1}t^{n_1}L] & \text{if } m_1 < 0. \end{cases}$$

Now if $m_1 > 0$, $R <_s sts^{-1}P <_s s^{m_1}t^{n_1}s^{-1}P <_s s^{m_1}t^{n_1}L$, so $sts^{-1}P \in B(I, uI)$; if $m_1 < 0$, $s^{m_1}t^{n_1}L <_s s^{-1}t^{n_1}L <_s s^{-1}t^{-1}P <_s L$, because $t^{n_1}L <_s t^{-1}R <_s t^{-1}P$, again because $\Delta > (n_1+1)\ell(t)$, so $s^{-1}t^{-1}P \in B(I, uI)$.

Suppose $n_1 < 0$. Then $B(I, t^{n_1}s^{-1}t^{n_2}I) = [t^{n_1}L, L]$, and it follows that if $m_1 < 0$, then $s^{-1}t^{-1}P \in B(I, uI)$, if $m_1 = 1$ then $st^{-1}L \in B(I, uI)$ and if $m_1 > 1$ then $sts^{-1}P \in B(I, uI)$.

Case 3. Suppose that $m_2 < -1$. Then $B(I, s^{m_2}t^{n_2}I) = [L, s^{m_2}R]$. If $n_1 > 0$, then $B(I, t^{n_1}s^{m_1}t^{n_2}I) = [t^{n_1}L, t^{n_1}s^{m_2}R]$, hence (as in the previous case)

$$B(I, uI) = \begin{cases} [L, s^{-1}t^{-1}P, s^{m_1}t^{n_1}L, s^{m_1}t^{n_1}s^{m_2}R] & \text{if } m_1 < 0 \\ [R, sts^{-1}P, s^{m_1}t^{n_1}L, s^{m_1}t^{n_1}s^{m_2}R] & \text{if } m_1 > 0 \end{cases}$$

If $n_1 < 0$ then $B(I, t^{n_1}s^{m_1}t^{n_2}I) = [L, t^{n_1}s^{m_2}R]$ and

$$B(I, uI) = \begin{cases} [L, s^{m_1}L, s^{m_1}t^{n_1}L, s^{m_1}t^{n_1}s^{m_2}R] & \text{if } m_1 < 0 \\ [R, s^{m_1}L, s^{m_1}t^{n_1}L, s^{m_1}t^{n_1}s^{m_2}R] & \text{if } m_1 > 0 \end{cases}$$

and as in the previous case, if $m_1 < 0$, then $s^{-1}t^{-1}P \in B(I, uI)$, if $m_1 = 1$ then $st^{-1}L \in B(I, uI)$ and if $m_1 > 1$ then $sts^{-1}P \in B(I, uI)$.

This completes the proof for the case $k = 2$ when $n_2 \geqq 0$. But the case $k = 2$, $n_2 \leqq 0$ now follows by symmetry, replacing s, t by s^{-1}, t^{-1}, which interchanges L and R, and interchanges P with $s^{-1}P$.

Now suppose $k \geqq 3$ and the result is true for $u' = s^{m_2}t^{n_2} \dots s^{m_k}t^{n_k}$. Then $B(I, uI)$ contains one of the four listed points.

Case 1. Suppose $sts^{-1}P \in B(I, u'I)$. If $n_1 > 0$, then $B(I, t^{n_1}u'I) = [R, t^{n_1}R] \cup t^{n_1}B(I, u'I)$, hence if $m_1 > 0$ then $[R, sts^{-1}P] \subseteq B(I, uI)$, if $m_1 = -1$ then $[L, s^{-1}tR] \subseteq B(I, uI)$ and if $m_1 < -1$ then $[L, s^{-1}t^{-1}P] \subseteq B(I, uI)$.

If $n_1 < 0$, then $B(I, t^{n_1}u'I) = t^{n_1}B(I, uI)$. Thus if $m_1 < 0$, we have $[s^{m_1}t^{n_1}R, s^{-1}t^{-1}P, L] \subseteq B(I, uI)$, while if $m_1 > 0$, we have $[s^{m_1}t^{n_1}R, sts^{-1}P, R] \subseteq B(I, uI)$.

Case 2. Suppose $st^{-1}L \in B(I, u'I)$. If $n_1 > 0$, then $[R, P, t^{n_1}P, t^{n_1}st^{-1}L] \subseteq B(I, t^{n_1}u'I)$, and again if $m_1 > 0$ then $[R, sts^{-1}P] \subseteq B(I, uI)$, if $m_1 = -1$ then $[L, s^{-1}tR] \subseteq B(I, uI)$ and if $m_1 < -1$ then $[L, s^{-1}t^{-1}P] \subseteq B(I, uI)$.

If $n_1 < 0$, then $B(I, t^{n_1}u'I) = t^{n_1}B(I, u'I)$ and we are finished as in the previous case. The remaining cases ($s^{-1}tR \in B(I, u'I)$, $s^{-1}t^{-1}P \in B(I, u'I)$) follow by symmetry just as in the case $k = 2$. This completes the proof. □

Lemma 22. *Under the hypotheses of Lemma 21, for any* $1 \neq g \in G$ *and any* $Q \in [L, R], d(Q, gQ) \geqq c$.

Proof. Suppose that $g = u$ as in Lemma 21. Then the result follows from Lemma 21 if $k \geqq 2$, so we may assume $k = 1$. If $g = t^n u$ where $k \geqq 2$ then the result follows from the arguments used in the inductive step in Lemma 21 (in all cases $B(I, t^{n_1} u' I)$ contains an interval of length at least c). Also, if $g = t^n$, with $n \neq 0$, then $d(gQ, Q) \geqq \ell(g) = |n| \ell(t) \geqq c$, for any $Q \in X$. There remains the case $g = t^n s^m t^{n_1}$ where $m \neq 0$. Again using symmetry as in Lemma 21 we may assume that $n_1 \geqq 0$. Put $u = s^m t^{n_1}$. We consider two cases.

Case 1. Suppose that $t^{n_1} Q \in I$. If $n_1 = 0$, then $d(Q, gQ) = |m| \ell(s) \pm n\ell(t) > \ell(t) > c$. Assume $n_1 \neq 0$. If $m > 0$, then since $tL \leqq_s tQ \leqq_s t^{n_1} Q$, we have $[Q, uQ] = [Q, t^{n_1} Q, R, stL, uQ]$, and $d(Q, uQ) = d(Q, t^{n_1} Q) + d(t^{n_1} Q, uQ) = m\ell(s) + n_1 \ell(t) > c$. If $n \geqq 0$, then $d(Q, gQ) = d(Q, uQ) + n\ell(t) > c$, while if $n < 0$ then $[Q, gQ] = [Q, t^n R, t^n stL, gQ]$, so $d(Q, gQ) \geqq d(t^n R, t^n stL) = d(R, stL) = \ell(t) > c$.

If $m < 0$, then since $tQ \leqq_s t^{n_1} Q \leqq_s R$, we have $Q \leqq_s t^{-1} R$, hence $[Q, uQ] = [Q, L, s^{-1} t^{-1} R, uQ]$, and $d(Q, uQ) = -m\ell(s) - n_1 \ell(t) \geqq \ell(s) - n_1 \ell(t) > \ell(t) > c$. If $n \leqq 0$, then $d(Q, gQ) = d(Q, uQ) + |n| \ell(t) > c$. If $n > 0$, then $[Q, gQ] = [Q, t^n L, t^n s^{-1} t^{-1} R, uQ]$, so $d(Q, gQ) \geqq d(t^n L, t^n s^{-1} t^{-1} R) = d(L, s^{-1} t^{-1} R) = c$.

Case 2. Suppose that $t^{n_1} Q \notin I$. Then $d(Q, R) \leqq n_1 \ell(t)$. Using arguments similar to those in Case 1, we see that if $m < -1$, $[Q, gQ]$ contains $[t^n L, t^n s^{-2} R]$, and if $m > 0$, $[Q, gQ]$ contains $[t^n R, t^n sR]$, so that $d(Q, gQ) \geqq \ell(s) > c$. If $m = -1$, then $[Q, gQ] = [Q, L, gQ]$ if $n < 0$, and $[Q, gQ] = [Q, t^n L, gQ]$ if $n \geqq 0$. Thus $d(Q, gQ) \geqq d(t^r L, Q)$ for some $r \geqq 0$. But $t^r L <_s Q$ and $d(t^r L, Q) > \ell(t)$ since $\Delta > (n_1 + r + 1)\ell(t)$. □

Proposition 6. *Under the hypotheses of Lemma 21, for all* $x \in X$ *and all* $1 \neq g \in G$ *we have* $d(x, gx) \geqq c$. *Hence the action of X on G is properly discontinuous. Also, s and t freely generate G.*

Proof. As in the previous three propositions, this follows from Lemmas 7 and 22, together with Lemma 6 in [3]. □

We are now in a position to prove the main theorem, which we restate, giving more detail in Case (iv).

Theorem 1. *Let* $G = <s, t>$ *act on a* Λ*-tree* (X, d). *Then one of the following possibilities occurs:*

(i) *G acts freely and properly discontinuously on X, and s, t freely generate G.*

(ii) *By a finite succession of elementary Nielsen transformations, s and t can be transformed into new generators u and v such that at least one of* $\ell(u)$, $\ell(u)$ *is zero.*

(iii) $\ell(s) > 0$, $\ell(t) > 0$ *but* $sts^{-1}t^{-1}$ *has a fixed point.*

(iv) *By a finite succession of elementary Nielsen transformations, s and t can be transformed into new generators u and v such that* $\ell(u) > 0$, $\ell(v) > 0$, *u and v meet* *coherently, $A_u \cap A_v$ is a segment of length $\ell(u)$, $\ell(v^n) < \ell(u)$ for all integers n, and $uvu^{-1}v$ has a fixed point.*

If Λ is archimedean then one of (i), (ii), (iii) *occurs.*

Proof. If one of $\ell(s)$, $\ell(t)$ is zero then (ii) occurs, so we may assume $\ell(s) > 0$ and $\ell(t) > 0$. We consider several cases. In some cases we shall make use of results in Section 8 of [**1**], where it is assumed that the action is without inversions, and we note that the conclusions we draw are valid without that assumption.

Case 1. $A_s \cap A_t = \emptyset$. By Proposition 2, s and t act freely and properly discontinuously on X, and s and t freely generate G. Thus Case (i) occurs.

In the following cases, we put $\Delta = \Delta(s,t)$.

Case 2. $\Delta < \min\{\ell(s), \ell(t)\}$. Replacing t by t^{-1} if necessary, we assume that s and t meet coherently. Then similarly using Proposition 1, Case (i) occurs.

In the remaining cases we can assume that $\ell(t) \leqq \ell(s)$.

Case 3. $\Delta = \ell(t) \leqq \ell(s)$. Again replacing t by t^{-1} if necessary, we assume that s and t meet coherently.

If $\Delta = 0$, then $\ell(t) = 0$ and Case (ii) occurs, so assume $\Delta > 0$. We are then in Case 4 of (8.3) in [**1**]. Define e and d as in Case 4 of (8.3). Referring to the proof of (8.3) (but with minor modifications to allow for the possibility of an action with inversions), we see that if $\ell(s) = \ell(t) = \Delta$, then $\ell(st^{-1}) = 0$, so Case (ii) occurs. Assume $\Delta = \ell(t) < \ell(s)$. If $e \leqq d$, then again $\ell(st^{-1}) = 0$, so assume $e > d$. If $d = 0$, then $\Delta(st^{-1}, t) = 0$, so by Case 2, st^{-1} and t generate a free group, hence so do s and t, and the action is free and properly discontinuous, so Case (i) occurs. If $d > 0$, then $A_{st}r^{-1} \cap A_t = \emptyset$, similarly reducing to Case 1.

Case 4. $\ell(t) < \Delta$, $\ell(t) \leqq \ell(s)$ and $\Delta < \ell(s) + \ell(t)$. In this case we also assume that s and t meet coherently, replacing t by t^{-1} if necessary. Then from [**1**], (8.3)(c) and the table given in the proof of (8.3), we see that $\ell(st^{-1}) = \ell(s) - \ell(t)$, $\Delta(st^{-1}, t) = \Delta - \ell(t)$, and st^{-1}, t meet coherently. If $\ell(st^{-1}) > \ell(t)$, put $t_1 = t$ and $s_1 = st^{-1}$, while if $\ell(st^{-1}) \leqq \ell(t)$, put $t_1 = st^{-1}$ and $s_1 = t$. Also, let $\Delta_1 = \Delta(s_1, t_1) = \Delta - \ell(t)$. Thus $\Delta_1 < \ell(s_1) + \ell(t_1)$, $\ell(t_1) \leqq \ell(t)$, $\ell(s_1) \leqq \ell(s)$ and $\ell(t_1) \leqq \ell(s_1)$. Also, we have $\ell(s) + \ell(t) - \Delta = \ell(s_1) + \ell(t_1) - \Delta_1$. If $\ell(t_1) = 0$, then Case (ii) occurs and we stop. Otherwise if a previous case applies to s_1, t_1 in place of s, t, either Case (i) or Case (ii) occurs and again we stop. Otherwise, s_1, t_1 satisfy the hypotheses of Case 4, and we can then repeat this procedure on s_1, t_1 to obtain s_2, t_2, and again we stop if $\ell(t_2) = 0$ or a previous case applies, otherwise we remain in Case 4 and we repeat the procedure to obtain s_3, t_3 etc. If this procedure stops after finitely many steps, then for some n, we can apply a previous case to s_n, t_n to see that either (i) or (ii) occurs.

Suppose the procedure does not stop. Let A be the subgroup of Λ generated by $\ell(s)$, $\ell(t)$ and Δ, and let M be the subgroup generated by $\ell(s)$ and $\ell(t)$. Inductively, M is generated by $\ell(s_n)$ and $\ell(t_n)$, and $(\ell(t_n))_{n \geqq 1}$, $(\ell(s_n))_{n \geqq 1}$ are decreasing sequences of elements of M with $0 < \ell(t_n) < \ell(s_n)$ for all n. We put $\Delta_n = \Delta(s_n, t_n)$, so inductively $\Delta_{n+1} = \Delta_n - \ell(t_n)$, and $(\Delta_n)_{n \geqq 1}$ is a decreasing sequence of positive elements of A. Since $(\ell(s_n) + \ell(t_n))_{n \geqq 1}$ is a strictly decreasing sequence of positive elements of M, it follows that M is not cyclic, hence $\ell(s)$ and $\ell(t)$ are $\mathbb{Z}$-linearly independent. Suppose that M is archimedean, and put $c=\ell(s)+\ell(t)-\Delta$. Let D be the intersection of all convex subgroups of A containing $\ell(s)$, and let C be the union of all convex subgroups of A not containing $\ell(s)$. For $d\in D$ let $\bar{d}$ denote the image of d under the canonical map $D\to D/C$. Now $0\leq\ell(t)\leqq\ell(s)$ implies that $\ell(t)\in D$, and $0\leqq\Delta\leqq\ell(s)+\ell(t)$ implies $\Delta\in D$, hence $c\in D$, and $D=A$. Also, by their definition, there are no convex subgroups of A strictly between C and D, hence $D/C=A/C$ is archimedean, so may be viewed as an additive subgroup of $\mathbb{R}$. It follows that, under the canonical map, M embeds in D/C. For if $m=\alpha\ell(s)+\beta\ell(t)\in C$, where α,β are integers, then since $\ell(s)\notin C$ we have $0=\overline{km}<\overline{\ell(s)}$ for all integers k, so $km<\ell(s)$ for all such k, hence $m=0$ since M is archimedean.

Let $a_n=\overline{\ell(t_n)}$, $b_n=\overline{\ell(s_n)}$ and $c_n=\overline{\Delta_n}$. Then we have three decreasing sequences of real numbers, $(a_n)_{n\geqq 1}$, $(b_n)_{n\geqq 1}$ and $(c_n)_{n\geqq 1}$. They are bounded below by 0, so we may define $\alpha=\lim_{n\to\infty}a_n$, $\beta=\lim_{n\to\infty}b_n$ and $\gamma=\lim_{n\to\infty}c_n$. Thus $\gamma\geqq 0$, and since $0<a_n\leqq b_n$ for all n, $0\leqq\alpha\leqq\beta$. Also, for all n, $c_{n+1}=c_n-a_n$, so taking limits gives $\gamma=\gamma-\alpha$, i.e. $\alpha=0$.

If $\beta>0$, we can therefore choose n such that $0<a_n<\beta$. Then s_{n+1} cannot be t_n since $b_{n+1}\geqq\beta$, so $s_{n+1}=s_nt_n^{-1}$ and $t_{n+1}=t_n$. Inductively $t_{n+r}=t_n$ for all $r\geqq 0$, so $a_n=\alpha=0$, a contradiction. Hence $\alpha=\beta=0$.

Inductively, $\bar{c}=a_n+b_n-c_n\geqq 0$ for all n, and taking limits gives $\bar{c}=-\gamma\leqq 0$, hence $\bar{c}=0$, that is, $c\in C$. It follows that $A=M\oplus C$ with the lexicographic ordering, and that C is the cyclic subgroup generated by c. Hence c is the smallest positive element of A. Further, $\Delta\notin M$. By Proposition 3 the action of G is free and properly discontinuous, and s and t freely generate G, so Case (i) occurs.

If M is non-archimedean, by an easy special case of a result of Zaitseva **[6]** (see Lemma 1 in **[2]**), M is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ with the lexicographic ordering. There is a non-trivial order-preserving homomorphism ϕ from M to $\mathbb{Z}$ whose kernel is infinite cyclic. For some n, we have $\phi\ell(t_n)=0$, otherwise $(\phi(\ell(s_n)+\ell(t_n)))_{n\geqq 1}$ would be a strictly decreasing sequence of positive integers. Also, $\phi\ell(s_n)>0$, since M is non-cyclic. It follows that $t_{n+r}=t_n$, and so $\Delta_{n+r}=\Delta_n-r\ell(t_n)$ for $r\geqq 0$, hence $\Delta_n>r\ell(t_n)$ for all $r\geqq 0$. By Propositions 4, 5 and 6, either the action is free and properly discontinuous, and s_n,t_n freely generate G, hence so do s,t, and (i) occurs, or else (iv) occurs with $u=s_n$, $v=t_n$.

Case 5. $\Delta\geqq\ell(s)+\ell(t)$. Then $A_s\cap A_t$ contains a closed segment of length at least $\ell(s)+\ell(t)$, say $[P,Q]$ where $P\leqq_s Q$. If s,t meet coherently, $s^{-1}t^{-1}Q\in[P,Q]$, at distance $\ell(s)+\ell(t)$ from Q, and so $sts^{-1}t^{-1}Q=Q$, because s and t act as translations on their axes. If s and t do not meet coherently, then $s^{-1}t^{-1}(sP)\in[P,Q]$, at distance $\ell(t)$ from P, so $ts^{-1}t^{-1}(sP)=P$, hence sP is a fixed point of $sts^{-1}t^{-1}$. Thus we have conclusion (iii).

Finally, note that if Λ is archimedean, the procedure in Case 4 stops after finitely many steps (otherwise the argument shows A is non-archimedean) so one of (i), (ii), (iii) occurs. □

As an application of Proposition 1 we shall show that an analogue of Theorem 2.8 in **[4]** holds for arbitrary Λ-trees. However, we shall adopt the terminology of **[1]**. Thus if G is a group acting on a Λ-tree X, we say that the action is abelian if $\ell(gh)\leqq\ell(g)+\ell(h)$ for all g, $h\in G$, where ℓ denotes hyperbolic length. The action is called dihedral if it is non-abelian and $\ell(gh) \leqq \ell(g) + \ell(h)$ for all hyperbolic g, $h \in G$. An action which is neither abelian nor dihedral is said to be of general type. In [4], abelian actions are the same as those described as "fixed end" and actions of general type are the same as irreducible actions. However, to justify this and to establish our analogue of Theorem 2.8, we first need to observe that there are analogues of Cor. 2.4 and Theorem 2.6 in [4]. (Also, for general Λ there is the possibility of an abelian action of cut type; see Theorem 7.5 in [1]). We note that in [4], the notation C_g is used instead of A_g to denote axes.

First, observe that 1.7 in [4] is valid for arbitrary Λ-trees, provided that the notion of closed subtree is defined as in 2.7 and 2.10 in [1], and we assume the isometry g is not an inversion. This follows using 6.1 and 6.3 in [1]. Any Λ-metric space (X,d) has a topology with basis the open balls $B(x,r) = \{y \in X;\ d(x,y) < r\}$, where $x \in X$ and $r \in \Lambda$, $r > 0$, just as in the case $\Lambda = \mathbb{R}$. It follows at once from the Bridge Proposition [1; 2.17] that any closed subtree of a Λ-tree, in the sense of [1], is closed in this topology, but the converse is false in general. For example, let $\Lambda = \mathbb{Z} \times \mathbb{Z}$ with the lexicographic ordering, let $X = \Lambda$ with $d(x,y) = |x - y|$, and let $A = \{0\} \times \mathbb{Z}$. Then A is a subtree of X closed in this topology, but is not convex closed.

We use $[g,h]$ to denote the commutator $ghg^{-1}h^{-1}$; this should not cause confusion with the notation for segments. We begin with the analogue of [4; Cor. 2.4].

Proposition 7. *Let G be a group acting on a Λ-tree (X,d). The following are equivalent:*

(1) *The action is abelian.*

(2) *The hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$ where $\rho: G \to \Lambda$ is a homomorphism.*

(3) *For all $g, h \in G$, $\ell([g,h]) = 0$.*

Proof. It follows from Theorem 7.6 in [1] that (1) implies (2), and it is obvious that (2) implies (3), so we assume (3) and show the action is abelian. If the action is without inversions, the argument of Cor. 2.4 in [4], using 7.4 of [1] in place of 1.5 in [4], shows that for all $g, h \in G$, $A_g \cap A_h \neq \varnothing$, so the action is abelian by Theorem 7.4 in [1].

Suppose there is an inversion $h \in G$. Assume that $g \in G$ and $\ell(g) > 0$. By (7.1) in [1], there is an action of G on the Λ-tree $X' = \frac{1}{2}\Lambda \otimes_{\Lambda} X$, without inversions and with the same hyperbolic length function. By what has already been proved, $A'_g \cap A'_h \neq \varnothing$, where the dashes denote axes in X'. But by 6.3 in [1], $A'_h = \{p\}$, a single point, so $A'_{ghg^{-1}} = gA'_h = \{gp\}$. By 8.1 in [1],

$$\begin{aligned} \ell([g,h]) &= \ell(ghg^{-1}) + \ell(h^{-1}) + 2d(p, gp) \\ &= 2\ell(g) > 0 \end{aligned}$$

a contradiction. Thus $\ell(g) = 0$ for all $g \in G$ and the action on X is abelian. □

Next we have the analogue of Theorem 2.6 in [4]. Readers should be aware of the classification of isometries of Λ given in 2.5 of [1].

Proposition 8. *Let G be a group acting on a Λ-tree (X,d). The following are equivalent*:

(1) *The action is dihedral.*

(2) *The hyperbolic length function is given by $\ell(g)=|\rho(g)|$ for $g\in G$, where ρ: $G\to\mathrm{Isom}(\Lambda)$ is a homomorphism whose image contains a reflection and a non-trivial translation, and the absolute value signs denote hyperbolic length for the action of* Isom(Λ).

(3) *For all hyperbolic $g,h\in G$, $\ell([g,h])=0$, but there exist $g,h\in G$ such that $\ell([g,h])\neq 0$.*

Proof. Assume the action is dihedral. Again there is an induced action on $X'=\frac{1}{2}\Lambda\otimes_{\Lambda}X$ without inversions and with the same hyperbolic length function, so this action is also dihedral. By 7.15 in [**1**], the minimal invariant subtree for the action on X', which we denote by A, is linear. By 7.13 of [**1**], the action of G on A has the same hyperbolic length function as the action on X, and by 6.1(a)(ii) in [**1**], if $\ell(g)=0$, g has a fixed point in A. Since there is some $g\in G$ with $\ell(g)\neq 0$, A has at least two elements, so by 2.5 in [**1**], the action of G on A extends to an action as isometries on $\frac{1}{2}\Lambda$. If $\ell(g)>0$, g acts as a translation, either $x\mapsto x+\ell(g)$ or $x\mapsto x-\ell(g)$, and $\ell(g)\in\Lambda$. If $\ell(g)=0$, then g is a reflection with a fixed point, say c, so g: $x\mapsto 2c-x$, and $2c\in\Lambda$. Thus the action of G on $\frac{1}{2}\Lambda$ restricts to an action on Λ, still with the same hyperbolic length function, and it follows that (1) implies (2).

The rest of the proof is similar to that of Theorem 2.6 in [**4**] and details are omitted. As in proof of Prop. 7 use (7.4) in [**1**] for the proof that (3) implies (1). □

Finally, we have the analogue of Theorem 2.8 in [**4**].

Proposition 9. *Let G be a group acting on a Λ-tree (X,d). The following are equivalent*:

(1) *The action is of general type.*

(2) *There exist hyperbolic elements $g,h\in G$ such that $\ell([g,h])\neq 0$.*

(3) *There exist hyperbolic elements $g,h\in G$ such that $A_g\cap A_h$ is a segment of length less than $\ell(g)+\ell(h)$.*

(4) *G contains a free subgroup of rank 2 which acts freely and properly discontinuously on X.*

Proof. By the previous two propositions, (1) implies (2). Assume (2) and let g,h be hyperbolic elements of G such that $\ell([g,h])\neq 0$. If $A_g\cap A_h\neq\emptyset$, then arguing as in Case (5) of the Main Theorem, this intersection is a segment of length less than $\ell(g)+\ell(h)$. If $A_g\cap A_h=\emptyset$, then by 8.1 in [**1**], $A_{gh}\cap A_h$ is a segment and gh is hyperbolic. Also, $[gh,h]=[g,h]$, so again this segment is of length less than $\ell(gh)+\ell(h)$, hence (3) holds.

Assume (3), and take hyperbolic $g,h\in G$ with $A_g\cap A_h\neq\emptyset$ such that $\Delta=\Delta(g,h)<\ell(g)+\ell(h)$. We may assume g,h meet coherently and $\ell(h)\leq\ell(g)$. By Prop. 8.3(a) in [**1**], $\Delta':=\Delta(gh,g)=\Delta+\ell(g)$ and $\ell(gh)=\ell(g)+\ell(h)$ (and for this we need not assume the action of G is without inversions). Thus $\Delta'<2\ell(g)+\ell(h)\leq 3\ell(g)=\ell(g^3)$ and $2\ell(g)+\ell(h)<2\ell(gh)=\ell(gh)^2)$. Also, g^3 and $(gh)^2$ have the same axes as g and gh, so by Prop. 1, g^3

and $(gh)^2$ freely generate a free group of rank 2 which acts freely and properly discontinuously on X and (3) implies (4).

Finally, if g, h freely generate a free group acting freely and properly discontinuously on X, the action is of general type just as in the proof of Theorem 2.8 in [**4**], using Propositions 7 and 8 above. □

Condition (3) in Prop. 9 is not an exact analogue of the corresponding condition in Theorem 2.8 of [**4**], because of the condition on the length of the intersection. As noted in [**4**], if $A_g \cap A_h$ is a single point, we can replace g by gh and so we can add the condition that the intersection has positive length. However, we have not been able to remove the restriction that the length is less than $\ell(g)+\ell(h)$, because of the possibility that both $\ell(g^n)<\Delta$ and $\ell(h^n)<\Delta$ for all integers n.

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