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THE HEREDITY MEASURE OF AN ALGEBRA

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To Professor B.H. Neumann in honour of his eightieth birthday

The concept of the heredity measure of a semiprimary ring (or finite-dimensional algebra) is introduced and some of its elementary properties are studied.

0. INTRODUCTION

The concept of a quasi-hereditary algebra was introduced by Cline, Parshall and Scott [**1**] in connection with their study of highest weight categories arising in the representation theory of complex Lie algebras and algebraic groups. More recently, these algebras have been linked to the structure of the categories of perverse sheaves by Mirollo and Vilonen [**6**] and Ringel and the author [**3**]. Some of the basic features of these algebras can be found in [**2**]. In the present note, we define for every finite-dimensional algebra, or more generally, for every semiprimary ring A, a rational number $\mu(A)$ satisfying $0 \leqslant \mu(A) \leqslant 1$. We call it the heredity measure of A and formulate some of its elementary properties.

THEOREM 1. *Let A be a semiprimary ring. Then*

(i) $\mu(A) > 0$ *if and only if A is quasi-hereditary.*
(ii) $\mu(A) = 1$ *if and only if A is hereditary.*

THEOREM 2. *Let A and B be semiprimary rings. Then*

(i) $\mu(A \times B) = \mu(A)\mu(B)$;
(ii) $\mu(A) = 1/n \sum_{I \in \mathcal{I}} \mu(A/I)$, *where $\mathcal{I}$ denotes the set of all minimal heredity ideals of A and n is the number of non-isomorphic simple A-modules.*

THEOREM 3. *Given a rational number r, $0 < r < 1$, there is a finite-dimensional algebra A such that $\mu(A) = r$.*

In Section 1, we introduce the notation, the definition of the hereditary measure and some examples. The proofs of Theorems, together with additional remarks, will be given in Section 2.

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1. Preliminaries

Let A be a semiprimary ring, N its (Jacobson) radical and $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ a *fixed* complete set of primitive orthogonal idempotents of A. It is well-known that there is a bijective correspondence between the set of all idempotent (two-sided) ideals of A and the subsets of the set $\mathcal{E}$. Indeed, every idempotent ideal I' can be written in the form $I' = A\varepsilon A$ with an idempotent ε and the above mentioned correspondence can be expressed as follows

$$I' \longleftrightarrow \mathcal{E}' = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\} \subseteq \mathcal{E},$$

where $\mathcal{E}' = \{e \in \mathcal{E} \mid e\varepsilon \neq 0\}$ and $I' = A(e_{i_1} + e_{i_2} + \ldots + e_{i_t})A$. As a consequence, we can formulate the following statement.

Proposition. *There is a bijective correspondence between the set of all saturated chains of idempotent ideals of* A

$$(*) \qquad 0 = I_0 \subset I_1 \subset \ldots \subset I_{t-1} \subset I_t \subset \ldots \subset I_n = A, \; I_{t-1} \neq I_t \text{ for } 1 \leqslant t \leqslant n,$$

and the set of all permutations of $\mathcal{E}$.

Proof: The bijection is given as follows: The saturated chain $(*)$ corresponds to the permutation

$$(**) \qquad \left(e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(n)}\right),$$

where $I_t = A\left(e_{\pi(n-t+1)} + e_{\pi(n-t+2)} + \ldots + e_{\pi(n)}\right)A$.

Following [2], an idempotent ideal $J = A\varepsilon A$, $\varepsilon^2 = \varepsilon$, of A is said to be a *heredity ideal* if the endomorphism ring of the right module εA is semisimple and the (right) A-module $A\varepsilon A$ is projective. Furthermore, a chain of idempotent ideals

$$0 = J_0 \subset J_1 \subset \ldots \subset J_{t-1} \subset J_t \subset \ldots \subset J_\ell = A$$

is called a *heredity chain* if for every t, $1 \leqslant t \leqslant \ell$, J_t/J_{t-1} is a heredity ideal of A/J_{t-1}. Trivially, every idempotent ideal J' contained in a heredity ideal J is alone a heredity ideal, and thus every heredity chain can be refined to a *saturated heredity chain* of the form $(*)$, that is, such that no proper refinement of the chain is possible. The length of such a chain is n and, by the Proposition, an ordering $(**)$ of $\mathcal{E}$ is attached to it. □

Definition: A permutation $(**)$ of $\mathcal{E}$ such that

$$\left\{A\left(e_{\pi(n-t+1)} + e_{\pi(n-t+2)} + \ldots + e_{\pi(n)}\right)A \mid 1 \leqslant t \leqslant n\right\}$$

defines a heredity chain will be called a *heredity sequence* of A.

In view of the fact that there is a bijection between the heredity sequences and the heredity chains of A, the following concept is well-defined.

DEFINITION: Let $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ be a (fixed) complete set of primitive orthogonal idempotents of A. Denote by $h(A)$ the number of heredity sequences of A and by $\mu(A)$ heredity measure of A:

$$\mu(A) = \frac{h(A)}{n!}.$$

Thus, $\mu(A)$ is a rational number and $0 \leqslant \mu(A) \leqslant 1$.

Example 1. Let T_n be the (infinite dimensional) path algebra of the "complete" graph on n vertices $\{1, 2, \dots, n\}$; denote by α_{ij} the arrow from the vertex i to the vertex j. Let D_n, P_n and S_n be the deep, peaked and shallow algebras over this graph (see [4, 5])

$$D_n = T_n / \langle \alpha_{i_1 i_2} \alpha_{i_2 i_3} \dots \alpha_{i_{t-1} i_t} \mid (\exists\, r, s, 1 \leqslant r < s \leqslant t)\ (i_r = i_s)\ \&\ (i_j < i_r)(\forall\, r < j < s) \rangle,$$

$$P_n = T_n / \langle \alpha_{i_1 i_2} \alpha_{i_2 i_3} \dots \alpha_{i_{t-1} i_t} \mid (\exists\, r, 1 < r < t)(i_{r-1} > i_r < i_{r+1}) \rangle$$

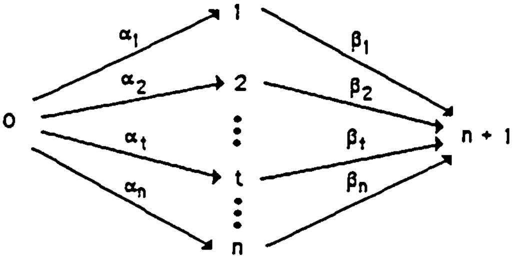
and

$$S_n = T_n / \langle \alpha_{i_1 i_2} \alpha_{i_2 i_3} \dots \alpha_{i_{t-1} i_t} \mid (\exists\, r, 1 < r < t)(i_r < \max(i_{r-1}, i_{r+1})) \rangle.$$

Let e_i be the (primitive) idempotent corresponding to the vertex $i, 1 \leqslant i \leqslant n$ and $\mathcal{E} = \{e_i \mid 1 \leqslant i \leqslant n\}$ for each of these algebras. Then, in each case, $(e_1, e_2, \dots, e_n)$ is the only heredity sequence. It follows that

$$\mu(D_n) = \mu(P_n) = \mu(S_n) = \frac{1}{n!}.$$

Example 2. Let A_n be the path algebra of the graph



over a field k, modulo the ideal generated by the element $\sum_{t=1}^{n} \alpha_t\beta_t$. Thus,

$$A_{nA_n} = \underbrace{\begin{matrix} & 0 & \\ & 1\,2\dots n & \\ (n+1) & \dots & (n+1)\end{matrix}}_{(n-1)-\text{times}} \oplus \begin{matrix}1\\(n+1)\end{matrix} \oplus \dots \oplus \begin{matrix}n\\(n+1)\end{matrix} \oplus (n+1).$$

Again, write $\mathcal{E} = \{e_0, e_1, \dots, e_n, e_{n+1}\}$. Clearly, $\dim_k A_n = 4n+1$. It is easy to verify that, for $n = 1$, (e_0, e_1, e_2), (e_1, e_2, e_0) and (e_2, e_1, e_0) are the only heredity sequences; thus $h(A_1) = 4$ and $\mu(A_1) = 2/3$. Assume, by induction, that

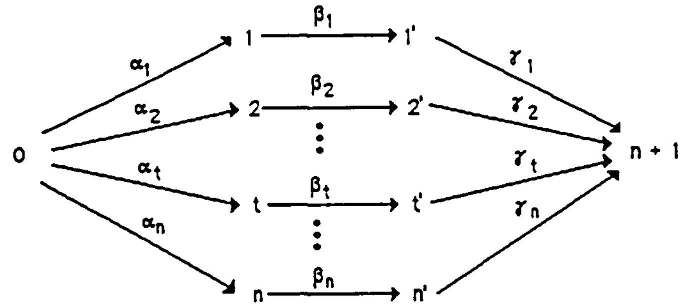
$$h(A_{n-1}) = (n-1)!(n-1)(n+2) \text{ and thus } \mu(A_{n-1}) = \frac{(n-1)(n+2)}{n(n+1)}.$$

Now, for A_n, there are $(n+1)!$ heredity sequences $(\dots, e_{n+1}), (n+1)!$ heredity sequences $(\dots, e_0)$ and, for each i, $1 \leqslant i \leqslant n$, $h(A_{n-1})$ heredity sequences $(\dots, e_i)$; consequently,

$$h(A_n) = 2(n+1)! + n\ h(A_{n-1}) = n!n(n+3),$$

which implies $\mu(A_n) = (h(A_n))/((n+2)!) = (n(n+3))/((n+1)(n+2))$, as required.

Example 3. Let B_n be the path algebra of the graph



over a field k, modulo the ideal generated by the element $\sum_{t=1}^{n} \alpha_t\beta_t\gamma_t$. We have

$$B_{nB_n} = \underbrace{\begin{matrix} & 0 & \\ & 1\,2\dots n & \\ & 1'2'\dots n' & \\ (n+1) & \dots & (n+1)\end{matrix}}_{(n-1)-\text{times}} \oplus \begin{matrix}1\\1'\\(n+1)\end{matrix} \oplus\dots\oplus \begin{matrix}n\\n'\\(n+1)\end{matrix} \oplus \begin{matrix}1'\\(n+1)\end{matrix} \oplus\dots\oplus \begin{matrix}n'\\(n+1)\end{matrix} \oplus (n+1);$$

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write $\mathcal{E} = \{e_0, e_1, \dots, e_n, e_{1'}, \dots, e_{n'}, e_{n+1}\}$. Clearly $\dim_k B_n = 8n+1$. As in the previous example, we are going to calculate $\mu(B_n)$ by induction. For $n = 1$, the heredity sequences are $(e_0, e_1, e_{1'}, e_2)$, $(e_0, e_{1'}, e_1, e_2)$, $(e_1, e_0, e_{1'}, e_2)$, $(e_1, e_{1'}, e_0, e_2)$, $(e_1, e_{1'}, e_2, e_0)$, $(e_1, e_2, e_{1'}, e_0)$, $(e_{1'}, e_0, e_1, e_2)$, $(e_{1'}, e_1, e_0, e_2)$, $(e_{1'}, e_1, e_2, e_0)$, $(e_{1'}, e_2, e_1, e_0)$, $(e_2, e_1, e_{1'}, e_0)$ and $(e_2, e_{1'}, e_1, e_0)$. Thus, $h(B_1) = 12$ and $\mu(B_1) = 1/2$. Assume that

$$h(B_{n-1}) = (2n)!\frac{n-1}{n}, \text{ and thus } \mu(B_{n-1}) = \frac{n-1}{n}.$$

Again, count the number of the heredity sequences of B_n ending in e_{n+1}, e_0, e_i and $e_{i'}$, $1 \leqslant i \leqslant n$. One can see easily that there are $(2n+1)!$ heredity sequences $(\dots e_{n+1}), (2n+1)!$ heredity sequences $(\dots e_0), (2n+1)h(B_{n-1})$ heredity sequences $(\dots e_i)$ and $(2n+1)h(B_{n-1})$ heredity sequences $(\dots e_{i'})$ for each $1 \leqslant i \leqslant n$. In the last two cases, we get all heredity sequences by taking a heredity sequence of the (canonically embedded) algebra B_{n-1} (on the subgraph with the vertices $0, 1, \dots, (n-1), 1', \dots, (n-1)', (n+1)$) and inserting the remaining idempotent $e_{i'}$, or e_i, into any of the possible $2n+1$ positions; this way, we get the numbers $(2n+1)h(B_{n-1})$. From here, it turns out that

$$h(B_n) = 2(2n+1)! + 2n(2n+1)h(B_{n-1}) = (2n+1)!2n$$

and consequently,

$$\mu(B_n) = \frac{h(B_n)}{(2n+2)!} = \frac{n}{n+1}.$$

2. Proofs

In this section, we present the proofs of Theorems formulated in the introduction.

Proof of Theorem 1: The statement (i) is trivial.

In order to prove (ii), assume first that A is hereditary. Since the quotient of a hereditary ring by an idempotent ideal is again hereditary, it is sufficient to show that, for every primitive idempotent e of A, AeA is a (minimal) heredity ideal. However, this is obvious.

To complete the proof, assume that $\mu(A) = 1$. Let $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A and write $P(i) = e_iA$. Observe that, due to the fact that every ideal Ae_iA, $1 \leqslant i \leqslant n$, is a heredity ideal, the set $\mathcal{E}$ is partially ordered by setting $e_i \prec e_j$ whenever $e_iNe_j \neq 0$, $1 \leqslant i, j \leqslant n$. Indeed, since there is no proper embedding of $P(i)$ into itself, $e_iNe_j \neq 0$ implies $e_jNe_i = 0$ for any i, $j (1 \leqslant i, j \leqslant n)$. Throughout the entire proof, we shall assume that the order $(e_1, e_2, \dots, e_n)$ is a completion of this partial order.

We are going to give an indirect proof. Recall that A is hereditary if and only if the radical e_iN of every indecomposable projective $P(i)$, $1 \leqslant i \leqslant n$, is projective.

Assume that A is a non-hereditary semiprimary ring with $\mu(A) = 1$ which is minimal in the sense that the quotient ring A/Ae_iA is hereditary for every i, $1 \leqslant i \leqslant n$. It follows immediately that e_1N is not projective; for, $Ae_1A = e_1A$ and thus, by minimality, all the remaining e_iN, $2 \leqslant i \leqslant n$, are projective. Moreover, $P(n)$ is simple; in fact, it turns out that $P(n)$ is the only projective module which is simple. For, if $Q = Q_2 \oplus Q_3 \oplus \ldots \oplus Q_n$ is a projective cover of e_1N, where Q_i denotes, for each i, $2 \leqslant i \leqslant n$, a (possibly zero) direct sum of copies of $P(i)$, then the kernel in the respective short exact sequence

$$0 \to K \to Q \to e_1N \to 0$$

is, by minimality, a semisimple homogeneous A-module. Thus, making use of the fact that all e_iN, $2 \leqslant i \leqslant n$, are projective, K is a direct sum of copies of $P(n)$ and there are no other simple projective A-modules $P(i)$, $i \neq n$.

Now, since every Ae_iA is a heredity ideal, we may write

$$e_1N = Q_2' + Q_3' + \ldots + Q_n' \text{ with } Q_i' \simeq Q_i,\ 2 \leqslant i \leqslant n.$$

Denoting the right ideal $Q_3' + \ldots + Q_n'$ by X and considering the quotient $\overline{A} = A/Ae_2A$, we observe that

$$e_1N/(e_1N \cap Ae_2A) = (Q_2' + X)/Q_2' \simeq X/(Q_2' \cap X);$$

for, there is no homomorphism from $P(2)$ into $P(i)$ for i, $3 \leqslant i \leqslant n$. Furthermore, clearly

$$Q_2' \cap X \subseteq e_1N^2,$$

and thus $\widetilde{Q} = Q_3 \oplus \ldots \oplus Q_n$ is a projective cover of the A-module X. Now, both $\widetilde{Q}$ and X are, in fact, $\overline{A}$-modules and, moreover, X is, by minimality, a projective $\overline{A}$-module. Thus, the respective short exact sequence

$$0 \to K' \to \widetilde{Q} \to X \to 0$$

is a split sequence of $\overline{A}$-modules. Hence, $\widetilde{Q}$ has a direct summand isomorphic to K' which is contained in rad $\widetilde{Q}$, and consequently $K' = 0$. This yields

$$Q_2' \cap X = 0;$$

therefore, by minimality, $X = Q_3' + \ldots + Q_n'$ is the direct sum $Q_3' \oplus \ldots \oplus Q_n'$, and thus

$$e_1N = Q_2' \oplus Q_3' \ldots \oplus Q_n',$$

a contradiction. The proof of Theorem 1 is completed. ∎

Remark. Let us point out that the property $\mu(A) = 1$ which means that any order of a complete set of primitive orthogonal idempotents defines a heredity chain is clearly equivalent to the property that every chain of idempotent ideals can be refined to a heredity chain. Hence, the above argument provides an alternative proof of Theorem 1 of [**2**].

Remark. It may be worthwhile to note that every hereditary semiprimary ring, and, more generally, every quotient of a hereditary semiprimary ring has a heredity sequence $(e_1, e_2, \dots, e_n)$ such that the respective Weyl modules

$$\Delta(i) = e_i A / e_i A(e_{i+1} + e_{i+2} + \dots + e_n)A$$

are all simple. In fact, the latter property is equivalent to the fact that the (oriented) graph of A has no cycles, that is, arrows define a partial order on the set of its vertices.

PROOF OF THEOREM 2: (i) If $\{e_{i_1}, e_{i_2}, \dots, e_{i_n}\}$ is a heredity sequence of A and $\{f_{j_1}, f_{j_2}, \dots, f_{j_m}\}$ a heredity sequence of B, then a sequence $\{g_1, g_2, \dots, g_{n+m}\}$ which is a permutation π of the sequence

$$(***) \qquad \{(e_{i_1}, 0), (e_{i_2}, 0), \dots, (e_{i_n}, 0), (0, f_{j_1}), (0, f_{j_2}), \dots, (0, f_{j_m})\}$$

preserving the order of e_i's as well as of f_j's is a heredity sequence of $A \times B$, and every heredity sequence of $A \times B$ is obtained in this way. In order to be more specific, let us formulate the phrase that π preserves the order of e_i's and f_j's explicitly: Every permutation π of $(***)$ defines two integral functions p from $\{r \mid 1 \leqslant r \leqslant n\}$ and q from $\{s \mid 1 \leqslant s \leqslant m\}$ to $\{t \mid 1 \leqslant t \leqslant n+m\}$ by

$$\pi(e_{i_r}) = g_{p(r)} \quad \text{and} \quad \pi(f_{j_s}) = g_{q(s)},$$

and these functions are required to be increasing. The number of the sequences obtained in this way is

$$h(A)h(B)\binom{m+n}{n};$$

here, we have made use of the elementary fact that the number of the subdivisions of a sequence of length m into $n+1$ segments is $\binom{m+n}{n}$. Now, we may easily complete the calculations:

$$\mu(A \times B) = \frac{h(A \times B)}{(m+n)!} = \frac{1}{(m+n)!} h(A)h(B)\frac{(m+n)!}{m!n!} = \mu(A)\mu(B).$$

(ii) If $I = Ae_iA$ is a minimal heredity ideal of A, then $h(A/I) = \mu(A/I) \cdot (n-1)!$ is the number of the hereditary sequences of A ending by e_i. Thus $h(A) = (n-1)! \sum_{I \in \mathcal{I}} \mu(A/I)$, where $\mathcal{I}$ is the set of all minimal heredity ideals of A, and therefore

$$\mu(A) = \frac{1}{n} \sum_{I \in \mathcal{I}} \mu(A/I).$$

Finally, let us turn our attention to Theorem 3; we are going to present a proof using the algebras A_n of Example 2 in Section 1, and then simplify it by using the algebras B_n of Example 3. ∎

PROOF OF THEOREM 3: Denote by R the set of the heredity measures of all (finite) products of algebras $A_n, n \geqslant 1$ (allowing repetitions). Thus, in view of Theorem 2, R is the set of all possible products of the numbers

$$r_n = \mu(A_n) = \frac{n(n+3)}{(n+1)(n+2)}.$$

We claim that $R = \{r \in \mathbb{Q} \mid 0 < r < 1\}$. Indeed, firstly

$$r_n = \frac{q_n - 1}{q_n} \quad \text{with} \quad q_n = \frac{(n+1)(n+2)}{2}.$$

Secondly, if $s - 1/s \in R$, then

$$\left(\forall t \leqslant \frac{s}{2} - 1\right)\left(\frac{s-2t-1}{s-2t} = r_{s-2t-1} r_{s-2t+1} \dots r_{s-3} \frac{s-1}{s} \in R\right).$$

Now, since $q_{n+1} - q_n = n+2$, there are both even and odd numbers amongst any three consecutive q_n, q_{n+1}, q_{n+2}, and thus $(s-1)/s \in R$ for all $s \geqslant 2$. Finally, if $0 < p < q$,

$$\frac{p}{q} = \frac{p}{p+1} \cdot \frac{p+1}{p+2} \cdots \frac{q-2}{q-1} \cdot \frac{q-1}{q},$$

and we conclude that any rational number r, $0 < r < 1$, belongs to R. ∎

Remark. We might have used the algebras B_n of Example 3 to facilitate a shorter (and more explicit) proof of Theorem 3. Again, using Theorem 2,

$$\mu(B_p \times B_{p+1} \times \dots \times B_{q-2} \times B_{q-1}) = \frac{p}{q}$$

for any $0 < p < q$. Hence, if r is a rational number, $0 < r < 1$, then writing $r = p/q$, $(p,q) = 1$, there is a k-algebra A whose heredity measure $\mu(A) = r$ and

$$\dim_k A = \prod_{t=p}^{q-1} (8t+1).$$

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