ON THE LATTICE OF σ-ALGEBRAS

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1. Introduction. In this paper I consider the relations between the lattice of topologies on a fixed space and the lattice of σ-algebras on that space. It is found that the intersection of these two lattices is the lattice of complete Boolean algebras, and that this lattice is anti-atomically generated. Some sufficient conditions for a topology to contain a maximal σ-algebra are noted.

2. Preliminary remarks. Fröhlich (**1**) characterized the ultraspaces (anti-atoms) in the lattice Σ of topologies as being of the form $G(x, U) = P(X\backslash\{x\}) \cup U$, where U is an ultrafilter on X other than the principal ultrafilter at x. If U is a principal ultrafilter at some other point, say $U(y)$, then $G(x, U(y))$ is called a principal ultraspace. A principal topology is a topology which is the intersection of principal ultraspaces. Steiner (**7**) characterized principal topologies as those which have a base of open sets minimal at each point, or equivalently, those which are closed under arbitrary intersection.

On a finite space, every topology is principal. Hence, it is possible to discuss any topology on a finite space by explicitly exhibiting its base of open sets minimal at each point.

An $n \times n$ "K-matrix" is a 0-1 matrix (a_{ij}) such that

$$a_{ii} = 1 \quad \text{for all } i,$$
$$a_{ij} = 1 \Rightarrow [(a_{jk} = 1) \Rightarrow (a_{ik} = 1)] \quad \text{for all } i, j.$$

Krishnamurthy (**3**) established a one-to-one correspondence between these matrices and the topologies on a space of n points. Indeed, for each $x_i \in X$, the ith row of the K-matrix corresponds to the minimal open set containing x_i. This minimal open set contains x_j if $a_{ij} = 1$. In (**5**), it was shown that those symmetric with respect to the main diagonal correspond to the σ-algebras.

3. Complete Boolean algebras.

THEOREM 1. *Every σ-algebra over X is a complete Boolean algebra if and only if X is countable.*

Proof. If X is uncountable, let B be the countable, co-countable sets of X;

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i.e., $B = \{T: T \text{ is countable or } X\backslash T \text{ is countable}\}$. It is easy to check that B is a σ-algebra, but is not closed under arbitrary unions.

For the converse, the only point perhaps not immediately obvious is that an uncountable σ-algebra, B, over a countably infinite set is closed under arbitrary unions. Let $\{B_\alpha | \alpha \in \Lambda\}$ be an arbitrary family of sets in B. However, $\bigcup_\Lambda B_\alpha \subseteq X$, thus,

$$\bigcup_{\Lambda} B_\alpha = \bigcup_{i=1}^{\infty} \{p_i\}, \qquad p_i \in X.$$

For each $i = 1, 2, \ldots$, choose one B_i in the family for which $p_i \in B_i$. Then

$$\bigcup_{i=1}^{\infty} \{p_i\} \subseteq \bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{\Lambda} B_\alpha.$$

Since B is closed under countable unions, the result follows.

THEOREM 2. *The lattice of complete Boolean algebras is precisely the intersection of the lattice of σ-algebras with the lattice of topologies.*

The proof is immediate from De Morgan's laws.

COROLLARY. *The complete Boolean algebras on X are principal topologies.*

Proof. The complete Boolean algebras on X are precisely the closed-open topologies on X. Hence, in them, arbitrary intersections of open sets are open. However, this characterizes the principal topologies.

4. A generating map of σ-algebras.

Every topology can be generated by a Kuratowski closure operator from $P(X)$ to $P(X)$ (**2**). By analogy with topology, we ask about operators from $P(X)$ to $P(X)$ which generate σ-algebras. The method of outer measures is close to this approach: If $m^*: P(X) \to \{\text{real numbers}\}$, such that $m^*(\emptyset) = 0$, and m^* is monotone and subadditive, then $B = \{T: \forall A \subseteq X, m^*(A) \geqq m^*(A \cap T) + m^*(A\backslash T)\}$ is a σ-algebra on X (**4**, p. 87).

Partial success is given by the following approach. Let the complementation operator $C: P(X) \to P(X)$ be given by $C(A) = X\backslash A$.

Definition. Let $k: P(X) \to P(X)$ be such that for all A, B in $P(X)$,
(1) $A \subseteq k(A)$;
(2) $k(A) = k^2(A)$;
(3) $k(A \cup B) = k(A) \cup k(B)$;
(4) $k \circ C = C \circ k$.

THEOREM 3. *If $B = \{A: k(A) = A\}$, then B is a complete Boolean algebra.*

Proof. From (1) and (4), X and $\emptyset$ are in B. Since $k = C \circ k \circ C$, if $k(A) = A$, then $C \circ k \circ C(X\backslash A) = X\backslash A = k(X\backslash A)$. Therefore, $A \in B$ if and only if $X\backslash A \in B$. Thus, k is a Kuratowski closure operator, and B is a closed-open topology. However, these are the complete Boolean algebras.

COROLLARY. *On a countable space, every σ-algebra can be so obtained.*

5. Topological σ-algebras and topologies from σ-algebras. Let Σ be the lattice of topologies and Δ the lattice of σ-algebras over a fixed space X. Suppose that $\tau: \Sigma \to \Delta$ is given by $\tau(T) = \bigcap\{B \in \Delta: T \subseteq B\}$.

Notice that τ comes close to being a Kuratowski operator.

THEOREM 4. (a) $\tau(\emptyset, X) = \{\emptyset, X\}$.

(b) *$T \subseteq \tau(T)$ and $T = \tau(T)$ if and only if T is a complete Boolean algebra.*

(c) *If $T_1 \subseteq T_2$, then $\tau(T_1) \subseteq \tau(T_2)$.*

(d) *If $\tau(T)$ is a complete Boolean algebra, then $\tau^2(T) = \tau(T)$.*

(e) $\tau(T_1 \wedge T_2) \subseteq \tau(T_1) \wedge \tau(T_2)$.

(f) $\tau(T_1 \vee T_2) \supseteq \tau(T_1) \vee \tau(T_2)$.

The proofs are immediate. Note that in (d), it suffices that the space be countable. To see, e.g., that (e) is sharp, let T_1 and T_2 be the two Sierpiński (proper) topologies on a space of two points.

It is equally easy to define a map from Δ to Σ in exactly the same manner.

Definition. Let $\mu: \Delta \to \Sigma$ be given by $\mu(B) = \bigcap\{T \in \Sigma: B \subseteq T\}$. Since for any B in Δ, $B \subseteq P(X) \in \Sigma$, this is well-defined.

THEOREM 5. *The same results hold as in Theorem* 4, *when Δ and Σ are interchanged, and τ is replaced by μ everywhere.*

There are several unanswered questions concerning these maps, e.g., in parts (e) and (f) of Theorem 4, under what conditions will equality hold? Furthermore, since $P(X)$ is a complete Boolean algebra and the intersection of any family of complete Boolean algebras is a complete Boolean algebra, every topology T is contained in a unique smallest complete Boolean algebra, $A(T)$. For what T in Σ will the ascending chain:

$$T \to \tau(T) \to \mu\tau(T) \to \ldots \to A(T)$$

terminate in a finite number of steps? For atomic topologies, ultraspaces, door spaces, and T_1 topologies (see Theorem 17 below), the chain terminates in not more than two steps. What about, e.g., extremally disconnected spaces?

6. Components. For each point p in the topological space (X, T), let $C(p)$ be the T-component of p. Note that each component is closed and

$$\{C(p): p \in X\}$$

is a partition of X.

Definition. (a) Let $C[T]$ be the σ-algebra generated by $\{C(p): p \in X\}$. Note: $C[T] \subseteq \tau(T)$.

(b) The "component topology" T_c of T is given by the base

$$\{\emptyset, C(p): p \in X\}.$$

Note: $T_c \subseteq \mu C[T]$.

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THEOREM 6. *If T has a finite number of components, or if T is locally connected, then $C[T] \subseteq T$.*

Proof. In either case, each $C(p)$ in $C[T]$ is T-open as well as T-closed.

COROLLARY. *In these cases, T is a complete Boolean algebra if and only if $C[T] = \tau[T]$.*

THEOREM 7. *If $C[T] \subseteq T$ and T has at most a countable number of components, then $C[T]$ is the largest σ-algebra contained in T.*

Proof. Let $B \in \Delta$ and $B \subseteq T$. Then each $H \in B$ is closed and open in T. Let $p \in H$. Claim: $C(p) \subseteq H$. For suppose not. Then $H \cap C(p) \neq \emptyset$ and is closed and open in $C(p)$. Moreover, $(X \backslash H) \cap C(p) \neq \emptyset$ and is closed and open in $C(p)$. These disconnect $C(p)$, a contradiction. Hence, $H = \bigcup\{C(p): p \in H\}$. Since this is a countable union, $H \in C[T]$. Hence, $B \subseteq C[T]$.

COROLLARY 1. *If $C[T]$ is the largest σ-algebra contained in T, then T and T_c are Borel-equivalent* (see § 8 below).

Proof. Since $C[T] \subseteq T$, then every component of T is open in T. Indeed, $C[T] \subseteq \mu C[T] = T_c \subseteq T \subseteq \tau(T)$.

Notice that if T is connected or if X is finite, then T has only a finite number of components.

COROLLARY 2. *If T is a connected topology, then it contains no proper σ-algebras.*

In (**5**), it was shown that if X is finite and T has K-matrix (a_{ij}), then $\tau(T)$ has K-matrix (b_{ij}), where $b_{ij} = 1$ if and only if $a_{ij} = a_{ji} = 1$.

COROLLARY 3. *If X is finite and T has K-matrix (a_{ij}), then $C[T]$ has K-matrix (c_{ij}), where $c_{ij} = 1$ if and only if $a_{ij} = 1$ or $a_{ji} = 1$.*

7. Atoms and anti-atoms in Δ.

THEOREM 8. *Δ is an atomically generated lattice.*

Proof. Let $\{\emptyset, X\} \neq B \in \Delta$. If $\emptyset \neq A \in B$, then clearly

$$B_A = \{\emptyset, X, A, X \backslash A\} \subseteq B$$

and is atomic. Claim: $B = \vee\{B_A: \emptyset \neq A \in B\}$. For $\bigcup B_A \subseteq B$, thus $\vee B_A \subseteq B$. If $\emptyset \neq C \in B$, then $C \in B_C \subseteq \vee B_A$. Therefore, $B \subseteq \vee B_A$.

THEOREM 9. *If B is a proper complete Boolean algebra over X, then B must miss at least two singletons.*

Proof. Suppose the contrary. Suppose that $\{p\}$ is the only singleton not in B. Then B contains all other singletons, hence contains their union, and also the complement of that union, namely, $\{p\}$. Hence, B contains all singletons, and is closed under arbitrary union. But then B must be $P(X)$, contradicting B proper.

COROLLARY. *If X is a countable space, and B is a proper σ-algebra over X, then B must miss at least two singletons.*

Definition. Suppose $\operatorname{card}(X) \geqq 3$ and that $p \neq q$ in X. Suppose that $\{p, q\} \subseteq A$. Then we let $G(p, q, U(A)) = P(X \backslash \{p, q\}) \cup U(A)$, where $U(A)$ is the family of all subsets of X which contain A. If $A = \{p, q\}$, write $G(P, q, U(p, q))$.

THEOREM 10. *$G(p, q, U(A))$ is a principal topology.*

Proof. $X \in U(A)$ and $\emptyset \in P(X \backslash \{p, q\})$. Clearly, each of P and U are closed under arbitrary unions and arbitrary intersections. Let $D \in P$ and $E \in U$. Then $D \cup E \in U$ and $D \cap E \subseteq D$, thus $D \cap E \in P$. Hence, G is closed under arbitrary unions and arbitrary intersections.

THEOREM 11. *The σ-algebras generated by the $G(p, q, U(A))$ are of the form $G(p, q, U(p, q))$, and are anti-atomic in the lattice of complete Boolean algebras.*

Proof. It follows from the definition that $G(p, q, U(p, q))$ is a σ-algebra, and contains $G(p, q, U(A))$. Notice that for any set B, $\{p, q\} \subseteq B \subseteq A$, we see that $X \backslash B \in P(X \backslash \{p, q\}) \subseteq G(p, q, U(A))$. Therefore, the smallest family containing $G(p, q, U(A))$ and closed under complementation is $G(p, q, U(p, q))$. Hence, this is the generated σ-algebra.

Now suppose that B is a complete Boolean algebra and

$$G(p, q, U(p, q)) \subset B \subseteq P(X)$$

(in this paper, $\subset$ denotes proper containment). However,

$$P(X) \backslash G(p, q, U(p, q)) = \{H: H \subseteq X, p \in H \text{ and } q \notin H\} \cup \{K: K \subseteq X, q \in K \text{ and } p \notin K\}.$$

Thus, B must contain at least one of these, say H with $p \in H$. Since $H \backslash \{p\} \in P(X \backslash \{p, q\})$, we find $\{p\} \in B$. Therefore, B contains all but at most one singleton, and $B = P(X)$.

COROLLARY. *If X is countable, the σ-algebras generated by the $G(p, q, U(A))$ are anti-atomic in Δ.*

LEMMA. $G(p, q, U(p, q)) = G(p, U(q)) \cap G(q, U(p))$.

The proof is immediate upon expanding by the definitions. Notice that $G(p, U(q))$ and $G(q, U(p))$ are principal ultraspaces in Σ.

THEOREM 12. *The lattice of complete Boolean algebras is anti-atomically generated.*

Proof. By the corollary to Theorem 2, if B is a complete Boolean algebra, then it is a principal topology. Hence, $B = \wedge\{G(x, U(y)) : B \subseteq G(x, U(y))\}$. Claim: if $B \subseteq G(x, U(y))$, then $B \subseteq G(y, U(x))$, thus

$$B \subseteq G(x, U(y)) \cap G(y, U(x)) = G(x, y, U(x, y)).$$

For, let $T \in B$. Then $X \backslash T \in B$, thus $x \notin X\backslash T$ or $y \in X\backslash T$. Hence, $x \in T$ or $y \notin T$, and $T \in G(y, U(x))$.

COROLLARY. *On a countable space, the lattice Δ of σ-algebras is anti-atomically generated.*

THEOREM 13. *If* card$(X) = n$, *then Δ has precisely $n(n-1)/2$ anti-atoms.*

Note the anti-atom $G(x_i, x_j, U(x_i, x_j))$ has K-matrix (a_{km}), where $a_{kk} = 1$, $a_{ij} = a_{ji} = 1$, and $a_{km} = 0$ otherwise. Call a matrix of this type an *i-j* elementary K-matrix. It is easy to decompose a symmetric K-matrix as the meet of its *i-j* elementary K-matrices; e.g.,

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

8. A sublattice of Δ. In § 5, given a topology T on X, $\tau(T)$ was defined to be the generated σ-algebra. In a previous paper (**5**), I defined two topologies to be "Borel-equivalent", $T_1 \sim T_2$, if $\tau(T_1) = \tau(T_2)$.

It follows from a result of Sharp (**6**), that on a finite space, columns in a K-matrix can be interpreted as the closures of the points. In (**5**), it was shown that if (a_{ij}) is a K-matrix, then $|a_{ij}| = 0$ or 1, and $|a_{ij}| = 1$ if and only if (a_{ij}) generates the K-matrix of the power set $P(X)$.

Noting that distinct points have distinct closures with respect to a topology over a finite space if and only if the K-matrix of the topology has pairwise distinct columns, we deduced that over a finite space, topology T is Borel-equivalent to $P(X)$ if and only if T is T_0.

My thanks are due to the referee for his remark that the same result holds true if the space is countable, and indeed obtain a more general result.

THEOREM 14. *Over an arbitrary space, if $T \sim P(X)$, then T is T_0.*

Proof. Suppose that T is not T_0. Then there are distinct points p and q such that each open set containing p contains q and vice versa. Consider the σ-algebra $G(p, q, U(p, q))$. Plainly, $T \subseteq G(p, q, U(p, q))$, hence

$$T \subseteq \tau(T) \subseteq G(p, q, U(p, q)) \subset P(X).$$

Hence, $\tau(T) \neq P(X)$ and T and $P(X)$ are not Borel equivalent.

COROLLARY. *If X is countable, then $T \sim P(X)$ if and only if T is T_0.*

Proof. Suppose that T is T_0 and let $p \in X$. Write $X\backslash\{p\} = \{q_n: n = 1, 2, \ldots\}$. Then for each n, there exists U_n (which is either open or closed in T) such that $p \in U_n$ and $q_n \notin U_n$. Note that $U_n \in \tau(T)$. Hence, $\bigcap_{n=1}^{\infty} U_n = \{p\} \in \tau(T)$. This is true for each point of X, thus $\tau(T)$ contains all singletons and is closed under countable unions. Thus, $\tau(T) = P(X)$ and $T \sim P(X)$.

Note that the converse of the corollary is also true. If the space is uncountable, then there are certainly T_0 topologies which do not generate $P(X)$.

Definition. Let S_B be the family of all singletons in a σ-algebra B, let $S = S_{P(X)}$, and let W be the family of all σ-algebras B for which $S_B = S$.

THEOREM 15. *W is a complete sublattice of Δ, with the countable, co-countable σ-algebra as 0-element, and $P(X)$ as* I*-element.*

Proof. Clearly, if $B \in W$, then the countable, co-countable σ-algebra is contained in B. Let $B, C \in W$. Then $S \subseteq B$ and $S \subseteq C$, hence, $S \subseteq B \cap C = B \wedge C$ and $S \subseteq B \cup C \subseteq B \vee C$. This argument clearly holds for arbitrary families of elements of W.

COROLLARY. *On a countable space, the only element of W is $P(X)$.*

THEOREM 16. *If T is a T_1 topology, then $\tau(T) \in W$. Moreover, the 0-element of W is generated by the 0-element of the lattice of T_1 topologies, namely the co-finite topology.*

THEOREM 17. *If T is a T_1 topology, then $\mu_\tau(T) = P(X)$.*

Proof. If T is T_1, then each point is closed. Hence, every singleton is an element of $\tau(T)$, and so of $\mu_\tau(T)$. However, $\mu_\tau(T)$ is a topology. Thus, $\mu_\tau(T) = P(X)$.

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