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HOMOGENEOUS STRUCTURES ON KÄHLER SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACES*

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In this paper we give a differential characterization of homogeneous Kähler submanifolds of complex projective spaces in terms of the existence of a tensor field, the homogeneous structure S. We show that for any $m \in M$, S_m determines a unitary representation whose orbit at m is a compact, complete Kähler submanifold which extends M. We consider the $U(n) \times U(N-n)$ $(n = \dim_{\mathbb{C}} M)$ module of the space of these tensors and we find its irreducible factors.

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1. Introduction

Let M be a Kähler submanifold of $\mathbb{CP}^N$. We shall denote by TM and νM the tangent and normal bundle of M, respectively; by ∇ and $\nabla^\perp$ the Levi-Civita and the normal connection on M. α will be the second fundamental form of M. We let J be both the complex structure on M and on $\mathbb{CP}^N$ (cf. Section 2 below). We denote by ξ the bundle $TM \oplus \nu M$, i.e. the pull-back on M of the tangent bundle of $\mathbb{CP}^N$. The complex structure of $\mathbb{CP}^N$ induces on M a tensor field $J \in TM^* \otimes \xi^* \otimes \xi$.

Following **[24]** and **[16]** (cf. also **[5]**), we introduce the notion of *homogeneous structure* on M.

Definition. A *homogeneous structure* on M is a tensor field $S \in TM^* \otimes \xi^* \otimes \xi$ such that

(1) TM and νM are parallel subbundles of ξ with respect to the connection on ξ given by $\tilde{\nabla} := \nabla \oplus \nabla^\perp - S$.

(2) $\tilde{\nabla}$ is a metric connection.

(3) $\tilde{\nabla} J = 0$.

(4) $\tilde{\nabla} \alpha = 0$.

(5) $\tilde{\nabla} S = 0$.

Our main result is the following differential characterization of homogeneous Kähler submanifolds.

Theorem A. *A connected Kähler submanifold M of $\mathbb{CP}^N$ is an open subset of a*

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globally homogeneous Kähler submanifold of $\mathbb{CP}^N$ *if and only if it admits a homogeneous structure* S.

The proof will be given in Sections 3 and 4.

Note that all examples of homogeneous Kähler submanifold of $\mathbb{CP}^N$ can be obtained by means of the Borel–Weil construction ([2], see also Section 2).

Theorem A plays the same role, in the intrinsic case, of a Theorem of K. Sekigawa [20]. Actually the restriction of $\tilde{\nabla}$ to TM is an Ambrose–Singer connection on M compatible with the complex structure J. An analogue of Theorem A, for submanifolds of $\mathbb{R}^N$ was stated by C. Olmos in [16].

A particular case of Theorem A was proved by H. Nakagawa and R. Takagi in [14] for Hermitian symmetric Kähler submanifolds of $\mathbb{CP}^N$. These spaces are characterized by the fact that they admit the null tensor as homogeneous structure. Nakagawa and Takagi also classified these submanifolds.

If S is a homogeneous structure on a Kähler submanifold of $\mathbb{CP}^N$ and $m=[p]\in M$, the triple (S_m, α_m, J_m) determines the submanifold uniquely up to isometries (see Theorem 4.1). Hence, a classification of the tensors which can arise as homogeneous structures gives rise to a classification of the extrinsic geometry of the homogeneous Kähler submanifolds of $\mathbb{CP}^N$. To this aim, in Section 5, we will study the space of tensors with the same symmetries as S_m. Set $V:=T_mM$, $W:=v_mM$, if $n=\dim V$ and $h=\dim W$, then one has a (canonically defined) action of $U(n)\times U(h)$ on this space. This $U(n)\times U(h)$ module will be denoted by $\mathscr{D}(V,W)$. We split $\mathscr{D}(V,W)$ into its irreducible components. This decomposition is done following the methods of S. M. Salamon ([19, Chapter 3]; see also [6]). These methods, compared with the ones of Weyl [25] have the advantage that one does not need to prove the irreducibility of the various components, thus avoiding the computation of the quadratic invariants.

Theorem B. *The* $U(n)\times U(h)$ *module* $\mathscr{D}(V,W)$ *splits as*

$$\mathscr{D}(V,W)\cong\mathscr{T}_+(V)\oplus\mathscr{N}(V,W),$$

where $\mathscr{T}_+(V)$ *and* $\mathscr{N}(V,W)$ *are respectively a* $U(n)$ *module and a* $U(n)\times U(h)$*-module which correspond to the* V*-component and the* W*-component of a tensor in* $\mathscr{D}(V,W)$. *Moreover* $\mathscr{T}_+(V)$ *and* $\mathscr{N}(V,W)$ *have the following decomposition into irreducible factors*

$$\mathscr{T}_+(V)\cong 2[\![\lambda_V^{1,0}]\!]\oplus[\![B_V]\!]\oplus[\![(\lambda_V)_0^{2,1}]\!]$$

$$\mathscr{N}(V,W)\cong([\![\lambda_V^{1,0}]\!]\otimes[(\lambda_W)_0^{1,1}])\oplus([\![\lambda_V^{1,0}]\!]\otimes\mathbb{R}_W).$$

We refer to Section 5 (cf. also [6] and [7]) for the definitions of $[\![\lambda_V^{1,0}]\!]$, $[\![B_V]\!]\dots$

Theorem B will be proved in Section 5 as a consequence of Theorems 5.1 and 5.2.

Note that $\mathscr{T}_+(V)$ is the Kähler part in the decomposition of homogeneous structures on almost Hermitian manifolds obtained by E. Abbena and S. Garbiero in [1]. More precisely, the space of homogeneous structures on almost Hermitian manifolds $\mathscr{T}(V)$ splits into $\mathscr{T}_+(V)\oplus\mathscr{T}_-(V)$, where $\mathscr{T}_+(V)$ corresponds to the Kähler structures. We remark that the components in $\mathcal{T}_+(V)$ we obtain here agree with the ones in [1] (there is just a different notation).

The component $\mathcal{N}(V, W)$ obviously comes from the existence of a normal space.

The homogeneous Kähler submanifolds admitting a homogeneous structure in $\mathcal{N}(V, W)$ will be characterized in Section 6.

Moreover, in Section 6 some applications and basic examples will be given. By Lemma 3.1 the homogeneous structure and the second fundamental form at m of an orbit $G\cdot m$ of a unitary representation can be determined in a simple algebraic way. In particular, some geometrical properties of the orbit can be read from the weight lattice of the representation.

2. Preliminaries

Let $i:(M,g,J_M)\to(\mathbb{CP}^N(c),\bar{g},J)$ be a Kähler submanifold. We denote by g and $\bar{g}$ respectively the Kähler metrics on M and the Fubini-Study metric on $\mathbb{CP}^N(c)$ (complex projective space with constant holomorphic sectional curvature c), and by $J_M(J)$ the complex structures of $M(\mathbb{CP}^N)$. Let $\nabla^M(\nabla^{\mathbb{CP}^N})$ be the Levi-Civita connection on $M(\mathbb{CP}^N)$. Then

$$\begin{gathered} i^*\bar{g}=g, \qquad i^*J=J_M, \\ J\nabla^{\mathbb{CP}^N}=\nabla^{\mathbb{CP}^N}J \text{ which implies } \begin{cases} J_M\nabla^M=\nabla^M J_M, \\ \alpha(X,J_M Y)=J\alpha(X,Y), \end{cases} \end{gathered} \tag{2.1}$$

where X, Y are vector fields on M and α is the second fundamental form of M.

To simplify the notation, in view of (2.1), we denote by $\langle\,,\rangle$ both the Kähler metric of M and the Fubini-Study metric on $\mathbb{CP}^N$ and by the same letters the complex structures on M and $\mathbb{CP}^N$.

The rigidity theorem of E. Calabi [3] plays a fundamental role in the study of Kähler submanifolds of $\mathbb{CP}^N$.

Theorem 2.1. (Calabi's Rigidity Theorem). *Let $f:M\to\mathbb{CP}^N(c)$ and $f':M\to\mathbb{CP}^{N'}(c)$ be two full Kähler immersions of the same Kähler manifold M. Then $N=N'$ and there exists a unique holomorphic isometry Φ of $\mathbb{CP}^N$ such that $\Phi f=f'$.*

As a straightforward corollary, any homogeneous Kähler submanifold is extrinsic homogeneous. Indeed, if M is homogeneous and G is a Lie group acting transitively on M as a group of isometries, any $g\in G$ extends to a unique holomorphic isometry of $\mathbb{CP}^N$. Hence M is an orbit of a representation of G in the isometry group of $\mathbb{CP}^N$.

There is a classical construction due to Borel and Weil (cf. [2]) which provides all examples of homogeneous Kähler submanifolds of $\mathbb{CP}^N$ (cf. [23] and Theorem 2.2 below). Here we sketch such a construction.

Let G be a compact semisimple Lie group and Λ a suitable (see [23]) linear combination of the fundamental weights of G. Let ρ be the irreducible representation of

G whose highest weight is Λ. Denote by V the eigenspace of ρ corresponding to Λ. Since $\dim_{\mathbb{C}} V=1$, V determines a point $[V]\in\mathbb{CP}^N$. The orbit $M=G\cdot[V]$ is a compact homogeneous, simply connected, Kähler submanifold of $\mathbb{CP}^N(c)$. The same construction can be done equivalently starting from a simply connected complex simple Lie group G' (see [**8**]; the connection between the two approaches is that the Lie algebra of G is a compact real form of the Lie algebra of G').

For example, if $G=SU(n)$ ($G'=SL(n,\mathbb{C})$) one obtains embeddings of the complex Grassmannian $G(k,n)$ of the k dimensional subspaces of $\mathbb{C}^n$. Plücker embeddings of the Grassmannian and the Veronese embedding also arise in the same way (for more details see [**8**, Section 23.3]).

Theorem 2.2. [**23**]. *Let $f:M\to\mathbb{CP}^N(c)$ be a Kähler immersion of a globally homogeneous Kähler manifold M. Then*

(1) *M is compact and simply connected,*
(2) *f is an embedding,*
(3) *M is the orbit in $\mathbb{CP}^N$ of the highest weight in an irreducible unitary representation of a compact semisimple Lie group.*

Let $m=[p]$ be a point in $\mathbb{CP}^N$. We remark that $T_m\mathbb{CP}^N$ can be identified with the orthogonal complement $\langle m\rangle^{\perp}$ of the plane $\langle m\rangle$ in $\mathbb{C}^{N+1}\cong\mathbb{R}^{2N+2}$. Since the quotient map

$$\pi: S^{2N+1}\subset\mathbb{C}^{N+1}\cong\mathbb{R}^{2N+2}\to\mathbb{CP}^N$$

is a Riemannian submersion, using the fundamental equations of submersions [**18**], we have

Lemma 2.3. *Let $\nabla^{\mathbb{CP}^N}$ denote the Levi-Civita connection of $\mathbb{CP}^N$ (endowed with the Fubini-Study metric) and $\nabla^{\mathbb{R}^{2N+2}}$ the Levi-Civita connection of $\mathbb{R}^{2N+2}$ (endowed with the euclidean metric). Then*

$$\nabla^{\mathbb{CP}^N}_u Y=\nabla^{\mathbb{R}^{2N+2}}_{\tilde u}\tilde Y-\langle u, Y\rangle p+\langle u, JY\rangle Jp,$$

where $u\in T_m\mathbb{CP}^N$, $m=[p]=\pi(p)$, Y is a vector field on $\mathbb{CP}^N$ and $\tilde u$ and $\tilde Y$ are the horizontal lifts of u and Y respectively.

Throughout the paper we will always identify tangent vectors to $\mathbb{CP}^N$ with their horizontal lifts.

3. The canonical homogeneous structure

Let $\tilde M\to\mathbb{CP}^N$ be a homogeneous Kähler submanifold. As remarked in the previous Section, $\tilde M$ is the orbit of a point $m=[p]\in\mathbb{CP}^N$ in a representation $\rho: G\to U(N+1)$. We recall how one can define on $\tilde M$ a homogeneous structure S^c, which is canonical as soon as a reductive decomposition of the Lie algebra of G is given. It is known (cf. [**23**], [**9**]) that if $\tilde{M}$ is an almost Hermitian homogeneous manifold, then there exists a reductive decomposition of $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ (where $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup at m) compatible with the complex structure, i.e.

$$[\mathfrak{h},\mathfrak{m}]\subseteq\mathfrak{m},\quad \mathfrak{m}\cong T_m\tilde{M} \tag{3.1}$$

and (via the isomorphism above)

$$J\mathfrak{m}\subseteq\mathfrak{m}\quad\text{and}\quad ad_{\mathfrak{h}}J=Jad_{\mathfrak{h}}. \tag{3.2}$$

Imitating a construction due to K. Nomizu (cf. [**15**]) one can associate to this decomposition a *canonical connection* $\tilde{\nabla}^c$. The difference tensor $S^c=\nabla\otimes\nabla^{\perp}-\tilde{\nabla}^c$ will be called the *canonical homogeneous structure*. The connection $\tilde{\nabla}^c$ can be characterized by the fact that its geodesics through m are orbits of 1-parameter subgroups, i.e. $\gamma(t)=\exp(tx)\cdot m$, $x\in\mathfrak{m}$ and that the parallel displacement along the geodesics coincides with the differential of the action of $\exp(tx)$. The "only if" part of Theorem A is then straightforward, since one verifies readily that J, α and S^c are $\tilde{\nabla}^c$-parallel.

Given a representation $\rho: G\to U(N+1)$ and an orbit of ρ, $\tilde{M}=G\cdot m$, S^c_m and the value at m of the second fundamental form can be expressed in terms of the representation of Lie algebras which corresponds to ρ. If Y is a tangent vector to $\mathbb{CP}^N$ at m, let $Y(t):=(\exp(tx))_{*m}\cdot Y$ be the corresponding $\tilde{\nabla}^c$ parallel tangent vector field along γ. Since $\rho(\exp tx): \mathbb{C}^{N+1}\cong\langle m\rangle\oplus\langle m\rangle^{\perp}\to\mathbb{C}^{N+1}\cong\langle\gamma(t)\rangle\oplus\langle\gamma(t)\rangle^{\perp}$ is linear, we get

$$(\exp tx)_{*m}\cdot Y=(\exp tx)\cdot Y, \tag{3.3}$$

hence

$$x\cdot Y=\frac{d}{dt}_{|t=0}(\exp tx)\cdot Y=\frac{d}{dt}_{|t=0}(\exp tx)_{*m}\cdot Y.$$

If $\mathrm{proj}_{\langle m\rangle^{\perp}}$ denotes the projection on $\langle m\rangle^{\perp}$, using Lemma 2.3, we get

$$\begin{aligned}(S^c_m)_xY&=[\nabla^{\mathbf{CP}^N}_x(\exp tx)_{*m}\cdot Y-\tilde{\nabla}^c_x(\exp tx)_{*m}\cdot Y]_{|m}\\&=[\nabla^{\mathbf{CP}^N}_x(\exp tx)_{*m}\cdot Y]_{|m}=\mathrm{proj}_{\langle m\rangle^{\perp}}[\nabla^{\mathbf{R}^{2N+2}}_x(\exp tx)_{*m}\cdot Y]_{|m}\\&=\mathrm{proj}_{\langle m\rangle^{\perp}}\frac{d}{dt}_{|t=0}(\exp tx)_{*m}\cdot Y=\mathrm{proj}_{\langle m\rangle^{\perp}}(x\cdot Y).\end{aligned}$$

Next we compute the second fundamental form α_m of $\tilde{M}$ at m. Let $x,y\in T_m\tilde{M}\cong\mathfrak{m}$ (we denote with the same letter the elements of $T_m\tilde{M}$ and $\mathfrak{m}$). As above, $y(t):=\exp(tx)_{*m}\cdot y$ is a ($\tilde{\nabla}^c$ parallel) vector field along $\gamma(t):=\exp(tX)\cdot m$. Then

$$\alpha_m(x,y)=[\nabla^{\mathbf{CP}^N}_x y(t)]^{\perp}=[\mathrm{proj}_{\langle m\rangle^{\perp}}\nabla^{\mathbf{R}^{2N+2}}_{\tilde{x}}\tilde{y}(t)]^{\perp}=[x\cdot y]^{\perp}$$

where $^\perp$ denotes the projection on $v_m\tilde{M}$. We remark that the isomorphism (3.1) identifies (with abuse of notation) y with $y\cdot m$, so (cf. [11])

$$\alpha_m(x,y)=(x\cdot y\cdot m)^\perp.$$

Hence we have proved the following

Lemma 3.1. *With the same notation and assumptions as above,*

$$(S^c_m)_x Y=\mathrm{proj}_{\langle m\rangle^\perp}(x\cdot Y), \tag{3.4}$$

$$\alpha_m(x,y)=(x\cdot y\cdot m)^\perp, \tag{3.5}$$

where $x,y\in\mathfrak{m}\cong T_m\tilde{M}$, $Y\in T_m\mathbb{CP}^N$.

We remark that the restriction of S^c to TM determines a homogeneous structure on M, whose torsion is given by $T_xy=S^c_yx-S^c_xy$ (cf. [24]). By (3.4) it follows readily that

$$T_xy=-[x,y]\cdot m. \tag{3.6}$$

4. The Lie subalgebra associated with a homogeneous structure

Let $M\to\mathbb{CP}^N$ be a Kähler submanifold which admits a homogeneous structure S and denote by $\tilde{\nabla}$ the corresponding metric connection. Let $m\in M$ be fixed and consider a curve $\gamma(t)$, with $\gamma(0)=m$, $\dot{\gamma}(0)=x$. Denote by $\tau_{\gamma(t)}$ the isomorphism of $T_m\mathbb{CP}^N=T_mM\oplus v_mM$ into $T_{\gamma(t)}\mathbb{CP}^N=T_{\gamma(t)}M\oplus v_{\gamma(t)}M$ determined by the parallel displacement with respect to $\tilde{\nabla}$ along $\gamma(t)$. Let $\tilde{\gamma}(t)$ be the horizontal lift of $\gamma(t)$ (in the Riemannian submersion $\pi\colon S^{2N+1}\to\mathbb{CP}^N$) such that $\pi(p)=m$. We identify $\mathbb{C}^{N+1}$ with $T_m\mathbb{CP}^N\oplus\mathbb{R}\{p\}\oplus\mathbb{R}\{Jp\}$. For any t, there exists a unique unitary transformation $F_t\in U(N+1)$ such that

$$F_t(p)=\tilde{\gamma}(t),\quad F_t(Jp)=J\tilde{\gamma}(t),\quad (F_t)_{*m}=\tau_{\gamma(t)}.$$

This gives a one parameter subgroup of $U(N+1)$, or, in other words, a curve based at the identity in $U(N+1)$. Hence the tangent vector at I to the curve F_t is an element φ_x of the Lie algebra $\mathfrak{u}(N+1)$.

To simplify the notation, we denote by α, A and $\bar{S}$ the value at m of the second fundamental form, the shape operator and homogeneous structure, respectively. Using Lemma 2.3 and the fundamental equations of an immersion, a straightforward computation shows that φ_x is described by

$$\varphi_x\colon\begin{cases} p\mapsto & x\\ Jp\mapsto & Jx\\ v\mapsto & \bar{S}_xv+\alpha(x,v^{\mathrm{T}})-A_{v^\perp}x+\langle x,Jv\rangle Jp-\langle x,v\rangle p,\end{cases}$$

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where $v \in T_m\mathbb{CP}^N$ and v^{T} ($v^{\perp}$) is the orthogonal projection of v on T_mM (v_mM).

Let $\tilde{R}_{xy}$ be the curvature tensor of $\tilde{\nabla}$, computed at m. By the Ambrose-Singer holonomy theorem, the Lie algebra of the holonomy group of $\tilde{\nabla}$ is generated by the elements of $\mathfrak{u}(N+1)$ which act as follows

$$\tilde{R}_{xy}: \begin{cases} p \mapsto & 0 \\ Jp \mapsto & 0 \\ v \mapsto & \tilde{R}_{xy}v. \end{cases}$$

Moreover to the Kähler form of $\mathbb{CP}^N$ correspond the operators

$$\rho_{xy}: \begin{cases} p \mapsto & \langle x, Jy\rangle Jp \\ Jp \mapsto & -\langle x, Jy\rangle p \\ v \mapsto & \langle x, Jy\rangle Jv. \end{cases}$$

The operators $\varphi_x, \rho_{yz}, \tilde{R}_{ut} (x, y, z, u, t \in T_mM)$ span a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{u}(N+1)$. Indeed, the Gauss, Ricci and Codazzi equations and the definition of homogeneous structure imply that the Lie brackets of these operators are:

$$[\varphi_u, \varphi_v] = \varphi_{\bar{S}_{uv} - \bar{S}_{vu}} + \tilde{R}_{uv} + 2\rho_{uv}, \tag{4.1}$$

$$[\tilde{R}_{uv}, \varphi_z] = \varphi_{\tilde{R}_{uv}z}, \tag{4.2}$$

$$[\tilde{R}_{uv}, \tilde{R}_{zw}] = \tilde{R}_{\tilde{R}_{uv}zw} + \tilde{R}_{z\tilde{R}_{uv}w}, \tag{4.3}$$

$$[\rho_{xy}, A] = 0, \quad \text{for any } A \in \mathfrak{g}. \tag{4.4}$$

Theorem 4.1. *Let M be a Kähler submanifold of $\mathbb{CP}^N$ that admits a homogeneous structure S and let G be the unique connected Lie subgroup of $U(N+1)$ whose Lie algebra is $\mathfrak{g}$. The orbit of $m \in \mathbb{CP}^N, \tilde{M} := G \cdot m$ is a complete Kähler submanifold of $\mathbb{CP}^N$ that extends M (up to isometries).*

In particular the values of S, α and J at m uniquely determine $\tilde{M}$ (up to isometries).

Proof. The Lie algebra $\mathfrak{g}$ admits the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where

$$\mathfrak{h} = \operatorname{span}\{\tilde{R}_{uv}, \rho_{wz}, u, v, w, z \in T_mM\}, \quad \mathfrak{m} = \operatorname{span}\{\varphi_u, u \in T_mM\}.$$

Note that $\mathfrak{m} \cong T_m\tilde{M}$ and that

$$T_m\tilde{M} = \operatorname{span}\{\varphi_u \cdot m, u \in T_mM\} \cong T_mM.$$

Let $\tilde{\nabla}^c$ be the canonical connection on $\tilde{M}$ associated to the reductive decomposition above. By Lemma 3.1 we get that $S^c_m = S_m$ and the second fundamental form at m of M and $\tilde{M}$ coincide.

Remark moreover that the complex structures of M and $\tilde{M}$ at m coincide via the isomorphisms $T_mM \cong \mathfrak{m} \cong T_m\tilde{M}$. A very similar argument as the one in the proof of Proposition 2.1 in [17], shows that there exists an isometry $F: \mathbb{CP}^N \to \mathbb{CP}^N$ such that $F(m) = m$, $F(M) \subseteq \tilde{M}$.

This proves Theorem 4.1, which is the "if" part of Theorem A.

5. Algebraic decomposition of the space of the homogeneous structures

Let $i: M \mapsto \mathbb{CP}^N$ be a Kähler submanifold. Let $m \in M$, put $V := T_mM$, $W = \nu_mM$, with $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} M = 2n$ and $\dim_{\mathbb{R}} W = \operatorname{codim}_{\mathbb{R}} M = 2h$. Using the hermitian metrics on V and W we shall make no distinction between covariant and contravariant tensors. In particular a homogeneous structure shall be considered as a tensor D_{xYZ} on $V^* \otimes (V \oplus W)^* \otimes (V \oplus W)^*$. By definition of homogeneous structure, the symmetries of D are

$$D_{xYZ} = -D_{xZY}, \tag{5.1}$$

$$D_{xyN} = 0, \tag{5.2}$$

$$D_{xYZ} = D_{xJYJZ}, \tag{5.3}$$

where x, y are vectors in V, N belongs to W, X, Y, Z are in $V \oplus W$. Hence

$$\mathcal{D}(V, W) = \{D \in V^* \otimes (V \oplus W)^* \otimes (V \oplus W)^* / D_{xYZ} = -D_{xZY}, D_{xyN} = 0,$$
$$D_{xYZ} = D_{xJYJZ}, x, y \in V, N \in W, Y, Z \in V \oplus W\}$$

is the space of tensors having the same symmetries as the homogeneous structures on Kähler submanifolds of $\mathbb{CP}^N$.

The inner product on V induces canonically an inner product on $\mathcal{D}(V, W)$ and determines an $U(n)$-equivariant isomorphism $V \cong V^*$ and an $U(h)$-equivariant isomorphism $W \cong W^*$.

The standard representation of $U(n) \times U(h)$ (regarded as a subgroup of $U(n+h)$) gives rise to a representation of $U(n)$ on V and of $U(h)$ on W and thus a representation of $U(n) \times U(h)$ on $\mathcal{D}(V, W)$ defined by

$$(gD)_{xYZ} = D_{g^{-1}xg^{-1}Yg^{-1}Z}, \quad g \in U(n) \times U(h), D \in \mathcal{D}(V, W).$$

It follows immediately that the above representation is completely reducible.

Because of (5.1) $\mathcal{D}(V, W)$ is $U(n) \times U(h)$-equivariantly included into

$$V \otimes \wedge^2(V \otimes W) \cong V \otimes (\wedge^2 V \oplus (V \otimes W) \oplus \wedge^2 W).$$

By (5.2) there is an $U(n)\times U(h)$-equivariant inclusion

$$\mathcal{D}(V,W)\subseteq(V\otimes\wedge^2V)\oplus(V\otimes\wedge^2W).$$

It is clear that $(J,\langle\,,\rangle)$ defines Kähler structures on V and W. The complexification of the dual spaces V^* and W^* are

$$V^*\otimes_{\mathbb{R}}\mathbb{C}=\lambda_V^{1,0}\oplus\lambda_V^{0,1},\quad W^*\otimes_{\mathbb{R}}\mathbb{C}=\lambda_W^{1,0}\oplus\lambda_W^{0,1},$$

where $\lambda_V^{1,0}$ is the annihilator of the space of complex tangent spaces vectors of the form $X+iJX$ and $\overline{\lambda_V^{1,0}}=\lambda_V^{0,1}$ is its conjugate (the same for $\lambda_W^{i,j}$). In the sequel we shall deal mainly with V and it shall be understood that what we say holds for W, too. We refer to **[19]**, [7] and [6] for more details.

The $(p+q)$-th exterior power of $\bar{V}^*$ contains a subspace $\lambda_V^{p,q}$ (which is isomorphic to $\wedge^p\lambda_V^{1,0}\otimes\wedge^q\lambda_V^{0,1}$) consisting of the so-called forms of type (p,q). Both $\lambda_V^{p,q}\oplus\lambda_V^{q,p}$, $(p\neq q)$ and $\lambda_V^{p,p}$ are the complexifications of real vector spaces which we denote by $[\![\lambda_V^{p,q}]\!]$ and $[\lambda_V^{p,p}]$ respectively, so that

$$[\![\lambda_V^{p,q}]\!]\otimes_{\mathbb{R}}\mathbb{C}=\lambda_V^{p,q}\oplus\lambda_V^{q,p},\quad(p\neq q)$$

and

$$[\lambda_V^{p,p}]\otimes_{\mathbb{R}}\mathbb{C}=\lambda_V^{p,p}.$$

The space of 2-forms decomposes as

$$\wedge^2\bar{V}^*=[\lambda_V^{1,1}]\oplus[\![\lambda_V^{2,0}]\!].\tag{5.4}$$

Here $[\lambda_V^{1,1}]$ equals the subspace of 2-forms σ for which $\sigma(X-iJX,Y-iJY)=0$ or equivalently $\sigma(JX,JY)=\sigma(X,Y)$, for all $X,Y\in V\oplus W$. Moreover, $\sigma\in[\![\lambda_V^{2,0}]\!]$ if and only if $\sigma(JX,JY)=-\sigma(X,Y)$, for all $X,Y\in V\oplus W$. One may identify $\wedge^2V$ with the Lie algebra $\mathfrak{so}(2n),[\lambda_V^{1,1}]$ with the subalgebra $\mathfrak{u}(n)$ and $[(\lambda_0)_V^{1,1}]$ with $\mathfrak{su}(n)$.

More generally, let ω_V be the Kähler form on V, i.e. $\omega_V=-i\sum_\alpha dz^\alpha\wedge d\bar{z}^\alpha$. Wedging with ω_V determines an $U(n)$-equivariant mapping $L_V:\lambda_V^{p-1,q-1}\to\lambda_V^{p,q}$. $(\lambda_0)_V^{p,q}$ is defined to be the orthogonal complement of the image of L_V with respect to the induced Hermitian metric. It is well known that the complex $U(n)$-modules $(\lambda_V)_0^{p,q}$ are irreducible.

Finally, we denote by $\mathbb{R}_V$ the module $[\lambda_V^{0,0}]$, i.e. the trivial representation on V.

Note now that, by (5.1),...,(5.3) we have the $U(n)\times U(h)$-equivariant isomorphism

$$\mathcal{D}(V,W)\cong(V\otimes[\lambda_V^{1,1}])\oplus(V\otimes[\lambda_W^{1,1}]).$$

Set

$$\mathcal{T}_+(V):=V\otimes[\lambda_V^{1,1}]\quad\mathcal{N}(V,W):=V\otimes[\lambda_W^{1,1}].$$

Every $D\in\mathcal{D}(V,W)$ splits into two components, i.e.

$$D=T+N, \quad T\in\mathscr{T}_+(V), \quad N\in\mathscr{N}(V,W).$$

Let B_V denote the kernel of the antisymmetrization $\lambda_V^{1,0}\otimes(\lambda_V)_0^{1,1}\mapsto\lambda_V^{2,1}$.

Theorem 5.1. *There is an isomorphism of $U(n)$-modules*

$$\mathscr{T}_+(V)\cong 2[\![\lambda_V^{1,0}]\!]\oplus[\![B_V]\!]\oplus[\![(\lambda_V)_0^{2,1}]\!].$$

Proof. We have $V=[\![\lambda_V^{1,0}]\!]$, so

$$\begin{aligned}\mathscr{T}_+(V)&\cong V\otimes[\lambda_V^{1,1}]\\ &\cong[\![\lambda_V^{1,0}]\!]\otimes([(\lambda_V)_0^{1,1}]\oplus\mathbb{R}_V)\\ &\cong[\![\lambda_V^{1,0}]\!]\otimes[(\lambda_V)_0^{1,1}]\oplus[\![\lambda_V^{1,0}]\!]\\ &\cong 2[\![\lambda_V^{1,0}]\!]\oplus[\![B_V]\!]\oplus[\![(\lambda_V)_0^{2,1}]\!].\end{aligned}$$

$[\![\lambda_V^{1,0}]\!]$, $[\![B_V]\!]$, $[\![(\lambda_V)_0^{2,1}]\!]$ are irreducible $U(n)$-modules. In fact, in Weyl's correspondence $[\![\lambda_V^{1,0}]\!]$, $[\![B_V]\!]$ and $[\![(\lambda_V)_0^{2,1}]\!]$ are associated to the dominant weights $(1,0,\dots,0)$, $(2,0,\dots,0,-1)$ and $(1,1,0,\dots,0,-1)$, respectively.

See [1] for the expressions of the projections of $D\in\mathscr{T}_+(V)$ on the various factors.

Theorem 5.2. *There is an isomorphism of $U(n)\times U(h)$-modules*

$$\mathscr{N}(V,W)\cong([\![\lambda_V^{1,0}]\!]\otimes[(\lambda_W)_0^{1,1}])\oplus([\![\lambda_V^{1,0}]\!]\otimes\mathbb{R}_W).$$

Proof.

$$\mathscr{N}(V,W)\cong V\otimes[\lambda_W^{1,1}]\cong V\otimes([(\lambda_W)_0^{1,1}]\oplus\mathbb{R}_W)\cong([\![\lambda_V^{1,0}]\!]\otimes[(\lambda_W)_0^{1,1}])\oplus([\![\lambda_V^{1,0}]\!]\otimes\mathbb{R}_W).$$

Now $[\![\lambda_V^{1,0}]\!]$ is an irreducible $U(n)$ module, $[(\lambda_W)_0^{1,1}]$ and $\mathbb{R}_W$ are irreducible $U(h)$ modules. The result follows from the fact that the tensor product of an irreducible $U(n)$ module and an irreducible $U(h)$ module is an irreducible $U(n)\times U(h)$ module.

Finally we determine the projection of $D\in\mathscr{N}(V,W)$ on the two irreducible factors. Let

$$c_{12}(D)(x):=\sum_\alpha\langle D_xe_\alpha,Je_\alpha\rangle.$$

where e_α is an orthonormal basis of W. Then

$$[\![\lambda_V^{1,0}]\!]\otimes[(\lambda_W)_0^{1,1}]=\{D\in\mathscr{N}(V,W)/c_{12}(D)=0\},$$

$$[\![\lambda_V^{1,0}]\!]\otimes\mathbb{R}_W=\left\{D\in\mathscr{N}(V,W)/\langle D_xY,Z\rangle=-\frac{1}{n}c_{12}(D)(x)\langle Y,JZ\rangle\right\}.$$

6. Examples

6.1. *2-symmetric Kähler submanifolds of* $\mathbb{CP}^N$.

Let $M \to \mathbb{CP}^N$ be a complex submanifold of $\mathbb{CP}^N$. We recall that M is then a Kähler submanifold.

Imitating [12] and [13] we give the following:

Definition 6.1. M is a *2-symmetric Kähler submanifold of* $\mathbb{CP}^N$ if there exists a family $\{\sigma_m\}_{m\in M}$ of involutive isometries of $\mathbb{CP}^N$ which leave the submanifold M invariant, such that any $m\in M$ is an isolated fixed point of $\sigma_{m|M}$, and for any $m,q\in M, \sigma_m\cdot\sigma_q=\sigma_r\cdot\sigma_m$, where $r=\sigma_m(q)$.

Note that the definition implies that M is a symmetric Kähler manifold. Indeed, $\{\sigma_{m|M}\}_{m\in M}$ is a family of symmetries of M.

We recall [21] that the *k-osculating space* to M at a point $m\in M$, $\overset{k}{O}_m$, is the span of

$$\{X_1, \nabla^{\mathbb{CP}^N}_{X_1} X_2, \dots, \nabla^{\mathbb{CP}^N}_{X_1}\nabla^{\mathbb{CP}^N}_{X_2}\dots\nabla^{\mathbb{CP}^N}_{X_{k-1}} X_k\},$$

computed at m, where X_i are vector fields on M. The orthogonal complement $\overset{k}{N}_m$ of $\overset{k}{O}_m$ in $\overset{k+1}{O}_m$ is called the *k-normal space*. If the dimension of every $\overset{k}{O}_m$ does not depend on m, the k-osculating and k-normal bundles $\overset{k}{O}$ and $\overset{k}{N}$ are defined. Their fibres at a point m are $\overset{k}{O}_m$ and $\overset{k}{N}_m$ respectively. If $\xi\in\overset{k}{N}$, then, for any vector field X on M, $\nabla_X\xi\in\overset{k-1}{N}\oplus\overset{k}{N}\oplus\overset{k+1}{N}$. The higher order second fundamental forms $\overset{k}{B}$ at m are defined inductively by

$$\overset{1}{B}(x_0,x_1):=\alpha(x_0,x_1),$$

$$\overset{k}{B}(x_0,\dots,x_k):=\operatorname{proj}_{(\overset{0}{N}_m\oplus\dots\oplus\overset{k-1}{N}_m)^\perp}\nabla^{\mathbb{CP}^N}_{x_0}\overset{k-1}{B}(X_1,\dots,X_k)$$

where X_i are vector fields extending x_i. A metric connection on any k-normal space is given by

$$\overset{k}{\nabla}_X\xi:=\operatorname{proj}_{\overset{k}{N}}\nabla^{\mathbb{CP}^N}_X\xi.$$

Lemma 6.1. *If M is a Kähler submanifold of $\mathbb{CP}^N$ then $J\overset{k}{N}\subseteq\overset{k}{N}$*

Proof. By induction on k. For $k=1$ we remark that $\alpha(x,Jy)=J\alpha(x,y)$ (cf. (2.1)) implies $J\overset{1}{N}\subset\overset{1}{N}$. Suppose $J\overset{k-1}{N}\subset\overset{k-1}{N}$ and that $\overset{k-1}{B}(x_0,\dots,Jx_i,\dots,x_{k-1})=J\overset{k-1}{B}(x_0,\dots,x_i,\dots,x_{k-1})$. Then

$$\overset{k}{B}(x_0,\ldots,Jx_l,\ldots,x_k)=\mathrm{proj}_{(\overset{0}{N}_m\oplus\ldots\oplus\overset{k-1}{N}_m)^\perp}\nabla^{\mathbb{C}P^N}_{x_0}\overset{k-1}{B}(X_1,\ldots,JX_l,\ldots,X_k))$$

$$=\mathrm{proj}_{(\overset{0}{N}_m\oplus\ldots\oplus\overset{k-1}{N}_m)^\perp}\nabla^{\mathbb{C}P^N}_{x_0}J\overset{k-1}{B}(X_1,\ldots,X_l,\ldots,X_k))$$

$$=\mathrm{proj}_{(\overset{0}{N}_m\oplus\ldots\oplus\overset{k-1}{N}_m)^\perp}J\nabla^{\mathbb{C}P^N}_{x_0}\overset{k-1}{B}(X_1,\ldots,X_l,\ldots,X_k))=J\overset{k}{B}(x_0,\ldots,x_l,\ldots,x_k)$$

since $(\overset{0}{N}_m\oplus\ldots\oplus\overset{k-1}{N}_m)^\perp$ is J invariant. As $\overset{k}{N}_m=\mathrm{span}\{\overset{k}{B}(x_0,\ldots,x_k)\}$, we get the proof of the lemma.

A direct consequence of Lemma 6.1 and $\nabla^{\mathbb{C}P^N}J=0$ is

Lemma 6.2. *If M is a Kähler submanifold of $\mathbb{CP}^N$, then $\overset{k}{\nabla}J=0$.*

Using Lemma 6.2 and the same techniques as in [**4**] (cf. also [**22**]) one can then prove

Theorem 6.3. *M is a 2-symmetric Kähler submanifold of $\mathbb{CP}^N$ if and only if*

$$\overset{k}{\nabla}\overset{k}{B}=0.$$

Let $\bar{\nabla}^\perp$ the metric connection on $\nu(M)$ given by

$$\bar{\nabla}^\perp:=\sum_{k\geqq 1}\overset{k}{\nabla}.$$

A straightforward computation shows that

Lemma 6.4. *M is a 2-symmetric Kähler submanifold of $\mathbb{CP}^N$ if and only if $\bar{S}:=\nabla\oplus\nabla^\perp-\nabla\oplus\bar{\nabla}^\perp$ is a homogeneous structure on M.*

By Lemma 6.1 it is easy to see that $c_{12}(\bar{S})=0$, so $\bar{S}\in[\![\lambda_V^{1,0}]\!]\otimes[(\lambda_W)_0^{1,1}]$. Hence a 2-symmetric Kähler submanifold admits a homogeneous structure belonging to $[\![\lambda_V^{1,0}]\!]\otimes[(\lambda_W)_0^{1,1}]$.

On the other hand, suppose that a Kähler submanifold M admits a homogeneous structure in $\mathcal{N}(V,W)$ and let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ be the Lie subalgebra of $\mathfrak{u}(N+1)$ constructed in Section 4. By Lemma 3.1 in [**10**], if $x\in\mathfrak{m}$, then

$$x\cdot\overset{k}{N}_m\subset\overset{k-1}{N}_m\oplus\overset{k+1}{N}_m \tag{6.1}$$

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(note that our $\overset{k}{N}_m$ coincide with the modules V_k in [10] for $k \geqq 1$). In the same vein as in the proof of Lemma 3.1, we have

$$(\overset{k}{\nabla}_x \overset{k}{B})(x_0, \dots, x_k) = \mathrm{proj}_{\overset{k}{N}_m}[x \cdot \overset{k}{B}(x_0, \dots, x_k)] = 0,$$

by (6.1). Applying Theorem 6.3 we get that M is 2-symmetric. Hence

Theorem 6.5. *M is a 2-symmetric Kähler submanifold of $\mathbb{CP}^N$ if and only if it admits a homogeneous structure belonging to $S \in [\![\lambda_V^{1,0}]\!] \otimes [(\lambda_W)_0^{1,1}]$.*

Remark. The above results imply that, if M admits a homogeneous structure belonging to $\mathcal{N}(V, W)$, then the $[\![\lambda_V^{1,0}]\!] \otimes \mathbb{R}_W$ factor can be eliminated. Indeed, if M has a homogeneous structure in $\mathcal{N}(V, W)$, then it is 2-symmetric and by Lemma 6.4 it has a homogeneous structure in $[\![\lambda_V^{1,0}]\!] \otimes [(\lambda_W)_0^{1,1}]$.

6.2. *Illustration of some examples.*

We recall that any homogeneous Kähler submanifold is an orbit in a unitary representation of compact Lie group G. Hence, using Lemma 3.1, one can recover the canonical homogeneous structure and the second fundamental form at m of the orbit $G \cdot m$ starting from the weight lattice of the Lie algebra $\mathfrak{g}$ of G.

Example 6.1. Let $\mathfrak{g} = \mathfrak{su}(2)$. Let ρ_α be the representation with highest weight $\alpha \in \mathbb{Z}_+$. Let v_α be a weight vector relative to α and consider the orbit of $[v_\alpha]$. The isotropy subgroup at $[v_\alpha]$ is isomorphic to $U(1)$ and the Kähler submanifold is $\mathbb{CP}^1 \to \mathbb{CP}^{N(\alpha)}$. It is easy to see that if $u \in \overset{k}{N}_{[p]}$, then $S^c_{mY} u \in \overset{k+1}{N}_m$ (where Y is a root vector relative to the root -2). This shows that $\mathbb{CP}^1 \to \mathbb{CP}^{N(\alpha)}$ is a 2-symmetric Kähler submanifold.

Example 6.2. Let $\mathfrak{g} = \mathfrak{su}(3)$ and Λ be a weight which lies in the fundamental Weyl chamber (including its walls). We consider the representation having Λ as highest weight.

(*a*) Let Λ belong to a wall of the fundamental Weyl chamber (which one is immaterial). Let v be an eigenvector relative to Λ and consider the orbit of $[v]$. The isotropy subgroup at $[v]$ is isomorphic to $S(U(1) \times U(2))$ and the orbit is $\mathbb{CP}^2$. Drawing the picture of the weight lattice one can visualize the normal spaces. In particular one can see that the orbit is a 2-symmetric Kähler submanifold.

(*b*) Let Λ lie in the interior of the fundamental Weyl chamber. In this case the orbit is not symmetric. (Indeed the orbit is the manifold of all flags in $\mathbb{C}^3$; cf. [8, page 383].) To see this directly from the weight lattice diagram, using homogeneous structures, we now give an explicit example. To this aim it is simpler to consider the complexified Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ (cf. what remarked in Section 2). Let L_i denote the functional

$$L_i \colon \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \mapsto a_i,$$

and choose as positive simple roots L_1-L_2 and L_2-L_3. We now take the irreducible representation ρ having highest weight $2L_1-L_3$. Let v belong to the weight space of $2L_1-L_3$ and consider the orbit M of v. One can readily see from the weight lattice diagram that the tangent space is (complex) three dimensional. Let $X\in\mathfrak{sl}(3,\mathbb{C})$ lie in the root space of L_3-L_2, Y in the root space of L_2-L_1 and set $Z:=[X,Y]$ (thus Z lies in the root space of L_3-L_1). Then $x:=\rho(X)v$, $y:=\rho(Y)v$ and $z:=\rho(Z)v$ are respectively in the weight space of $2L_1-L_2$, $-2L_3$ and L_1 and they span the tangent space of M at $[v]$. By (3.6) in Section 3 we have

$$T_x y=-\rho([X,Y])v=-\rho(Z)v=-z\neq 0.$$

This shows that the canonical homogeneous structure on M has non-vanishing torsion, which clearly implies that the orbit is not symmetric.

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