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ARITHMETIC PROPERTIES OF CERTAIN FUNCTIONS IN SEVERAL VARIABLES II

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Abstract

If $T = (t_{ij})$ is an $n \times n$ matrix with non-negative integer entries, we define a transformation $T: \boldsymbol{C}^n \to \boldsymbol{C}^n$ by $z' = Tz$ where

$$z'_i = \prod_{j=1}^{n} z_j^{t_{ij}} \qquad (1 \leqq i \leqq n).$$

We consider functions $f_1(z), \cdots, f_p(z)$ of n complex variables which satisfy functional equations of the form

$$f_i(z) = a_i f_i(Tz) + b_i(z) \qquad (1 \leqq i \leqq p)$$

and we obtain conditions under which the values of these functions at algebraic points are algebraically independent.

1. Introduction

This paper is a sequel to our earlier paper under this title and which we refer to hereafter as (I). We extend the results of Mahler (1930) and prove the algebraic independence of the values of functions in several complex variables satisfying a certain type of functional equation. Recent related work by the authors and, independently, by Kenneth K. Kubota and Kurt Mahler is described in our survey paper on transcendence and algebraic independence by a method of Mahler (1977). It is inappropriate to attempt to detail here the most general result we obtain and accordingly we mention only some more or less amusing instances covered by our main theorem.

The functions

$$G_k(z) = \sum_{h=0}^{\infty} (z^{2^h}/(1+z^{2^h}))^k \qquad (k = 1, 2, \cdots)$$

satisfy the respective functional equations

$$G_k(z^2) = G_k(z) - (z/(1+z))^k.$$

It follows that the functions $G_k(z)$ are algebraically independent over the field of rational functions in z and, if $\alpha_1, \alpha_2, \cdots, \alpha_m$ are multiplicatively independent algebraic numbers satisfying $0 < |\alpha_j| < 1$, then the numbers

$$G_k(\alpha_j) \qquad (j = 1, 2, \cdots, m; \; k = 1, 2, \cdots)$$

are algebraically independent over the field of rational numbers. For example, if $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \cdots$ is the sequence of rational primes, then the numbers $p_1^{-1}, p_2^{-1}, \cdots$ are appropriate multiplicatively independent algebraic numbers, and so the numbers

$$\sum_{h=0}^{\infty} (p_l^{2^h} + 1)^{-k} \qquad (k, l = 1, 2, \cdots)$$

are algebraically independent. In particular, if $F_h = 2^{2^h} + 1$ is the h-th Fermat number, then the numbers

$$\sum_{h=0}^{\infty} F_h^{-k} \qquad (k = 1, 2, \cdots)$$

are algebraically independent. (This last result is already implied by Mahler (1930).)

Let f_h be the h-th Fibonacci number, defined by $f_0 = 0$, $f_1 = 1$ and $f_{h+2} = f_{h+1} + f_h$ for $h \geqq 0$. The functions

$$H_k(z_1, z_2) = \sum_{h=0}^{\infty} \{z_1^{f_{h+1}} z_2^{f_h} / (1 - (-1)^{f_h} z_1^{2f_{h+1}} z_2^{2f_h})\}^k \qquad (k = 1, 2, \cdots)$$

satisfy the respective functional equations

$$H_k(z_1^3 z_2^2, z_1^2 z_2) = H_k(z_1, z_2) + \frac{z_1}{1 - z_1^2} + \frac{z_1 z_2}{1 + z_1^2 z_2^2} + \frac{z_1^2 z_2}{1 + z_1^4 z_2^2}.$$

As before, the functions $H_k(z_1, z_2)$ are algebraically independent over the field of rational functions in z_1 and z_2. If α_j, β_j $(j = 1, 2, \cdots, m)$ are pairs of non-zero algebraic numbers such that the series for $H_k(\alpha_j, \beta_j)$ converge and, in addition, there is no choice of the $2m$ rational integers $\mu_1, \cdots, \mu_m$, $\nu_1, \cdots, \nu_m$ not all zero such that

$$\left| \prod_{j=1}^{m} \alpha_j^{\mu_j f_{3h+1} + \nu_j f_{3h}} \beta_j^{\mu_j f_{3h} + \nu_j f_{3h-1}} \right| = 1$$

for infinitely many non-negative integers h, then the numbers

$$H_k(\alpha_j, \beta_j) \qquad (j = 1, 2, \cdots, m \,;\, k = 1, 2, \cdots)$$

are algebraically independent. In particular, we can take $\alpha = 1$ and $\beta = \frac{1}{2}(\sqrt{5} - 1)$, so that $f_l^{-1} = \sqrt{5}\beta^l/(1-(-1)^l\beta^{2l})$, and it follows that the numbers

$$\sum_{h=0}^{\infty} f_{f_h}^{-k} \qquad (k = 1, 2, \cdots),$$

are algebraically independent. In a similar spirit, we can show by considering the series

$$\sum_{h=0}^{\infty} (z_1^{f_{h+1}} z_2^{f_h})^k \qquad (k = 1, 2, \cdots),$$

that the numbers

$$\sum_{h=0}^{\infty} p_l^{-kf_h} \qquad (k, l = 1, 2, \cdots)$$

are algebraically independent.

Let ω be a quadratic irrational. Mahler (1929) shows that for algebraic α satisfying $0 < |\alpha| < 1$, the number

$$\sum_{h=1}^{\infty} [h\omega]\alpha^h$$

is transcendental. It follows from our results that if $\alpha_1, \cdots, \alpha_m$ are algebraic numbers satisfying $0 < |\alpha_j| < 1$ and an appropriate independence condition, namely that their absolute values are multiplicatively independent, then the numbers

$$\sum_{h=1}^{\infty} [h\omega]\alpha_j^k \qquad (j = 1, 2, \cdots, m)$$

are algebraically independent.

The remainder of the paper is set out as follows. Sections 2 to 4 contain a number of definitions and preliminary lemmas which describe the ingredients of the main algebraic independence theorem. The main theorem itself appears in section 5 and its proof is given in sections 6 to 8. Finally, section 9 contains some remarks justifying the examples instanced above.

2. Some properties of non-negative matrices

Let $T = (t_{ij})$ be an $n \times n$ non-negative integer matrix. We define the spectral radius $r(T)$ of T to be, as usual, the maximum of the absolute values of the eigenvalues of T. By a theorem of Frobenius (see, for example, Gantmacher (1959), Chapter 3), $r(T)$ is itself an eigenvalue of T. We make the following hypotheses about the matrix T.

DEFINITION 1. An $n \times n$ matrix T with non-negative integer entries is of *class* $\mathcal{T}$ if it is non-singular, it has a positive eigenvector belonging to the eigenvalue $r(T)$, and none of its eigenvalues is a root of unity.

We remark that the spectral radius of such a matrix T is greater than 1, by a well-known theorem of Kronecker (1857). This observation and the existence of a positive eigenvector of T are the most crucial properties of T required later. An explicit necessary and sufficient conditon for the existence of a positive eigenvector of T can be found in Gantmacher (1959). (In (I), we took this property as the defining property of class $\mathcal{T}$, but here we need the 2 additional conditions of Definition 1).

Suppose T is a matrix of class $\mathcal{T}$ and let $\lambda_1, \cdots, \lambda_s$ be the eigenvalues of T of maximum absolute value $r(T)$. As in (I), Lemma 3, there is a partial spectral decomposition

$$T = \sum_{j=1}^{s} \lambda_j E_j + F,$$

where E_j is the canonical projection onto the eigenspace belonging to λ_j $(1 \leqq j \leqq s)$ and F has spectral radius $r(F) < r(T)$. Set

$$U = \bigoplus_{j=1}^{s} \operatorname{im} E_j \quad \text{and} \quad V = \bigcap_{j=1}^{s} \ker E_j,$$

so that $U \oplus V$ is a decomposition of the underlying space as a direct sum of invariant subspaces of T. We call U the *dominant eigenspace of* T and we call the projection on U along V the *projection on the dominant eigenspace of* T.

Given an $n \times n$ non-negative integer matrix T, we define a transformation $T: \boldsymbol{C}^n \to \boldsymbol{C}^n$ as follows: If $z = (z_1, \cdots, z_n)$ is a point of $\boldsymbol{C}^n$, then $z' = Tz$ is the point with coordinates

$$z'_i = \prod_{j=1}^{n} z_j^{t_{ij}} \qquad (1 \leqq i \leqq n).$$

We adopt the usual vector notation for multi-indices, that is, if $\mu = (\mu_1, \cdots, \mu_n)$, then we write

$$|\mu| = |\mu_1| + \cdots + |\mu_n|$$

and

$$z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n} \qquad (z \text{ in } \boldsymbol{C}^n).$$

Note that

$$(Tz)^\mu = z^{\mu T} \qquad (z \text{ in } \boldsymbol{C}^n),$$

where the exponent μT on the right is the usual product of the row vector μ and the matrix T.

We denote by $\boldsymbol{C}^{*n}$ the set of points $z = (z_1, \cdots, z_n)$ in $\boldsymbol{C}^n$ with $z_1 z_2 \cdots z_n \neq 0$ and, for z in C^{*n}, we define $L(z)$ to be the real vector

$$L(z) = (-\log|z_1|, \cdots, -\log|z_n|).$$

DEFINITION 2. For a matrix T of class $\mathcal{T}$, we denote by $\mathcal{U}(T)$ the set of all points z in $\boldsymbol{C}^{*n}$ such that the projection of the vector $L(z)$ on the dominant eigenspace of T is positive. Thus $\mathcal{U}(T)$ is an open neighbourhood of the origin in $\boldsymbol{C}^{*n}$.

LEMMA 1. *Let T be a matrix of class $\mathcal{T}$ with spectral radius r. If α is a point of $\mathcal{U}(T)$ and μ is a non-negative integer vector, then there is a positive constant c_1, independent of k, such that*

$$\log|(T^k\alpha)^\mu| \leqq -c_1 r^k |\mu|,$$

for all sufficiently large positive integers k.

(See (I), Lemma 4).

3. The functional equations

Throughout this section, we suppose T is a non-singular $n \times n$ non-negative integer matrix, none of whose eigenvalues is a root of unity. In the sequel, we shall study functions of n complex variables which are regular in some neighbourhood of the origin and satisfy functional equations of the shape

$$f(z) = af(Tz) + b(z),$$

where a is a non-zero constant and $b(z)$ is a rational function of z. The function $b(z)$ is also necessarily regular in some neighbourhood of the origin. We remark two lemmas concerning the existence and uniqueness of solutions of such equations.

LEMMA 2. *Let a be a non-zero constant. If $f(z)$ can be expressed as the quotient of 2 functions regular in some neighbourhood of the origin and satisfies the functional equation $f(z) = af(Tz)$, then $f(z)$ is constant.*

(This is Lemma 1 of our paper on hypertranscendental functions).

LEMMA 3. *Suppose a is a non-zero constant and $b(z)$ is regular in some neighbourhood of the origin. Then the general solution of the functional equation $f(z) = af(Tz) + b(z)$ which is regular in some neighbourhood of the origin is given by*

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$$f(z)=f(0)+\sum_{k=0}^{\infty} a^k\{b(T^k z)-b(0)\}, \qquad (1-a)f(0)=b(0).$$

PROOF. It is easy to check that the formula of the lemma does define a regular solution of the functional equation and Lemma 2 guarantees that it gives all regular solutions.

4. Admissible points

The notion of an admissible point sums up the remaining conditions needed in order to apply the main theorem. Throughout this section, T denotes a matrix of class $\mathcal{T}$.

DEFINITION 3. A point α of C^n has *Property* (A) if, for every function $f(z)$ of n complex variables which is regular in some neighbourhood of the origin and is not identically zero, there are infinitely many positive integers k such that $f(T^k\alpha)\neq 0$.

The analysis of (I) suggests that, roughly speaking, any algebraic point α in $\mathcal{U}(T)$, whose coordinates satisfy an appropriate independence condition, has Property (A). More precisely, we can state the following 2 criteria which were practically proved in (I), but escaped our notice at the time. We showed in Theorem 2 of (I) that, if $|\alpha_1|,\cdots,|\alpha_n|$ are multiplicatively independent, then α has Property (A). The hypotheses can be slightly weakened as follows.

PROPOSITION 1. *Let T be a matrix of class $\mathcal{T}$ and let α be an algebraic point of $\mathcal{U}(T)$. If there is no non-zero integer n-tuple μ such that $|\alpha^{\mu T^k}|=1$ for infinitely many k, then α has Property* (A).

PROOF. Choose $\varepsilon>0$ and let μ be a non-zero integer n-tuple. By hypothesis, $|\alpha^{\mu T^k}|\neq 1$ for all sufficiently large k, so by an inequality of Baker for linear forms in logarithms (see (I), Lemma 12, for the formulation appropriate to the present case), we have

$$||\alpha^{\mu T^k}|-1|>e^{-\varepsilon r(T)^k}$$

for all sufficiently large k. Now, if $f(z)=\sum a_\mu z^\mu$ is a power series convergent in some neighbourhood of the origin, it follows as in (I), Lemma 11, that the series for $f(T^k\alpha)$ has a single dominant term for all sufficiently large k and so α has Property (A).

The much more complicated argument of Theorem 4 of (I) allows us to remove the absolute value in the criterion of Proposition 1 in case the matrix T is triangular. This gives the following criterion.

PROPOSITION 2. *Let T be a triangular matrix of class $\mathcal{T}$ and let α be an*

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algebraic point of $\mathcal{U}(T)$. *If there is no non-zero integer n-tuple* μ *such that* $\alpha^{\mu T^k} = 1$ *for infinitely many k, then* α *has Property* (A).

PROOF. The proposition is implicit in the argument of sections 10 and 11 of (I).

Now suppose $f_1(z), \cdots, f_p(z)$ are functions regular in some neighbourhood of the origin and satisfying the respective functional equations

$$f_i(z) = a_i f_i(Tz) + b_i(z) \qquad (1 \leqq i \leqq p), \tag{1}$$

where $a_1, \cdots, a_p$ are non-zero constants and the functions $b_1(z), \cdots, b_p(z)$ are rational.

DEFINITION 4. A point α of C^n is *admissible* (more explicitly, admissible with respect to the matrix T and the functional equations (1)) if α is in $\mathcal{U}(T)$ and has Property (A) and the numbers

$$b_i(T^k\alpha) \qquad (i = 1, 2, \cdots, p;\ k = 0, 1, 2, \cdots)$$

are all defined.

5. Statement of the main theorem

We are now in a position to give a precise statement of the algebraic independence theorem.

THEOREM. *Let T be a matrix of class* $\mathcal{T}$. *Suppose that* $f_1(z), \cdots, f_p(z)$ *are functions regular in some neighbourhood of the origin, that their power series expansions about the origin have algebraic coefficients and that they satisfy the respective functional equations*

$$f_i(z) = a_i f_i(Tz) + b_i(z) \qquad (1 \leqq i \leqq p),$$

where $a_1, \cdots, a_p$ *are non-zero algebraic numbers and* $b_1(z), \cdots, b_p(z)$ *are rational functions with algebraic coefficients. Finally, let* α *be an admissible algebraic point. If the functions* $f_1(z), \cdots, f_p(z)$ *are algebraically independent and* $f_1(\alpha), \cdots, f_p(\alpha)$ *are defined, then the numbers* $f_1(\alpha), \cdots, f_p(\alpha)$ *are algebraically independent over the rationals.*

We remark that the algebraic independence of the functions $f_1(z), \cdots, f_p(z)$ in the theorem is easily recognised because, essentially, the only possible algebraic dependences are linear ones. Indeed, suppose $f_1(z), \cdots, f_p(z)$ satisfy the respective functional equations (1) and are algebraically dependent and put

$$S(i) = \{j : a_j = a_i,\ 1 \leqq j \leqq p\}.$$

Then there is an index i $(1 \leqq i \leqq p)$ and constants d_j not all 0 such that the function

$$\sum_{j \text{ in } S(i)} d_j f_j(z)$$

is rational. (This is Theorem 2 of our paper on hypertranscendental functions).

The proof of the theorem is quite difficult and depends on a number of technical lemmas. However, as usual, the basic step consists of constructing an auxiliary function which we then show to have properties incompatible with the assumption that the numbers $f_1(\alpha), \cdots, f_p(\alpha)$ are algebraically dependent.

6. The auxiliary function

Let the matrix T, the functions $f_1(z), \cdots, f_p(z)$ and the point α satisfy all the requirements of the theorem. Assume, in addition, that the numbers $f_1(\alpha), \cdots, f_p(\alpha)$ are algebraically dependent. Thus there is a relation

$$F(\alpha; \omega) = \sum_{\mu} \omega_\mu f_1(\alpha)^{\mu_1} \cdots f_p(\alpha)^{\mu_p} = 0, \tag{2}$$

where the components ω_μ of ω are rational numbers, not all 0, indexed by the p-tuples $\mu = (\mu_1, \cdots, \mu_p)$ of non-negative integers with $0 \leqq \mu_i \leqq m_i$, say. By Lemma 3 and our hypotheses, the coefficients of the power series expansions of each of the functions $f_1(z), \cdots, f_p(z)$ about the origin all lie in some fixed algebraic number field, so there is an algebraic number field K of finite degree, d say, over $\boldsymbol{Q}$ which contains all the coefficients of the power series expansions of $f_1(z), \cdots, f_p(z)$, the numbers $a_1, \cdots, a_p$ and the coefficients of the rational functions $b_1(z), \cdots, b_p(z)$ appearing in the functional equations, and all the coordinates of $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\omega = (\omega_\mu)$. In the following work, $c_1, c_2, \cdots$ denote positive constants depending only on the quantities already introduced and, in particular, not depending on the parameters k and ρ which will appear shortly.

Now, introduce m (say) $= (m_1 + 1) \cdots (m_p + 1)$ new variables $w = (w_\mu)$ indexed in the same way as ω by the p-tuples $\mu = (\mu_1, \cdots, \mu_p)$ of non-negative integers with $0 \leqq \mu_i \leqq m_i$. We define a linear transformation $\Omega_k(u)$ of the variables $w = (w_\mu)$, for $u = (u_1, \cdots, u_p)$ and $k = 1, 2, \cdots$, by

$$(\Omega_k(u)w)_\mu = a^{k\mu} \sum_{\nu} \binom{\nu_1}{\mu_1} \cdots \binom{\nu_p}{\mu_p} u^{\nu - \mu} w_\nu,$$

the summation being over all p-tuples $\nu = (\nu_1, \cdots, \nu_p)$ of non-negative integers with $0 \leqq \nu_i \leqq m_i$. By iterating the functional equations (1), we obtain the further equations

$$f_i(z) = a_i^k f_i(T^k z) + b_i^{(k)}(z),$$

where

$$b_i^{(k)}(z) = \sum_{j=0}^{k-1} a_i^j b_i(T^j z)$$

$(i = 1, 2, \cdots, p;\ k = 1, 2, \cdots)$. It is convenient to adopt the abbreviation

(3) $$\omega^{(k)} = \Omega_k(b_1^{(k)}(\alpha), \cdots, b_p^{(k)}(\alpha))\omega \qquad (k = 1, 2, \cdots).$$

An easy calculation shows that the function

$$F(z; w) = \sum_{\mu} w_\mu f_1(z)^{\mu_1} \cdots f_p(z)^{\mu_p}$$

is invariant under the transformation $z \to T^k z$, $w \to \Omega_k(b_1^{(k)}(z), \cdots, b_p^{(k)}(z))w$, so that by (2), we have

(4) $$F(T^k\alpha; \omega^{(k)}) = 0 \qquad (k = 1, 2, \cdots).$$

Denote by $\mathscr{P}$ the ring of polynomials in $w = (w_\mu)$ with coefficients in K and denote by $\mathscr{P}(\omega)$ the subset of $\mathscr{P}$ comprising those polynomials $p(w)$ such that $p(\Omega_k(u)\omega)$ is identically zero in $u = (u_1, \cdots, u_p)$ for $k = 1, 2, \cdots$. Further, denote by $\mathscr{A}$ the ring of power series $E(z; w) = \Sigma\, p_\mu(w) z^\mu$ in $z = (z_1, \cdots, z_n)$ with coefficients $p_\mu(w)$ in $\mathscr{P}$ and converging in some neighbourhood of the origin. We define the *index* of $E(z; w)$ at ω to be the least non-negative integer h for which there is a coefficient $p_\mu(w)$ of $E(z; w)$ with $|\mu| = h$ and such that $p_\mu(w)$ is not in $\mathscr{P}(\omega)$. If all the coefficients of $E(z; w)$ are in $\mathscr{P}(\omega)$, we define the index of $E(z; w)$ at ω to be ∞.

LEMMA 4. *The index of the product of two functions in $\mathscr{A}$ is the sum of the indices of the two factors.*

PROOF. The lemma follows at once from the observation that $\mathscr{P}(\omega)$ is a prime ideal of $\mathscr{P}$. To prove this assertion, suppose $p(w)$ and $q(w)$ are polynomials in $\mathscr{P}$ such that $p(w)q(w)$ is in $\mathscr{P}(\omega)$. Using the definition of $\Omega_k(u)$, we can write

$$p(\Omega_k(u)\omega) = \sum_{j=1}^{N} d_j^k p_j(u), \qquad q(\Omega_k(u)\omega) = \sum_{j=1}^{N} d_j^k q_j(u),$$

where the d_j are distinct non-zero constants and the $p_j(u)$ and $q_j(u)$ are polynomials whose coefficients are independent of k. By restricting k to a suitable arithmetic progression, we may suppose the numbers $d_1^k, \cdots, d_N^k$ are distinct for each k. The hypothesis that $p(w)q(w)$ is in $\mathcal{P}(\omega)$ implies that either $\Sigma\, d_j^k p_j(u) = 0$, or $\Sigma\, d_j^k q_j(u) = 0$ for each k. By a well-known theorem of van der Waerden (1971), one of these alternatives, say the first, must hold for N consecutive values of k in arithmetic progression. The N equations $\Sigma\, d_j^k p_j(u) = 0$ for these N values of k, considered as equations for the coefficients of the $p_j(u)$, have non-zero determinant, so they imply that the $p_j(u)$ are identically 0, whence $p(w)$ is in $\mathcal{P}(\omega)$, as required.

For a non-negative integer ρ, let $\mathcal{P}_\rho$ and $\mathcal{P}_\rho(\omega)$ denote the sets of polynomials in $\mathcal{P}$ and $\mathcal{P}(\omega)$ respectively of degree at most ρ in each of the variables w_μ. Thus, $\mathcal{P}_\rho$ is a vector space of dimension $(\rho+1)^m$ over K and $\mathcal{P}_\rho(\omega)$ is a subspace of $\mathcal{P}_\rho$. We set $\delta_\rho(\omega) = \dim(\mathcal{P}_\rho/\mathcal{P}_\rho(\omega))$.

LEMMA 5. $\delta_{2\rho}(\omega) \leqq 2^m \delta_\rho(\omega)$.

PROOF. First observe that a polynomial of $\mathcal{P}_\rho$ is in $\mathcal{P}_\rho(\omega)$ if and only if its coefficients satisfy a certain set of $\delta_\rho(\omega)$ independent linear homogeneous conditions. Now any polynomial $p(w)$ in $\mathcal{P}_{2\rho}$ can be written in the form

$$p(w) = \sum_{j=1}^{2^m} e_j(w)^\rho p_j(w),$$

where the $e_j(w)$ are the 2^m monomials of degree at most 1 in each w_μ and the $p_j(w)$ are in $\mathcal{P}_\rho$. Clearly, if each $p_j(w)$ is in $\mathcal{P}_\rho(\omega)$, then $p(w)$ is in $\mathcal{P}_{2\rho}(\omega)$, so by the first remark of the proof, we have $\delta_{2\rho}(\omega) \leqq 2^m \delta_\rho(\omega)$.

The first step in the actual proof of the theorem is the construction of the auxiliary function $E_\rho(z; w)$.

LEMMA 6. *Let $f_1(z), \cdots, f_p(z)$ be functions satisfying the hypotheses of the theorem and put*

$$F(z; w) = \sum_\mu w_\mu f_1(z)^{\mu_1} \cdots f_p(z)^{\mu_p}.$$

Then, for each $\rho \geqq c_2$, there are $\rho + 1$ polynomials $p_0(z; w), \cdots, p_\rho(z; w)$ of degree at most ρ in each of the variables z_i and w_μ, with coefficients algebraic integers of K and with $p_0(z; w)$ having finite index at ω, such that the function

$$E_\rho(z; w) = \sum_{j=0}^{\rho} p_j(z; w) F(z; w)^j$$

is not identically zero, but its index at ω is at least $2^{-2-m/p}\rho^{1+1/p}$.

PROOF. The $\rho + 1$ polynomials $p_j(z; w)$, subject to the restriction that each has finite index at ω or vanishes identically, together possess $(\rho+1)^{p+1}\delta_\rho(\omega)$ independent coefficients. On the other hand, we can write

$$E_\rho(z;w)=\sum_{j=0}^{\rho} p_j(z;w)F(z;w)^j=\sum_\mu B_\mu(w)z^\mu,$$

where the coefficients $B_\mu(w)$ are in $\mathcal{P}_{2\rho}$, so the requirement that the index of $E_\rho(z;w)$ at ω exceed $c_3\rho^{1+1/p}$ gives at most $(c_3\rho^{1+1/p}+1)^p\delta_{2\rho}(\omega)$ homogeneous linear equations in the coefficients of the polynomials $p_j(z;w)$. If $c_3<2^{-1-m/p}$ and ρ is sufficiently large, the number of these equations is at most

$$(c_3\rho^{1+1/p}+1)^p\delta_{2\rho}(\omega)\leqq 2^{m+p}c_3^p(\rho+1)^{p+1}\delta_\rho(\omega)<(\rho+1)^{p+1}\delta_\rho(\omega),$$

by Lemma 5, so the system has a non-trivial solution in K which we can normalise so that it consists of algebraic integers. Let l be the least integer with $0\leqq l\leqq\rho$ such that the polynomial $p_l(z;w)$ has finite index at ω. The function $F(z;w)$ clearly has finite index, h say, at ω, so by Lemma 4, the index at ω of the function

$$\sum_{j=l}^{\rho} p_j(z;w)F(z;w)^{j-l}$$

exceeds

$$c_3\rho^{1+1/p}-hl\geqq \tfrac{3}{4}c_3\rho^{1+1/p},$$

providing ρ is sufficiently large. So the polynomials $p_l(z;w),\cdots,p_\rho(z;w)$, together with l polynomials identically zero, fulfil the requirements of the lemma. Since $f_1(z),\cdots,f_p(z)$ are algebraically independent, the function $E_\rho(z;w)$ obtained in this way is not identically zero.

The proof of the theorem will be completed in the next 2 sections as follows. Denote the spectral radius of T by r. We show first in Lemma 7 that

$$\log|E_\rho(T^k\alpha;\omega^{(k)})|\leqq -c_4r^k\rho^{1+1/p}, \tag{5}$$

providing $\rho\geqq c_2$ and k is sufficiently large compared to ρ. Then we show in Lemma 12 that providing $\rho\geqq c_2$, there is an infinite sequence of values of k such that

$$\log|E_\rho(T^k\alpha;\omega^{(k)})|\geqq -c_5r^k\rho. \tag{6}$$

Now, fixing the parameter ρ by

$$\rho=c_6(\text{say})>\max\{c_2,(c_5/c_4)^p\},$$

we find that the inequalities (5) and (6) conflict for some suitable large integer k and this contradiction proves the theorem.

7. An upper bound for $|E_\rho(T^k\alpha;\omega^{(k)})|$

LEMMA 7. *Suppose T is a matrix of class $\mathcal{T}$ with spectral radius r, α is an admissible point and $E_\rho(z;w)$ is the function constructed in Lemma 6. Then*

$$\log|E_\rho(T^k\alpha;\omega^{(k)})| \leqq -c_4 r^k \rho^{1+1/p},$$

providing $\rho \geqq c_2$ *and* k *is sufficiently large compared to* ρ.

Proof. As in the proof of Lemma 6, we write

$$E_\rho(z;w) = \sum_\mu B_\mu(w) z^\mu,$$

where the coefficients $B_\mu(w)$ are in $\mathscr{P}_{2\rho}$. By Lemma 1 and the definition (3) of the $\omega^{(k)}$, we have

$$\log|(T^k\alpha)^\mu| \leqq -c_1 r^k |\mu|, \qquad \log|\omega_\mu^{(k)}| \leqq c_7 k,$$

for all sufficiently large k, so that

$$\log|B_\mu(\omega^{(k)})(T^k\alpha)^\mu| \leqq -c_8 r^k |\mu|,$$

whenever k is sufficiently large compared to ρ and $B_\mu(\omega^{(k)}) \neq 0$. Thus the series for $E_\rho(T^k\alpha;\omega^{(k)})$ is convergent whenever k is sufficiently large compared to ρ and, by the construction of Lemma 6,

$$\log|E_\rho(T^k\alpha;\omega^{(k)})| \leqq -c_4 r^k \rho^{1+1/p}$$

whenever $\rho \geqq c_2$ and k is sufficiently large compared to ρ.

8. A lower bound for $|E_\rho(T^k\alpha;\omega^{(k)})|$

For each β in K, we can find a non-zero rational integer den β, a denominator for β, such that (den $\beta)\beta$ is an algebraic integer. It is convenient to write

$$\|\beta\| = \max_\sigma \{|\sigma\beta|, |\text{den } \beta|\},$$

where σ runs through the d distinct embeddings of K into $\mathbf{C}$. If β is a non-zero algebraic number in K, we have the fundamental inequality

$$\log|\beta| \geqq -2d \log\|\beta\|. \tag{7}$$

The inequality follows easily from the observation that the norm of (den $\beta)\beta$ has absolute value at least 1. (See, for example, Lang (1966), page 3).

Lemma 8. *Let T be a matrix of class $\mathscr{T}$ and α be an admissible point. Let $d_1,\cdots,d_N$ be distinct non-zero constants and let $g_1(z),\cdots,g_N(z)$ be functions regular in some neighbourhood of the origin and not all identically zero. Then there are infinitely many positive integers k such that*

$$\sum_{j=1}^N d_j^k g_j(T^k\alpha) \neq 0.$$

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PROOF. We prove the lemma by induction on N, the case $N = 1$ being just the definition of Property (A). Suppose that

$$\sum_{j=1}^{N} d_j^k g_j(T^k\alpha) = 0$$

for all sufficiently large k and that $g_N(z)$, say, is not identically zero. Then, for all sufficiently large k,

$$\sum_{j=1}^{N-1} (d_j/d_N)^k h_j(T^k\alpha) = 0$$

where

$$h_j(z) = g_j(z)g_N(Tz) - (d_j/d_N)g_j(Tz)g_N(z) \qquad (1 \leqq j \leqq N-1).$$

By the induction hypothesis, the $h_j(z)$ are all identically zero and so, by Lemma 2, the $g_j(z)$ $(1 \leqq j \leqq N-1)$ are all identically zero and we are back to the case $N = 1$.

LEMMA 9. *Assume the notation and hypotheses of the theorem. If $p(z; w)$ is a polynomial in the variables $z = (z_1, \cdots, z_n)$ and $w = (w_\mu)$, then the following assertions are equivalent*:

(i) *$p(z; w)$ has finite index at ω*;

(ii) *$p(T^k\alpha; \omega^{(k)}) \neq 0$ for infinitely many positive integers k.*

PROOF. Using the definition (3) of the $\omega^{(k)}$, with $b_i^{(k)}(z) = f_i(z) - a_i^k f_i(T^k z)$, we obtain the representations

$$p(T^k\alpha; \omega^{(k)}) = \sum_{j=1}^{N} d_j^k P_j(f_1(T^k\alpha), \cdots, f_p(T^k\alpha); T^k\alpha), \tag{8}$$

where the d_j are distinct non-zero constants and the $P_j(u; z)$ are polynomials in $u = (u_1, \cdots, u_p)$ and $z = (z_1, \cdots, z_n)$ whose coefficients are independent of k. If $p(T^k\alpha; \omega^{(k)}) = 0$ for all sufficiently large k, then the functions $P_j(f_1(z), \cdots, f_p(z); z)$ are all identically zero, by Lemma 8, and so, by the algebraic independence of $f_1(z), \cdots, f_p(z)$, the polynomials $P_j(u; z)$ are all identically zero. It now follows from (8) that

$$p(z; \Omega_k(f_1(\alpha) - a_1^k u_1, \cdots, f_p(\alpha) - a_p^k u_p)\omega) = \sum_{j=1}^{N} d_j^k P_j(u; z)$$

is identically zero in u and z for each k and, finally, that $p(z; \Omega_k(u)\omega)$ is identically zero in u and z for each k. Thus (i) implies (ii). The converse is immediate.

For a polynomial $p(z) = \Sigma\, p_\mu z^\mu$ with coefficients in K, we define $\|p\| = \max \|p_\mu\|$. Further, we say the polynomial $q(z) = \Sigma\, q_\mu z^\mu$ dominates $p(z)$, written $p(z) \prec q(z)$, if the coefficients of $q(z)$ are rational integers and $\|p_\mu\| \leqq q_\mu$ for each μ. These definitions extend in the obvious way to polynomials in z and w.

LEMMA 10. *Let T be a matrix of class $\mathcal{T}$ with spectral radius r and let $p(z)$ be a polynomial with coefficients in K and degree at most ρ in each variable. If α is a point with coordiantes in K, then*

$$\log \|p(T^k\alpha)\| \leqq \log \|p\| + c_9 r^k \rho.$$

(See (I), Lemma 8).

LEMMA 11. *Let T be a matrix of class $\mathcal{T}$ with spectral radius r and let α be an admissible point with coordinates in K. Let $p(z;w)$ be a polynomial in z and w with coefficients in K and degree at most ρ in each variable. Then*

$$\log \|p(T^k\alpha; \omega^{(k)})\| \leqq \log \|p\| + c_{10} r^k \rho.$$

PROOF. By applying Lemma 10 to the numerators and denominators of the rational functions $b_i(z)$, we have

$$\log \|b_i^{(k)}(\alpha)\| = \log \left\| \sum_{j=0}^{k-1} a_i^j b_i(T^k\alpha) \right\| \leqq c_{11} r^k.$$

As in the proof of Lemma 9, we write

$$p(z; \Omega_k(u)\omega) = \sum_{j=1}^{N} d_j^k P_j(u; z),$$

where the d_j are distinct elements of K and the $P_j(u;z)$ are polynomials with coefficients in K and degrees at most ρ in z and at most $m_0\rho$ in u, with $m_0 = m_1 + \cdots + m_p$. Further,

$$\log \|d_j\| \leqq c_{12}\rho, \qquad \log \|P_j\| \leqq \log \|p\| + c_{13}\rho.$$

Now

$$P_k(u;z) \prec \|P_k\| \prod_{i=1}^{p} (1+u_i)^{m_0\rho} \prod_{j=1}^{n} (1+z_j)^{\rho},$$

so by the previous remarks,

$$\log \|p(T^k\alpha; \omega^{(k)})\| \leqq \log \|p\| + c_{10} r^k \rho.$$

LEMMA 12. *Assume the notation and hypotheses of the theorem. Denote the spectral radius of the matrix T by r and let $E_\rho(z;w)$ be the function constructed in Lemma 6. If $\rho \geqq c_2$, then there is an infinite sequence of values of k such that*

$$\log |E_\rho(T^k\alpha; \omega^{(k)}| \geqq -c_5 r^k \rho.$$

PROOF. By (4) and the construction of Lemma 6, we have $E_\rho(T^k\alpha;\omega^{(k)})=p_0(T^k\alpha;\omega^{(k)})$. Since $p_0(z;w)$ has finite index at ω, $p_0(T^k\alpha;\omega^{(k)})\neq 0$ for infinitely many k, by Lemma 9. Finally, by (7) and Lemma 11, for each such k,

$$\log|p_0(T^k\alpha;\omega^{(k)})|\geqq -2d\log\|p\|-2c_{10}dr^k\rho\geqq -c_5r^k\rho,$$

providing that k is sufficiently large compared to ρ. This proves the assertion.

As explained in section 6, this completes the proof of the theorem.

9. Concluding remarks

There is a plethora of examples additional to those briefly mentioned in the introduction. We refer the reader to our papers cited below, as well as to Mahler (1929) and (1930). The principal generalisations of Mahler's earlier work that we effect is that our transformation matrices T need not satisfy the very restrictive conditions found necessary there, namely that the characteristic polynomial of T be irreducible and that T have a single eigenvalue whose absolute value is greater than 1 and greater than the absolute values of all the other eigenvalues of T. As indicated in the introduction, we can use our more general result to deduce the algebraic independence of the values of a suitable function at different points. This may be done by the following stratagem.

Let $f(z)$ be a transcendental function of the n complex variables $z=(z_1,\cdots,z_n)$ satisfying an appropriate functional equation

$$f(z)=af(Tz)+b(z) \tag{9}$$

of the shape described in the theorem. Let $\alpha^{(1)},\cdots,\alpha^{(p)}$ be distinct algebraic points of $\boldsymbol{C}^n$. We introduce the functions

$$F_i(Z)=F_i(z^{(1)},\cdots,z^{(p)})=f(z^{(i)})\qquad(1\leqq i\leqq p)$$

of the np complex variables $Z=(z^{(1)},\cdots,z^{(p)})$, and the $np\times np$ matrix S with p copies of T along its main diagonal. Then the functions $F_i(Z)$ satisfy the respective functional equations

$$F_i(Z)=aF_i(SZ)+b(z^{(i)}). \tag{10}$$

Also, the functions $F_i(Z)$ are clearly algebraically independent over the field of rational functions in Z. Now, if the np-tuple $\mathrm{A}=(\alpha^{(1)},\cdots,\alpha^{(p)})$ is admissible with respect to the matrix S and the functional equations (10) then it follows from the theorem that the numbers $F_1(\mathrm{A}),\cdots,F_p(\mathrm{A})$ are algebraically independent. But this is just to say that the numbers $f(\alpha^{(1)}),\cdots,f(\alpha^{(p)})$ are algebraically independent, which is what we set out to show.

We should remark that the point $\mathrm{A}=(\alpha^{(1)},\cdots,\alpha^{(p)})$ above is admissible with respect to the matrix S and the functional equations (10) if each $\alpha^{(i)}$ is admissible with respect to the matrix T and the original functional equation (9) and, in addition, the only set of integer n-tuples $\mu^{(1)}, \cdots, \mu^{(p)}$ such that

$$\left| \prod_{i=1}^{p} (\alpha^{(i)})^{\mu^{(i)} T^k} \right| = 1$$

for infinitely many k is the trivial solution with

$$\mu^{(1)} = \cdots = \mu^{(p)} = 0.$$

After Proposition 1, the last condition guarantees that the point $A = (\alpha^{(1)}, \cdots, \alpha^{(p)})$ has Property (A).

The examples of the introduction can now be readily demonstrated.

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