A note on the zeros of the Bessel function

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(*Received 2nd August*, 1938. *Read 4th November*, 1938.)

This note contains an elementary proof of the well-known result that *every solution of Bessel's equation*

(1) $$x^2 y'' + xy' + (x^2 - n^2) y = 0,$$

with a real parameter n has an infinite number of real positive zeros.

We first make the substitution $y = \sqrt{\left(\frac{2}{\pi x}\right)} u$ in (1), to remove the first derivative, giving

(2) $$u'' + [1 + (\tfrac{1}{4} - n^2)/x^2]\, u = 0.$$

This equation approximates, for x sufficiently large, to the form $u'' + u = 0$ associated with the circular functions. In fact, when $n^2 = \frac{1}{4}$, the solutions of (1) may be written

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos x, \quad J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x.$$

We employ the simple methods used by Bôcher[1] in his studies of the Sturm-Liouville equation to exploit these facts.

Suppose first that $n^2 - \frac{1}{4} < 0$, and consider the equation

(3) $$v'' + v = 0.$$

The solutions of this equation are of the form $A \cos (x + a)$ and the zeros are at intervals of π. Let x_1, x_2 be two consecutive positive zeros of $v(x)$. Multiplying (2) by v and (3) by u, and subtracting, we get

$$\frac{d}{dx}(u'v - uv') = u''v - uv'' = (n^2 - \tfrac{1}{4}) \frac{uv}{x^2}$$

or

(4) $$[u'v - uv']_{x_1}^{x_2} = -(\tfrac{1}{4} - n^2) \int_{x_1}^{x_2} uv \frac{dx}{x^2}.$$

Two cases are possible: either $v(x)$ is positive between x_1 and x_2 with $v'(x_1) > 0$ and $v'(x_2) < 0$ or $v(x)$ is negative between x_1 and x_2 with $v'(x_1) < 0$ and $v'(x_2) > 0$.

[1] M. Bôcher, *Leçons sur les Méthodes de Sturm* (Paris, 1917), Ch. III. and IV.

Let us assume that $u(x)$ does not vanish between x_1 and x_2. Then, from (4), in the first case the left-hand side has the same sign as $u(x)$ while the right-hand side has the opposite sign to that of $u(x)$. This contradiction shows that $u(x)$ *must vanish at least once between* x_1 *and* x_2. In the second case the left-hand side has the sign of $-u(x)$ while the right-hand side has the sign of $u(x)$, again leading to a contradiction.

Next suppose that $n^2 - \frac{1}{4} > 0$, and consider the equation

$$v'' + [1 - (n^2 - \tfrac{1}{4})/k^2]\, v = 0, \tag{5}$$

in which $k^2 > (n^2 - \frac{1}{4})$. The solutions of this equation have their zeros separated by intervals of $\pi k/\sqrt{(k^2 - n^2 + \frac{1}{4})}$. Applying to (2) and (5) the same treatment as before, we get

$$[u'v - uv']_{x_1}^{x_2} = -(n^2 - \tfrac{1}{4}) \int_{x_1}^{x_2} uv \left(\frac{1}{k^2} - \frac{1}{x^2}\right) dx \tag{6}$$

and, provided that $x_2 > x_1 > k > 0$, the same argument holds as before.

Thus, *the real positive zeros of the solutions of* (1) *and* (2) *are at least as numerous as those of a cosine function.*

Reverting to the case in which $n^2 - \frac{1}{4} < 0$ and making assumptions on $u(x)$ similar to those made on $v(x)$ (following (4)), we can now use (2), (5) and (6) to show that, *for x sufficiently large, $v(x)$ vanishes at least once between every pair of consecutive zeros of $u(x)$.*

Similarly, we can use (2), (3) and (4) to obtain the same result when $n^2 - \frac{1}{4} > 0$.

The zeros of (5) occur with less frequency than those of (2), when $n^2 - \frac{1}{4} > 0$ and more frequently when $n^2 - \frac{1}{4} < 0$, but, by taking k sufficiently large we can make the period of the solutions of (5) as near to 2π as we like. From this and the remarks following (6) we see that *the real and positive zeros of any solution of* (1) *approximate to those of* $\cos(x + \alpha)$, *for x sufficiently large*, a matter indicated by the nature of the asymptotic expansion[1] of $J_n(x)$.

[1] T. M. MacRobert, *Spherical Harmonics* (London, 1927), 274.

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https://doi.org/10.1017/S0013091500008506 Published online by Cambridge University Press