MEASURABLE SELECTIONS IN NORMED SPACES

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It is sometimes desirable to know in what circumstances a measurable set-valued function admits a measurable selector; this problem occurs regularly in the theory of optimal control (see for example (3) and (7)). In this paper we demonstrate the existence of measurable selectors in two particular cases where the choice of selector has a simple geometrical interpretation, namely that of being a " nearest-point " selector, as is explained in detail below. This work derives in part from that of C. Castaing, particularly from Théorème 3.4 of (2), of which this is an extension.

Following (1), (2), (6) and (8), we define a function Γ from a measurable space S (that is, a set on which is defined a σ-algebra of subsets called " measurable ") into the non-empty subsets of a topological space X to be *measurable* if and only if for each closed set F in X the set

$$\Gamma^-(F) = \{t \in S : \Gamma(t) \cap F \neq \phi\}$$

is measurable. Γ is described as a " measurable multifunction from S into X ". A function f from S into X is called a *selector* for Γ if $f(t) \in \Gamma(t)$ for all t in S; f is measurable if and only if $f^{-1}(F)$ is measurable for every closed subset F of X.

If Γ is a measurable multifunction from S into X, and if B is any closed set in X, then we define the *refinement of* Γ *by the set* B to be the multifunction Γ_B, where

$$\Gamma_B(t) = \Gamma(t) \cap B \quad \text{for} \quad t \in \Gamma^-(B)$$

and

$$\Gamma_B(t) = \Gamma(t) \qquad \text{otherwise.}$$

Γ_B is measurable, since if C is a closed subset of X,

$$\Gamma_B^-(C) = (\Gamma^-(B \cap C)) \cup (\Gamma^-(C) \backslash \Gamma^-(B)),$$

which is clearly measurable (see also (8)).

Consider now a descending sequence (Γ_n) of closed-valued measurable multifunctions from S into X. Then in order to show that the multifunction $\Gamma = \bigcap \Gamma_n$, defined by

$$\Gamma(t) = \bigcap_{n=1}^{\infty} \Gamma_n(t) \quad \text{for each } t \text{ in } S,$$

is also measurable, it is sufficient to show that we have

$$\Gamma^-(A) = \bigcap_{n=1}^{\infty} \Gamma_n^-(A)$$

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for each closed set A in X. Conditions which ensure this, and also that $\Gamma(t)$ be non-empty for each t, are

(i) for each t, $\Gamma_n(t)$ is compact for some n

or (ii) X is a complete metric space, and for each t the diameter of the set $\Gamma_n(t)$ converges to zero as $n \to \infty$.

If condition (ii) holds, then $\Gamma(t)$ contains but a single point $\gamma(t)$; γ is therefore a measurable selector for Γ_1. We shall make use of these facts in what follows.

Lemma. *Let S be a measurable space and (X, d) a metric space. Let Γ: $S \to X$ be a measurable multifunction with values compact and non-empty, and let x_0 be a (fixed) point of X. For each t, let*

$$\Delta(t) = \{x \in \Gamma(t)\colon \; d(x_0, x) = d(x_0, \Gamma(t))\}$$

the set of all points of $\Gamma(t)$ at shortest distance from x_0. Then Δ is also a measurable multifunction.

Proof. We note first that the function $x \to d(x, x_0)$ is continuous and so attains its minimum on $\Gamma(t)$. Hence, for all t, $\Delta(t)$ is compact and non-empty.

Arrange the non-negative rationals in a sequence (r_n) and let

$$B_n = \{x \in X\colon \; d(x, x_0) \leqq r_n\}.$$

Write $\Gamma_0 = \Gamma$ and define Γ_n recursively so that Γ_n is the refinement of Γ_{n-1} by the closed set B_n. Then the sequence (Γ_n) satisfies condition (i) mentioned above, and so $\bigcap \Gamma_n$ is a measurable multifunction. The proof of the lemma will be complete when we have shown that

$$\Delta(t) = \bigcap_{n=1}^{\infty} \Gamma_n(t) \tag{iii}$$

for all t.

To see this, if $x \in \Delta(t)$, then $d(x, x_0) = d(x_0, \Gamma(t))$, It follows by induction on n that $x \in \Gamma_n(t)$ for all n. Conversely, suppose that $x \in \Gamma_n(t)$ for all n. Then $r_m \geqq d(x, x_0) \geqq d(x_0, \Gamma(t))$ for every rational r_m greater than $d(x_0, \Gamma(t))$. (For if $r_m \geqq d(x_0, \Gamma(t))$, then B_m meets $\Gamma(t)$. Hence $\Gamma_p(t) \subseteq B_m$ for $p \geqq m$, and so $x \in B_m$.) Therefore $d(x, x_0) = d(x_0, \Gamma(t))$ and so $x \in \Delta(t)$. This proves (iii) and completes the proof of the Lemma.

The problem of finding a selector in a metric (or metrisable) space is now reduced to that of finding a selector for Δ. If Δ contains a single point $\delta(t)$ for all t, then we need look no further. This happens in a wide class of normed spaces when $\Gamma(t)$ is assumed to be convex.

A normed linear space E is said to be *rotund* (or *strictly convex*) if for any two distinct elements x, y in E such that $\| x \| = 1$ and $\| y \| = 1$, then $\| x+y \| < 2$.

A normed linear space E is said to be *strictly normable* if it has an equivalent strictly convex norm (for a discussion of strict convexity, see **(4)** p. 342 *et seq.*).

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In particular it is known that every reflexive Banach space is strictly normable; this is proved in (**5**).

Theorem 1. *Let S be a measurable space, and E a strictly normable space. Then every measurable multifunction Γ from S into E with values which are non-empty, convex and compact has a measurable selector.*

Proof. Using the notation of the Lemma, let $x_0 = 0$, the zero element; we may assume that the norm on E is strictly convex. Then

$$\Delta(t) = \{u \in \Gamma(t) \colon \| u \| = d(0, \Gamma(t))\}$$

and Δ is measurable, by the Lemma; moreover $\Delta(t)$ is compact, convex and non-empty for each t. Also $\Delta(t)$ contains just one point $\delta(t)$; for if $x \in \Delta(t)$ and $y \in \Delta(t)$, then $\frac{1}{2}(x+y) \in \Delta(t)$. Since the norm is strictly convex, $x \neq y$ would imply $\| \frac{1}{2}(x+y) \| < \| x \|$ strictly, which contradicts the definition of Δ.

Thus δ is the required measurable selector.

A normed linear space is said to be *uniformly convex* if it always follows from

$$\| x_n \| \leqq 1,\ \| y_n \| \leqq 1 \text{ and } \lim_{n\to\infty} \| x_n + y_n \| = 2 \text{ that } \lim_{n\to\infty} \| x_n - y_n \| = 0.$$

A normed linear space is said to be *uniformly normable* if it has an equivalent uniformly convex norm. Uniform convexity is discussed in (**4**), Section 26. In particular the standard Banach spaces l^p and L^p are uniformly convex for $1 < p < \infty$. We note the following: *if (A_n) is a sequence of convex sets in a uniformly convex space and (β_n) a sequence of non-negative real numbers, with limit α, such that for each n*

$$\alpha \leqq \| x \| \leqq \beta_n \quad \textit{for all} \quad x \in A_n,$$

then the diameter of A_n tends to zero as $n \to \infty$.

Proof. This is obvious if $\alpha = 0$; otherwise if $x_n, y_n \in B_n$ where $\beta_n B_n = A_n$, then $\| x_n \| \leqq 1$, $\| y_n \| \leqq 1$ and

$$\| \tfrac{1}{2}(x_n + y_n) \| \geqq \frac{\alpha}{\beta_n} \to 1.$$

Therefore $\| x_n - y_n \| \to 0$, so diam $(B_n) \to 0$, and so diam $(A_n) \to 0$.

Theorem 2. *Let S be a measurable space, and E a uniformly normable Banach space. Then every measurable multifunction from S into E with values closed, convex and non-empty has a measurable selector.*

Proof. We consider E with a uniformly convex norm; E is a Banach space with respect to this norm. We proceed as in the proof of the Lemma. The proof will be complete when we have shown that the sequence (Γ_n) obtained by this method satisfies the second of the two conditions discussed in the introduction. That is, it is sufficient to prove that for each t in S

$$\text{diam}\,(\Gamma_n(t)) \to 0 \quad \text{as} \quad n \to \infty.$$

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This follows from the preceding remark: take $A_n = \Gamma_n(t)$, $\alpha = d(0, \Gamma(t))$ and define (β_n) recursively as follows:

$$\beta_n = \min(\beta_{n-1}, r_n) \quad \text{if} \quad \alpha \leqq r_n,$$

and

$$\beta_n = \beta_{n-1} \qquad \text{otherwise, for } n > 1.$$

We take β_1 to be equal to r_k, where k is the first integer such that $r_k \geqq \alpha$. Then β_n satisfies the required condition for $n \geqq k$.

Hence $\Delta(t)$ contains just one point $\gamma(t)$, and γ is the required selector.

It is possible to relax the conditions on E in the " nearest-point " problem at the expense of placing greater restrictions on S. In (1), the authors have relaxed the uniform convexity condition, by restricting S to be a locally compact Hausdorff space in which every compact subset is metrisable and also by modifying the definition of measurability.

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