Proceedings of the Edinburgh Mathematical Society (1982) 25, 69–71 ©

A REMARK ON THE RADICAL OF A GROUP ALGEBRA

by PETER BROCKHAUS

(Received 5th July 1980)

This paper contains a new proof of a theorem of D. A. R. Wallace [**5**] in case of a p-solvable group. An alternative proof has been given by K. Motose [**4**].

Let G be a finite group, F a field with prime characteristic p, P a Sylow p-subgroup of G. JFG will denote the Jacobson radical of FG.

Lemma 1. *Let Q be a p-group acting on a p'-group H such that Q stabilises each conjugacy class of H. Then Q acts trivially.*

Proof. Let K be a conjugacy class of H. By assumption Q acts on K. Since $p \nmid |K|$ there exists $k \in K$ such that $k \in C_H(Q)$ implies

$$K = k^H \subseteq \bigcup_{h \in H} [C_H(Q)]^h$$

It follows $H = \bigcup_{h \in H} [C_H(Q)]^h$. By [**2**], p. 12, Aufgabe 4, $H = C_H(Q)$.

Lemma 2. *Assume $G = PN$ where $N := O_{p'}(G)$. Then*

$$\dim_F JFG = |G| - |N| \Rightarrow p \trianglelefteq G.$$

Proof. Since $FN \cap JFG \subseteq JFN = 0$ (by Maschke's theorem) the map

$$\varphi\colon FN \to FG/JFG \quad \alpha \mapsto \alpha + JFG$$

is an F-algebra monomorphism. φ is an isomorphism, since $\dim_F (FG/JFG) = |G| - \dim_F JFG = |N|$. Define $\Psi\colon FG \to FN$ by $\Psi(\alpha) := \varphi^{-1}(\alpha + JFG)$ $(\alpha \in FG)$. Ψ is an F-algebra homomorphism with property $\Psi|_{FN} = 1_{FN}$. Let K be a conjugacy class of N and $x \in P$.

$$c := \sum_{k \in K} k \in ZFN.$$

$c^x = \Psi(c^x) = \Psi(x^{-1})\Psi(c)\Psi(x) = c^u = c$, where $u := \Psi(x)$, a unit in $FN \Rightarrow K^x = K \Rightarrow P$ acts on K by conjugation. By Lemma 1 $N = C_N(P)$.

Lemma 3.
(a) $H \leqq G \Rightarrow \dim_F JFG \leqq \dim_F JFH + |G| - |H|$;
(b) $\dim_F JFG \leqq |G| - |G: P|$, *if G has a p-complement*;
(c) $H \trianglelefteq G$, $p \nmid |G: H| \Rightarrow \dim_F JFG = |G: H| \dim_F JFH$;
(d) $Q \trianglelefteq G$, *Q a p-group* $\Rightarrow \dim_F JFG = \dim_F JF(G/Q) + |G| - |G: Q|$.

Proof. (a) $x + JFG \cap FH \mapsto x + FH$ for all $x \in JFG$ defines an F-monomorphism from the F-vector space $JFG/JFG \cap FH$ into the F-vector space FG/FH. Therefore, since $JFG \cap FH \subseteq JFH$,

$$\dim_F JFG = \dim_F (JFG/JFG \cap FH) + \dim_F (JFG \cap FH)$$
$$\leqq \dim_F (FG/FH) + \dim_F JFH = |G| - |H| + \dim_F JFH.$$

(b) Let H be a p-complement. Apply (a) and Maschke's theorem.
(c) follows from [**3**], p. 524.
(d) follows easily from [**1**], p. 71–72.

Theorem (Wallace [**5**]). *Let G be p-solvable. Then*

$$P \trianglelefteq G \Leftrightarrow \dim_F JFG = |G| - |G: P|.$$

Proof. "$\Rightarrow$":

$$\dim_F JFG = \dim_F JF(G/P) + |G| - |G: P| \quad \text{(by Lemma 3(d))}$$
$$= |G| - |G: P| \quad \text{(by Maschke's theorem)}$$

"$\Leftarrow$": We use induction on $|G|$. The case $|G| = 1$ is clear. Assume that $||G| > 1$ and the result holds for p-solvable groups of smaller order. As G is p-solvable, there exists $H \lneqq G$: $p \nmid |G: H|$ or $|G: H| = p$.

Case 1: $p \nmid |G: H|$. Then $P \in \mathrm{Syl}_p(H)$.

$$\dim_F JFH = |G: H|^{-1} \dim_F JFG \quad \text{(by Lemma 3(c))}$$
$$= |G: H|^{-1}(|G| - |G: P|) = |H| - |H: P|$$

$\Rightarrow P \trianglelefteq H$ (by induction) $\Rightarrow P \trianglelefteq G$ (since P char $H \trianglelefteq G$).

Case 2: $|G: H| = p$.

Case 2a: $p \nmid |H|$. Then $H = O_{p'}(G)$ and, by Lemma 2, $P \trianglelefteq G$.

Case 2b: $p \mid |H|$. Let $Q \in \mathrm{Syl}_p(H)$.

$$\dim_F JFH \geqq \dim_F JFG + |H| - |G| \quad \text{(by Lemma 3(a))}$$
$$= |G| - |G: P| + |H| - |G| = |H| - |G: P| = |H| - |H: Q|$$

$$\Rightarrow \dim_F JFH = |H| - |H: Q| \quad \text{(by Lemma 3(b))}$$
$$\Rightarrow Q \trianglelefteq H \text{ (by induction)} \Rightarrow Q \trianglelefteq G \quad \text{(since } Q \text{ char } H \trianglelefteq G)$$
$$\Rightarrow \dim_F JF(G/Q) = \dim_F JFG + |G: Q| - |G| \quad \text{(by Lemma 3(d))}$$
$$= |G| - |G: P| + |G: Q| - |G|$$
$$= |G/Q| - |G/Q: P/Q|.$$

By induction $P/Q \trianglelefteq G/Q$ (since $P/Q \in \mathrm{Syl}_p(G/Q)$) and therefore $P \trianglelefteq G$.

In conclusion I wish to thank Professor G. Michler for helpful discussions, Dr. M. Lorenz for improvements of the original version and the referee for a useful hint.

REFERENCES

1. W. Hamernik. *Group Algebras of Finite Groups* Lectures given at the University of Giessen, Winter 1973/74.

2. B. Huppert, *Endliche Gruppen I* (Springer, Berlin, Heidelberg, New York, 1967).

3. G. Michler, *Blocks and Centers of Group Algebras* (Lecture Notes in Mathematics 246, Springer, 1972).

4. K. Motose, On a Theorem of Wallace and Tsushima, *Proc. Japan Acad.* **50** (1974), 572–575.

5. D. A. R. Wallace, On the Radical of a Group Algebra, *Proc. Amer. Math. Soc.* **12** (1961), 133–137.

Department of Mathematics
Essen University
4300 Essen
Federal Republic of Germany