

PARTITIONED TRANSVERSALS

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Let E be a finite set and let $\mathfrak{A} = (A_1, \dots, A_n)$ be a family of subsets of E . A subset T of E is called a *transversal* of \mathfrak{A} if there exists a bijection $\psi: T \rightarrow \{1, \dots, n\}$ such that $x \in A_{\psi(x)}$ for all $x \in T$. If $I \subseteq \{1, \dots, n\}$, we shall, for brevity, write

$$A(I) = \bigcup_{i \in I} A_i$$

(and similarly for families denoted by other letters). The cardinal of a set S will be denoted by $|S|$. If λ is a non-negative integer, we define λS as S or \emptyset according as $\lambda > 0$ or $\lambda = 0$.

A criterion for the existence of transversals was established in P. Hall's celebrated paper [2] which appeared in 1935.

THEOREM 1. *The family $\mathfrak{A} = (A_1, \dots, A_n)$ of sets possesses a transversal if and only if $|A(I)| \geq |I|$ for all $I \subseteq \{1, \dots, n\}$.*

Making use of the duality theorem of linear programming, Hoffman and Kuhn [3] obtained in 1956 the following striking generalization of Hall's criterion.

THEOREM 2. *Let (E_1, \dots, E_p) be a partition (into pairwise disjoint subsets) of a finite set E ; let $\mathfrak{A} = (A_1, \dots, A_n)$ be a family of subsets of E ; and let $0 \leq r_j \leq s_j$ ($1 \leq j \leq p$) be integers. Then \mathfrak{A} possesses a transversal, say T , such that*

$$r_j \leq |T \cap E_j| \leq s_j \quad (1 \leq j \leq p) \tag{1}$$

if and only if

$$|A(I) \cap E(J)| \geq |I| - \min\{n - \sum_{j \in J} r_j, \sum_{j \notin J} s_j\} \tag{2}$$

for all $I \subseteq \{1, \dots, n\}$, $J \subseteq \{1, \dots, p\}$.

For $p = 1$, $r_1 = s_1 = n$ this reduces, of course, to Theorem 1. In addition to the original treatment of Hoffman and Kuhn, there exist several alternative proofs. Most of these make use of ideas drawn from the theory of abstract independence and invoke at some stage Rado's theorem on independent transversals [7] (which is itself a generalization of Theorem 1). In this context, we may mention the proofs of D. J. A. Welsh [8], of Hazel Perfect [6, Chap. VII], and of the present writer [5, Chap. 9]. Another proof is to be found in Ford and Fulkerson's book [1, p. 75], where Theorem 2 is deduced from the "max-flow min-cut" theorem, a result intimately related to Menger's graph theorem. Here we adopt an altogether different approach and give a demonstration of Theorem 2 which presupposes no background other than a knowledge of P. Hall's fundamental criterion. This treatment provides yet a further illustration of the theme of "self-refinement" of Hall's criterion discussed in a recent expository article [4] by the author.

If \mathfrak{A} possesses a transversal T which satisfies (1), then obviously

$$r_j \leq |E_j| \quad (1 \leq j \leq p), \quad \sum_{j=1}^p r_j \leq n \leq \sum_{j=1}^p s_j. \tag{3}$$

But equally, these relations follow if (2) is assumed to hold for all I and J. Thus the validity of (3) can be assumed throughout the argument.

Let D_1, \dots, D_p be sets such that E, D_1, \dots, D_p are pairwise disjoint and $|D_j| = s_j - r_j$ ($1 \leq j \leq p$). Denote by \mathfrak{A}^* the family of subsets of $E \cup D_1 \cup \dots \cup D_p$ consisting of A_1, \dots, A_n together with $|E_j| - r_j$ copies of $E_j \cup D_j$ ($1 \leq j \leq p$) and $\sum_1^p s_j - n$ copies of $D_1 \cup \dots \cup D_p$. We note that the number of sets in \mathfrak{A}^* is equal to the cardinal of the ground set, namely

$$|E| + \sum_1^p (s_j - r_j).$$

Hence a transversal of \mathfrak{A}^* , if one exists, must be identical with the ground set.

Suppose, in the first place, that \mathfrak{A}^* has a transversal. Then a certain subset of this transversal, say $T (\subseteq E)$, is a transversal of \mathfrak{A} . Write

$$|T \cap E_j| = t_j \quad (1 \leq j \leq p) \tag{4}$$

so that

$$t_j \leq |E_j| \quad (1 \leq j \leq p), \quad \sum_{j=1}^p t_j = n.$$

Now t_j elements of E_j have been used in T ; hence the $|E_j| - r_j$ copies of $E_j \cup D_j$ are represented by $|E_j| - t_j$ elements of E_j (so that $|E_j| - t_j \leq |E_j| - r_j$) and $t_j - r_j$ elements of D_j (so that $t_j - r_j \leq s_j - r_j$). Thus $r_j \leq t_j \leq s_j$; and T satisfies (1).

Conversely, suppose that \mathfrak{A} has a transversal T which satisfies (1); and use the notation in (4), so that $r_j \leq t_j \leq s_j$ ($1 \leq j \leq p$). Let the $|E_j| - r_j$ copies of $E_j \cup D_j$ be represented by the $|E_j| - t_j$ elements of E_j not used in T and by $t_j - r_j$ elements of D_j ($1 \leq j \leq p$). We are then left with $s_j - t_j$ elements of D_j ($1 \leq j \leq p$), and these elements just suffice to represent the $\sum_1^p s_j - n$ copies of $D_1 \cup \dots \cup D_p$, since $\sum_1^p t_j = n$. Hence \mathfrak{A}^* has a transversal.

We have thus shown that \mathfrak{A} has a transversal T subject to (1) if and only if \mathfrak{A}^* has a transversal. By Hall's criterion (Theorem 1), this is the case if and only if

$$|A(I) \cup \bigcup_{j=1}^p \lambda_j (E_j \cup D_j) \cup \mu (D_1 \cup \dots \cup D_p)| \geq |I| + \sum_{j=1}^p \lambda_j + \mu$$

whenever $I \subseteq \{1, \dots, n\}$, $0 \leq \lambda_j \leq |E_j| - r_j$ ($1 \leq j \leq p$), $0 \leq \mu \leq \sum_1^p s_j - n$. Write

$$J = \{j : 1 \leq j \leq p, \lambda_j = 0\}.$$

Then the above requirement is that

$$|A(I) \cup \bigcup_{j \notin J} E_j| + |\bigcup_{j \notin J} D_j \cup \mu (D_1 \cup \dots \cup D_p)| \geq |I| + \sum_{j \notin J} (|E_j| - r_j) + \mu$$

whenever $I \subseteq \{1, \dots, n\}$, $J \subseteq \{1, \dots, p\}$, $0 \leq \mu \leq \sum_1^p s_j - n$. This means that

$$|A(I) \cap E(J)| + |\bigcup_{j \notin J} D_j \cup \mu (D_1 \cup \dots \cup D_p)| \geq |I| - \sum_{j \notin J} r_j + \mu \tag{5}$$

whenever $I \subseteq \{1, \dots, n\}$, $J \subseteq \{1, \dots, p\}$, $0 \leq \mu \leq \sum_1^p s_j - n$.

For $\mu = 0$, this condition becomes

$$|A(I) \cap E(J)| + \sum_{j \notin J} (s_j - r_j) \geq |I| - \sum_{j \notin J} r_j,$$

i.e. $|A(I) \cap E(J)| \geq |I| - \sum_{j \notin J} s_j,$ (6)

for all $I \subseteq \{1, \dots, n\}$, $J \subseteq \{1, \dots, p\}$. Again, for $\mu > 0$, condition (5) becomes

$$|A(I) \cap E(J)| + \sum_{j=1}^p (s_j - r_j) \geq |I| - \sum_{j \notin J} r_j + \sum_{j=1}^p s_j - n,$$

i.e. $|A(I) \cap E(J)| \geq |I| - n + \sum_{j \in J} r_j$ (7)

for all $I \subseteq \{1, \dots, n\}$, $J \subseteq \{1, \dots, p\}$. Thus \mathfrak{A} has a transversal T subject to (1) if and only if (6) and (7) hold for all I, J . This implies our assertion.

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