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A NOTE ON WHITNEY MAPS

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In his recent book [3] Nadler observes that the property of admitting a Whitney map is of fundamental importance in studying the internal structure of hyperspaces, especially their arc structure. Nadler presents three distinct methods of constructing a Whitney map on the hyperspace 2^X of nonempty closed subsets of a continuum.

A *partially ordered space* is a topological space X endowed with a partial order $\leq$ whose graph is a closed subset of $X \times X$. It is well-known (see, for example, [2], page 167) that if X is a regular Hausdorff space then 2^X is a partially ordered space with respect to inclusion. The author has felt for some time that a natural and fruitful avenue for attacking hyperspace problems is to consider such problems in the more inclusive setting of partially ordered spaces. In this note we give some substance to that feeling by extending the concept of a Whitney map to arbitrary partially ordered spaces and then proving by a brief argument that each member of a large class of partially ordered spaces (much larger than the hyperspaces of compacta) admits such a map.

An element m of a partially ordered space X is *minimal* (*maximal*) if, whenever $x \in X$ and $x \leq m$ ($m \leq x$), it follows that $x = m$. We denote the set of minimal elements of X by $\operatorname{Min} X$; the set of maximal elements is denoted $\operatorname{Max} X$. It is well-known [4] that if X is compact and $x \in X$ then x has a predecessor in $\operatorname{Min} X$ and a successor in $\operatorname{Max} X$.

A *Whitney map* for a partially ordered space X is a continuous function $\omega : X \to [0, 1]$ which satisfies

(i) *if* $x \in \operatorname{Min} X$ *then* $\omega(x) = 0$,

(ii) *if* $x \in \operatorname{Max} X$ *then* $\omega(x) = 1$,

(iii) *if* $x < y$ *in* X *then* $\omega(x) < \omega(y)$.

It is obvious that if $X = 2^Y$ for some compactum Y and if ω satisfies (i), (ii) and (iii), then ω is a Whitney map in the hyperspace sense. Moreover, if Y is a compactum then any Whitney map for 2^Y is, up to a constant factor, a Whitney map in the sense employed here.

THEOREM. *If X is a compact metric partially ordered space such that* $\operatorname{Min} X$ *and* $\operatorname{Max} X$ *are disjoint closed sets, then X admits a Whitney map.*

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Proof. Let X' be the quotient space which results from identifying Min X with an element 0 and Max X with an element 1, and let $\varphi : X \to X'$ be the natural map. Partially order X' by $\varphi(x) \leq \varphi(y)$ if and only if $x \in \text{Min } X$ or $y \in \text{Max } X$ or $x \leq y$ in X. It is routine to verify that X' is a compact metrizable partially ordered space and that $0 \leq y \leq 1$ for each $y \in X'$. Moreover, φ is order-preserving and φ is $1-1$ on $X - (\text{Min } X \cup \text{Max } X)$.

Let H denote the Hilbert cube $[0, 1]^N$, partially ordered by the coordinate-wise partial order: $x \leq y$ if and only if $x_n \leq y_n$ for each $n \in N$. Carruth has observed [1] that there is a homeomorphism $\psi : X' \to H$ which is also an order isomorphism. Moreover, $\psi(0) = 0$ and $\psi(1) = 1$.

It is almost immediate that H admits a Whitney map. We need only define

$$\omega_1(x) = \sum_{n=1}^{\infty} 2^{-n} x_n.$$

It follows that if $\omega : X \to [0, 1]$ is defined by $\omega = \omega_1 \psi \varphi$ then ω is a Whitney map for X.

COROLLARY. *If X is a non-degenerate compactum then 2^X admits a Whitney map.*

Proof. It is well-known ([3], page 7) that 2^X is a compact metric space, and we have already noted that 2^X is a partially ordered space with respect to inclusion. It is clear that Min $2^X = \{\{x\} : x \in X\}$ and Max $2^X = \{X\}$ are disjoint closed sets.

It is easy to find examples which show that the hypotheses on Min X and Max X are essential to the theorem. For example, let $X = [0, 1]$ and define $x \leq y$ if and only if $x = y$ or $y = 1$. Then X is a compact metric partially ordered space, Min $X = [0, 1)$ and Max $X = \{1\}$. Therefore, if $\omega : X \to [0, 1]$ satisfies (i) and (ii) then ω is not continuous.

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