6

Linear response theory

Suppose that a solid is hit with a hammer. Sound waves will propagate outwards from the point of contact. How is the frequency of the sound wave related to its wave number? How does a light wave propagate through plasma? What happens when a charge impurity is embedded in an electrically neutral medium? Is it screened, and if so how is that screening described quantitatively? If a medium is disturbed by a small amount one might expect its response also to be small. The quantitative formalism for dealing with small disturbances is called linear response theory. The beauty of the theory is that the response of the system can be expressed as a folding of the external source causing the disturbance with a response function that is computed using equilibrium correlation functions not dependent on the strength of the external source. Therefore, details of the internal dynamics of the thermodynamic system can be studied using weak external probes. Other areas of science where linear response theory has proven to be extremely useful are quite extensive, and include x-ray scattering from crystals and molecules, electron scattering from protons and nuclei, and sound waves generated by earthquakes propagating through the earth's interior.

6.1 Linear response to an external field

Suppose we apply some external field to our system, which is initially in equilibrium. The goal of linear response theory is to calculate the change in the ensemble average value of any operator $Y(\mathbf{x}, t)$ caused by the external field, to first order in that external field.

Let

$$H'(t) = H + H_{\rm ext}(t) \tag{6.1}$$

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where H is the unperturbed Hamiltonian (but which still contains interactions) and $H_{\rm ext}(t)$ is the perturbation that couples the external field to the system. We will imagine that $H_{\rm ext}(t)$ vanishes when $t < t_0$, so that the system has had plenty of time to achieve equilibrium in the past. The exact equation of motion for Y is

$$\frac{\partial Y(\mathbf{x},t)}{\partial t} = i\left[H'(t), Y(\mathbf{x},t)\right] \tag{6.2}$$

Let $|j\rangle$ be an eigenstate of H (in the Heisenberg picture). Then it follows that the time rate of change of the expectation value of Y in the state $|j\rangle$ is

$$\begin{aligned}\frac{\partial\langle j|Y(\mathbf{x},t)|j\rangle}{\partial t} &= i\langle j|\left[H'(t), Y(\mathbf{x},t)\right]|j\rangle \\ &= i\langle j|\left[H_{\rm ext}(t), Y(\mathbf{x},t)\right]|j\rangle\end{aligned} \tag{6.3}$$

Equation (6.3) is exact, but it is generally impossible to solve it in closed form. At this point we assume that $H_{\rm ext}$ causes only a small change in the expectation value of Y. Then to first order in $H_{\rm ext}$ we can integrate (6.3) as

$$\begin{aligned}\delta\langle j|Y(\mathbf{x},t)|j\rangle &= \langle j|Y(\mathbf{x},t)|j\rangle - \langle j|Y(\mathbf{x},t_0)|j\rangle \\ &= i\int_{t_0}^{t} dt'\langle j|\left[H_{\rm ext}(t'), Y(\mathbf{x},t)\right]|j\rangle\end{aligned} \tag{6.4}$$

Now take the (grand canonical) ensemble average,

$$\delta\langle Y(\mathbf{x},t)\rangle = \frac{\sum_j \mathrm{e}^{-\beta K_j}\delta\langle j|Y(\mathbf{x},t)|j\rangle}{\sum_j \mathrm{e}^{-\beta K_j}} \tag{6.5}$$

Here $K = H - \mu_i N_i$, where allowance is made for an arbitrary number of conserved charges. Using (1.1) and (6.4) in (6.5), we obtain

$$\delta\langle Y(\mathbf{x},t)\rangle = i\int_{t_0}^{t} dt'\ \mathrm{Tr}\left\{\hat{\rho}\left[H_{\rm ext}(t'), Y(\mathbf{x},t)\right]\right\} \tag{6.6}$$

This expresses the change in the ensemble-average value of Y in terms of the commutator of $H_{\rm ext}$ and Y evaluated in the unperturbed ensemble, represented by $\hat{\rho}$. We reiterate that (6.6) is correct to first order in $H_{\rm ext}$.

As an example, consider a real scalar field ϕ that is coupled to an external source $J(\mathbf{x},t)$ via

$$H_{\rm ext}(t) = \int d^3x\, J(\mathbf{x},t)\hat{\phi}(\mathbf{x},t) \tag{6.7}$$

We are interested in the change in the ensemble-average value of $\hat{\phi}$ when the external source is turned on. Putting (6.7) into (6.6) with $Y = \hat{\phi}$

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gives

$$\delta\langle\hat{\phi}(\mathbf{x},t)\rangle = -i\int_{t_0}^{t} dt' \int d^3x' J(\mathbf{x}',t')\ \mathrm{Tr}\left\{\hat{\rho}\left[\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t')\right]\right\} \tag{6.8}$$

At this point it is useful to introduce the following quantities:
the time-ordered propagator,

$$iD(\mathbf{x},t;\mathbf{x}',t') = \mathrm{Tr}\left\{\hat{\rho}T_t\left(\hat{\phi}(\mathbf{x},t)\hat{\phi}(\mathbf{x}',t')\right)\right\} \tag{6.9}$$

the retarded Green's function,

$$iD^{\mathrm{R}}(\mathbf{x},t;\mathbf{x}',t') = \mathrm{Tr}\left\{\hat{\rho}\left[\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t')\right]\right\}\theta(t-t') \tag{6.10}$$

the advanced Green's function,

$$iD^{\mathrm{A}}(\mathbf{x},t;\mathbf{x}',t') = -\mathrm{Tr}\left\{\hat{\rho}\left[\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t')\right]\right\}\theta(t'-t) \tag{6.11}$$

In (6.9), T_t is the time-ordering operator. Then (6.8) becomes

$$\delta\langle\hat{\phi}(\mathbf{x},t)\rangle = \int_{-\infty}^{\infty} dt' \int d^3x' J(\mathbf{x}',t') D^{\mathrm{R}}(\mathbf{x},t;\mathbf{x}',t') \tag{6.12}$$

Here we have let $t_0 \to -\infty$ and have set the upper limit of integration over t' to ∞ on account of (6.10).

Since the unperturbed system is in thermal equilibrium, D^{R} must depend only on $\mathbf{x}-\mathbf{x}'$ and $t-t'$ (the former would not be true for a solid or crystal, of course). We insert the Fourier transforms

$$D^{\mathrm{R}}(\mathbf{x}-\mathbf{x}',\ t-t') = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, \mathrm{e}^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]} D^{\mathrm{R}}(\omega,\mathbf{k}) \tag{6.13}$$

$$J(\mathbf{x}',t') = \int \frac{d^3p\, d\alpha}{(2\pi)^4}\, \mathrm{e}^{i(\mathbf{p}\cdot\mathbf{x}'-\alpha t')} \tilde{J}(\alpha,\mathbf{p}) \tag{6.14}$$

into (6.12) to obtain

$$\delta\langle\hat{\phi}(\mathbf{x},t)\rangle = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tilde{J}(\omega,\mathbf{k}) D^{\mathrm{R}}(\omega,\mathbf{k}) \tag{6.15}$$

or

$$\delta\langle\tilde{\phi}(\omega,\mathbf{k})\rangle = \tilde{J}(\omega,\mathbf{k}) D^{\mathrm{R}}(\omega,\mathbf{k}) \tag{6.16}$$

which is a very aesthetic form. The change in the ensemble average of the field, in frequency–momentum space, is equal to the external source times the retarded Green's function.

6.2 Lehmann representation

The question arises how the real time Green's functions required in the linear response approach to dynamical perturbations are obtained. Are they related to the imaginary time propagators studied in previous chapters? In fact they should be, since all dynamical information in a quantum theory is contained in the matrix elements of operators. If both the real time and imaginary time correlation functions can be expressed in terms of matrix elements then a connection can be made. These expressions are referred to as Lehmann representations. We shall work them out for a real scalar field. It is straightforward to do the same for complex scalar fields and for fields with spin, the main complication being the tensorial structures.

Consider the fully interacting ensemble average of a product of scalar field operators. Suppose that the states $|n\rangle$ form a complete set of eigenstates of the Hamiltonian and of the momentum operator. Starting with

$$iD^+(x,y) = \langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{1}{Z}\sum_n \mathrm{e}^{-\beta E_n}\langle n|\hat{\phi}(x)\hat{\phi}(y)|n\rangle \tag{6.17}$$

we insert a complete set of states between the field operators:

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}\langle n|\hat{\phi}(x)|m\rangle\langle m|\hat{\phi}(y)|n\rangle \tag{6.18}$$

Under the assumption that the system is translation invariant in both time and space, the matrix elements at x are related to the matrix elements at $x = 0$ as follows:

$$\langle n|\hat{\phi}(x)|m\rangle = \mathrm{e}^{i(p_n-p_m)\cdot x}\langle n|\hat{\phi}(0)|m\rangle \tag{6.19}$$

Thus the explicit representation of the average of the product of fields is

$$iD^+(x-y) = \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}\mathrm{e}^{i(p_n-p_m)\cdot(x-y)}\langle n|\hat{\phi}(0)|m\rangle\langle m|\hat{\phi}(0)|n\rangle \tag{6.20}$$

The Fourier transform (we use the same symbol D in coordinate space and momentum space for ease of notation)

$$D^+(k) = \int d^4z\, \mathrm{e}^{ik\cdot z}D^+(z) \tag{6.21}$$

can be expressed in terms of the spectral density

$$\rho^+(k) = \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}(2\pi)^3\delta(k-p_m+p_n)|\langle n|\hat{\phi}(0)|m\rangle|^2 \tag{6.22}$$

as

$$iD^+(k) = 2\pi\rho^+(k) \tag{6.23}$$

This spectral density is positive definite. The Dirac delta functions do not affect this, since one can always work in a large but finite box for which the energy and momentum modes are discrete, replacing the Dirac delta functions by Kronecker delta functions.

In a similar way we define

$$iD^-(x,y) = -\langle\hat{\phi}(y)\hat{\phi}(x)\rangle \tag{6.24}$$

whose Fourier transform is also expressed in terms of a spectral density:

$$iD^-(k) = 2\pi\rho^-(k) \tag{6.25}$$

where

$$\rho^-(k) = -\mathrm{e}^{-\beta k_0}\rho^+(k) \tag{6.26}$$

The minus sign comes from the definition and the Boltzmann factor comes from interchanging the labels m and n in the sum over states and using energy conservation. Obviously this spectral density is negative definite.

The ensemble average of the commutator is

$$D^{\mathrm{n}}(x-y) = -i\langle[\hat{\phi}(x),\hat{\phi}(y)]\rangle = D^+(x-y) + D^-(x-y) \tag{6.27}$$

where the superscript "n" denotes the "normal" commutator-defined Green's function. Its spectral density is

$$\begin{aligned}\rho^{\mathrm{n}}(k) &= \rho^+(k) + \rho^-(k) = \left(1-\mathrm{e}^{-\beta k_0}\right)\rho^+(k) = -\left(\mathrm{e}^{\beta k_0}-1\right)\rho^-(k) \\ &= \frac{1}{Z}\sum_{m,n}\left(\mathrm{e}^{-\beta E_n}-\mathrm{e}^{-\beta E_m}\right)(2\pi)^3\delta(k-p_m+p_n)|\langle n|\hat{\phi}(0)|m\rangle|^2\end{aligned} \tag{6.28}$$

For linear response theory the most relevant correlation function is the retarded propagator

$$D^{\mathrm{R}}(z) = \theta(z_0)D^{\mathrm{n}}(z) \tag{6.29}$$

Associated with it is the advanced propagator

$$D^{\mathrm{A}}(z) = -\theta(-z_0)D^{\mathrm{n}}(z) \tag{6.30}$$

Straightforward manipulations show that these can be expressed as integrals over the spectral density ρ^{n}:

$$D^{\mathrm{R}}(k) = -\int_{-\infty}^{\infty}\frac{d\omega}{\omega-k_0-i\varepsilon}\,\rho^{\mathrm{n}}(\omega,\mathbf{k}) \tag{6.31}$$

$$D^{\mathrm{A}}(k) = -\int_{-\infty}^{\infty}\frac{d\omega}{\omega-k_0+i\varepsilon}\,\rho^{\mathrm{n}}(\omega,\mathbf{k}) \tag{6.32}$$

The imaginary parts of these functions are proportional to the spectral density,

$$\mathrm{Im}\,D^{\mathrm{R}}(k) = -\mathrm{Im}\,D^{\mathrm{A}}(k) = -\pi\rho^{\mathrm{n}}(k) \tag{6.33}$$

and their real parts are equal,

$$\operatorname{Re} D^{\mathrm{R}}(k) = \operatorname{Re} D^{\mathrm{A}}(k) \tag{6.34}$$

under the assumption that k is real.

Now we come to the connection with the imaginary time propagator, for which the finite-temperature perturbation theory was developed. From (3.21) we know that

$$\begin{aligned}\mathcal{D}(\mathbf{x},\tau) &= \langle\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(0)\rangle \\ &= \frac{1}{Z}\sum_{n} \mathrm{e}^{-\beta E_n}\langle n|\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(0)|n\rangle \\ &= \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}\langle n|\hat{\phi}(\mathbf{x},\tau)|m\rangle\langle m|\hat{\phi}(0)|n\rangle \end{aligned} \tag{6.35}$$

Just as in (2.86), the field evolves in imaginary time according to

$$\hat{\phi}(\mathbf{x},\tau) = \mathrm{e}^{H\tau}\hat{\phi}(\mathbf{x},0)\mathrm{e}^{-H\tau} \tag{6.36}$$

which leads to

$$\mathcal{D}(\mathbf{x},\tau) = \frac{1}{Z}\sum_{m,n} \mathrm{e}^{-\beta E_n}\mathrm{e}^{\tau(E_n-E_m)}\mathrm{e}^{i(\mathbf{p}_m-\mathbf{p}_n)\cdot\mathbf{x}}\langle n|\hat{\phi}(0)|m\rangle\langle m|\hat{\phi}(0)|n\rangle \tag{6.37}$$

Following the conventions of Chapter 3, the Fourier transform is

$$\begin{aligned}\mathcal{D}(\omega_n,\mathbf{k}) &= \int_0^\beta d\tau\int d^3x\,\mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{x}+\omega_n\tau)}\mathcal{D}(\mathbf{x},\tau) \\ &= \frac{1}{Z}\sum_{m,n}(2\pi)^3\delta(\mathbf{k}-\mathbf{p}_m+\mathbf{p}_n)\langle n|\hat{\phi}(0)|m\rangle\langle m|\hat{\phi}(0)|n\rangle \\ &\quad\times\frac{\mathrm{e}^{-\beta E_m}-\mathrm{e}^{-\beta E_n}}{E_n-E_m-i\omega_n}\end{aligned} \tag{6.38}$$

which can be written in terms of the spectral density as

$$\mathcal{D}(\omega_n,\mathbf{k}) = \int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\omega_n}\,\rho^{\mathrm{n}}(\omega,\mathbf{k}) \tag{6.39}$$

Thus the advanced and retarded propagators can be obtained from the finite-temperature propagator by analytic continuation as follows:

$$D^{\mathrm{R}}(k) = -\mathcal{D}(\omega_n\to ik_0-\varepsilon) \tag{6.40}$$

$$D^{\mathrm{A}}(k) = -\mathcal{D}(\omega_n\to ik_0+\varepsilon) \tag{6.41}$$

The spectral density ρ^{n} determines both the real time and imaginary time propagators and is therefore a very important function.

A concrete example of these relations is provided by a free field. The imaginary time propagator is $\mathcal{D} = 1/(\omega_n^2+\mathbf{k}^2+m^2)$. From this one immediately obtains $\rho^{\mathrm{n}} = \mathrm{sign}(k_0)\,\delta(k_0^2 - \mathbf{k}^2 - m^2)$. This shows quite directly that all the weight is concentrated on the mass shell of the particle. Generally, for interacting particles in a medium, this weight gets spread out over a finite range of energies. The free-particle spectral density has two obvious properties that generalize to interacting systems. One is the symmetry in the sign of the energy and the other is an integral over the energy.

The spectral density ρ^{n} given in (6.28) has the symmetry

$$\rho^{\mathrm{n}}(-\omega, -\mathbf{k}) = -\rho^{\mathrm{n}}(\omega, \mathbf{k}) \tag{6.42}$$

Here $k_0 = \omega$. In a rotationally invariant system, for every state with energy E_n and momentum $\mathbf{p}_n$ there is a state with the same energy and the opposite momentum. Therefore

$$\rho^{\mathrm{n}}(\omega, -\mathbf{k}) = \rho^{\mathrm{n}}(\omega, \mathbf{k}) \tag{6.43}$$

Combining the above symmetries we conclude that ρ^{n} is an odd function of the energy:

$$\rho^{\mathrm{n}}(-\omega, \mathbf{k}) = -\rho^{\mathrm{n}}(\omega, \mathbf{k}) \tag{6.44}$$

The canonical commutation relation can be usefully employed to derive a sum rule on the spectral density. Take the spatial Fourier transform of

$$\lim_{x_0 \to 0} \frac{\partial}{\partial x_0} D^{\mathrm{n}}(x) = -i\langle[\hat{\pi}(0, \mathbf{x}), \hat{\phi}(0, \mathbf{0})]\rangle = -\delta(\mathbf{x}) \tag{6.45}$$

and use the Lehmann representation for D^{n}. One concludes that

$$\int_{-\infty}^{\infty} d\omega\, \omega \rho^{\mathrm{n}}(\omega, \mathbf{k}) = 1 \tag{6.46}$$

This sum rule is naturally obeyed by the free-particle spectral density. It also implies that interactions might modify the shape of the function ρ^{n} but that the total integrated weight is constant.

6.3 Screening of static electric fields

Let us apply an external static electric field $\mathbf{E}_{\mathrm{cl}}$, as might be generated by an imposed charge distribution, to a QED plasma and observe the response. The Hamiltonian density for this interaction is

$$\mathcal{H}_{\mathrm{ext}} = \mathbf{E} \cdot \mathbf{E}_{\mathrm{cl}} \tag{6.47}$$

The external field $\mathbf{E}_{\mathrm{cl}}$ is a classical field, not a quantum operator like $\mathbf{E}$ and $\mathbf{B}$. It depends on position but not on time.

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The change in the electric field caused by the introduction of the external field into the plasma can be computed using (6.6):

$$\delta\langle E_i(\mathbf{x},t)\rangle = -i\int_{-\infty}^{\infty} dt' \int d^3x' E_j^{\rm cl}(\mathbf{x}')\,\mathrm{Tr}\big\{\hat\rho\,\big[E_i(\mathbf{x},t), E_j(\mathbf{x}',t')\big]\big\}\,\theta(t-t') \tag{6.48}$$

Thus we need to know the commutator of two electric field operators. Using the expression for the electric field in terms of the vector potential and the canonical commutation relations, one readily finds that

$$\begin{aligned}\big\langle\big[E_i(\mathbf{x},t), E_j(\mathbf{x}',t')\big]\big\rangle\,\theta(t-t') = {}& \partial_i\partial_j'\left\{\langle[A_0(\mathbf{x},t),\ A_0(\mathbf{x}',t')]\rangle\theta(t-t')\right\}\\ &- \partial_i\partial_0'\left\{\langle[A_0(\mathbf{x},t),\ A_j(\mathbf{x}',t')]\rangle\theta(t-t')\right\}\\ &- \partial_0\partial_j'\left\{\langle[A_i(\mathbf{x},t),\ A_0(\mathbf{x}',t')]\rangle\theta(t-t')\right\}\\ &+ \partial_0\partial_0'\left\{\langle[A_i(\mathbf{x},t), A_j(\mathbf{x}',t')]\rangle\theta(t-t')\right\}\\ &- i\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\end{aligned} \tag{6.49}$$

The real time photon propagator is

$$D^{\rm R}_{\mu\nu}(\mathbf{x}-\mathbf{x}',\, t-t') = i\,\mathrm{Tr}\big\{\hat\rho\,\big[A_\mu(\mathbf{x},t), A_\nu(\mathbf{x}',t')\big]\big\}\,\theta(t-t') \tag{6.50}$$

where the sign is chosen to be compatible with the definition of the imaginary time propagator in Section 5.3. It depends only on the differences $\mathbf{x}-\mathbf{x}'$ and $t-t'$, owing to translation invariance in a plasma and to the assumption of thermal equilibrium. Combining (6.48) to (6.50) we obtain the net electric field in the medium,

$$\begin{aligned}E_i^{\rm net}(\mathbf{x},t) &= E_i^{\rm cl}(\mathbf{x}) + \delta\langle E_i(\mathbf{x},t)\rangle\\ &= \int_{-\infty}^{\infty} dt' \int d^3x' E_j^{\rm cl}(\mathbf{x}')\\ &\quad\times\left(-\partial_i\partial_j' D^{\rm R}_{00} + \partial_i\partial_0' D^{\rm R}_{0j} + \partial_0\partial_j' D^{\rm R}_{i0} - \partial_0\partial_0' D^{\rm R}_{ij}\right)\end{aligned} \tag{6.51}$$

where the arguments of $D^{\rm R}_{\mu\nu}$ are $\mathbf{x}-\mathbf{x}'$ and $t-t'$ as in (6.50). The Fourier transforms are

$$D^{\rm R}_{\mu\nu}(\mathbf{x}-\mathbf{x}',\, t-t') = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\mathrm{e}^{-i\omega(t-t')} D^{\rm R}_{\mu\nu}(\omega,\mathbf{k}) \tag{6.52}$$

and

$$\mathbf{E}_{\rm cl}(\mathbf{x}') = \int \frac{d^3p}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}'}\mathbf{E}_{\rm cl}(\mathbf{p}) \tag{6.53}$$

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Substitution in (6.51) gives

$$E_i^{\rm net}(\mathbf{x}) = -\int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} E_j^{\rm cl}(\mathbf{k}) \Big[k_i k_j D_{00}^{\rm R}(\omega,\mathbf{k}) + \omega k_i D_{0j}^{\rm R}(\omega,\mathbf{k}) + \omega k_j D_{i0}^{\rm R}(\omega,\mathbf{k}) + \omega^2 D_{ij}^{\rm R}(\omega,\mathbf{k}) \Big]_{\omega=0} \tag{6.54}$$

Note that the $\omega = 0$ limit is a consequence of the static nature of the applied field.

In covariant gauges the propagator is given in (5.46). In such gauges the last three terms in (6.51) vanish. Hence the net electric field in momentum space is

$$E_i^{\rm net}(\mathbf{k}) = \frac{k_i k_j E_j^{\rm cl}(\mathbf{k})}{\mathbf{k}^2 + F(\omega = 0,\, \mathbf{k})} \tag{6.55}$$

For a plasma, the net electric field must point in the same direction as the applied external field owing to rotational invariance. The magnitudes can be related by multiplying both sides of the above equation by k_i and summing over i. Thus

$$\mathbf{E}_{\rm net}(\mathbf{k}) = \frac{\mathbf{E}_{\rm cl}(\mathbf{k})}{\epsilon(\mathbf{k})} \tag{6.56}$$

where $\epsilon(\mathbf{k})$ is the static dielectric constant and is given by

$$\epsilon(\mathbf{k}) = 1 + \frac{F(0,\mathbf{k})}{\mathbf{k}^2} \tag{6.57}$$

This result may be obtained in other gauges as well. In the temporal axial gauge, $A_0(\mathbf{x}, t) = 0$, the propagator is given by

$$\begin{gathered} \mathcal{D}_{00} = 0 \qquad \mathcal{D}_{0i} = \mathcal{D}_{0i} = 0 \\ \mathcal{D}_{ij} = \frac{1}{G - k^2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) + \frac{1}{F - k^2}\,\frac{k^2}{k_0^2}\,\frac{k_i k_j}{\mathbf{k}^2} \end{gathered} \tag{6.58}$$

Insertion of (6.58) into (6.54) again yields (6.55), although it is interesting that in this gauge the contributing term is $[\omega^2 D_{ij}^{\rm R}(\omega,\mathbf{k})]_{\omega=0}$. In the Coulomb gauge, $\nabla\cdot\mathbf{A}(\mathbf{x}, t) = 0$, the propagator is given by

$$\mathcal{D}^{\mu\nu} = \frac{1}{G - k^2} P_{\rm T}^{\mu\nu} + \frac{1}{F - k^2}\,\frac{k^2}{\mathbf{k}^2}\, u^\mu u^\nu \tag{6.59}$$

where $u^\mu = (1, 0, 0, 0)$ defines the rest frame of the medium. The self-energy $\Pi^{\mu\nu}$ is related to F and G just as in (5.46). This may be verified by returning to the definition of the self-energy in terms of the full and bare propagators, which may be written as

$$\mathcal{D}^{\mu\nu} = \mathcal{D}_0^{\mu\nu} - \mathcal{D}_0^{\mu\alpha}\Pi_{\alpha\beta}\mathcal{D}^{\beta\nu} \tag{6.60}$$

Substitution of (6.59) into (6.54) again yields (6.55).

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It must be emphasized that (6.55) is an exact result, to be used with the exact expression for $F(0,\mathbf{k})$ or with the best available approximation to it. The only assumption is that the applied field $\mathbf{E}_{\rm cl}$ is weak enough to justify the linearity approximation.

The dielectric function is the screening factor. In the limit of no interactions, where $e \to 0$ and $F \to 0$, the net electric field equals the applied field. In the absence of matter $T = 0$ and $\mu = 0$, but with interactions turned on, $e \neq 0$, there is still a modification of the applied electric field known as vacuum polarization. When $|\mathbf{k}| \ll m_e$, one finds that

$$F_{\rm vac}(0,\mathbf{k}) \approx -\frac{\alpha}{15\pi}\frac{|\mathbf{k}|^4}{m_e^2} \tag{6.61}$$

When $|\mathbf{k}| \gg m_e$,

$$F_{\rm vac}(0,\mathbf{k}) \approx -\frac{\alpha}{3\pi}\mathbf{k}^2 \ln\left(\frac{\mathbf{k}^2}{M^2}\right) \tag{6.62}$$

where M is the renormalization energy scale. One may think of virtual electron–positron pairs continually popping out of and back into the vacuum to produce this modification of the applied field. Since $\epsilon(|\mathbf{k}| > 0) \neq 1$, one may in this sense think of the vacuum as a medium. Furthermore we may think of the dielectric constant as the ratio of the squared net observed charge at momentum transfer $\mathbf{k}$ to the squared ordinary electric charge at zero momentum transfer,

$$\frac{\alpha_{\rm net}(\mathbf{k})}{\alpha} = \frac{1}{1 + F_{\rm vac}(0,\mathbf{k})/\mathbf{k}^2} \tag{6.63}$$

Substitution of (6.62) into (6.63) gives exactly the lowest-order renormalization-group result, (5.82) with $\mu \to |\mathbf{k}| \gg m_e$, which is no coincidence.

The one-loop finite-temperature and finite-density contribution to F is in general a complicated function of $\mathbf{k}$. It is given in (5.51) since $F(0,\mathbf{k}) = -\Pi^{00}(0,\mathbf{k})$. At very short distances, $|\mathbf{k}| \gg T$ and μ, the vacuum contribution dominates, $F_{\rm vac} \gg F_{\rm mat}$. At very long distances, $|\mathbf{k}| \ll T$ and μ, the matter contribution dominates, $F_{\rm vac} \ll F_{\rm mat}$. At distances much less than the average interparticle spacing, many-body effects cannot be important and one recovers the vacuum. At distances much greater than the average interparticle spacing, many-body effects are most important. In fact $F_{\rm vac}/F_{\rm mat} \propto \mathbf{k}^2$ as $\mathbf{k} \to 0$, modulo logarithms. Recalling (5.66), (5.68), and (5.69), we define the QED electric mass $m_{\rm el}$ by $m_{\rm el}^2 = F(0,\,\mathbf{k} \to 0)$. Then, approximately,

$$\epsilon(\mathbf{k}) \approx 1 + \frac{F_{\rm vac}(0,\mathbf{k})}{\mathbf{k}^2} + \frac{m_{\rm el}^2}{\mathbf{k}^2} \tag{6.64}$$

Linear response theory gives both vacuum polarization and plasma screening.

6.4 Screening of a point charge

As a concrete demonstration of a situation commonly encountered, place a static charge Q_1 at $\mathbf{x}_1$ and another static charge Q_2 at $\mathbf{x}_2$. What is the change in free energy of the QED plasma as a function of separation? Analogous problems arise in condensed matter physics when treating an impurity or defect.

From Gauss's law,

$$\nabla \cdot \mathbf{E}_1^{\text{cl}} = Q_1 \, \delta(\mathbf{x} - \mathbf{x}_1) \tag{6.65}$$

we obtain

$$\mathbf{E}_1^{\text{cl}}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{E}_1^{\text{cl}}(\mathbf{k})$$

where

$$\mathbf{E}_1^{\text{cl}} = -i \frac{\mathbf{k}}{\mathbf{k}^2} \, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}_1} \, Q_1 \tag{6.66}$$

Similar equations are obtained for charge 2. The change in free energy is

$$V(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} \int d^3x \left[\mathbf{E}_1^{\text{cl}}(\mathbf{x}) \cdot \langle \mathbf{E}_2(\mathbf{x}) \rangle + \mathbf{E}_2^{\text{cl}}(\mathbf{x}) \cdot \langle \mathbf{E}_1(\mathbf{x}) \rangle \right]$$

where

$$\langle \mathbf{E}_1(\mathbf{x}) \rangle = \mathbf{E}_1^{\text{net}}(\mathbf{x}) \qquad \langle \mathbf{E}_2(\mathbf{x}) \rangle = \mathbf{E}_2^{\text{net}}(\mathbf{x}) \tag{6.67}$$

After some manipulation, this takes the form

$$V(\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2) = Q_1 Q_2 \int \frac{d^3k}{(2\pi)^3} \frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + F(0, \mathbf{k})} \tag{6.68}$$

When r is very large, the dominant contribution to the integral comes from $\mathbf{k} \approx 0$. For this case, we replace $F(0, \mathbf{k})$ by its infrared limit m_{el}^2. Then we get

$$V(r) = \frac{Q_1 Q_2}{4\pi} \frac{\mathrm{e}^{-m_{\text{el}} r}}{r} \tag{6.69}$$

which is a screened Coulomb potential with inverse screening length m_{el}.

At $T = \mu = 0$ one may compute the change in the form of Coulomb's law due to vacuum polarization by expanding (6.68) to first order in F and substituting in (6.61). The result is

$$\Delta V_{\text{C}} = \frac{\alpha}{15\pi m_e^2} Q_1 Q_2 \delta(\mathbf{r}) \tag{6.70}$$

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This result was first obtained by Uehling [1] and by Serber [2]. See also Bjorken and Drell [3]. Its effect has been measured in the Lamb shift in atomic hydrogen.

At low temperatures, $T \ll |\mu|$, the functional form of (6.69) is not correct even at long distances; it turns out that it is not a good approximation to replace $F(0, \mathbf{k})$ by its infrared limit $m_{\rm el}^2$ because of the sharp Fermi surface.

The formula (5.51) gives the matter part of F at one-loop order for arbitrary values of external energy, momentum, temperature, and chemical potential. Evaluating it at zero energy (which is the same as at zero Matsubara frequency) and $T = 0$ gives

$$\begin{aligned} F_{\rm mat}&(0, k) \\ &= \frac{e^2}{24\pi^2}\left[16\mu k_{\rm F} - 4k^2 \ln\left(\frac{\mu + k_{\rm F}}{m}\right) - \frac{\mu(4\mu^2 - 3k^2)}{k}\ln\left(\frac{k - 2k_{\rm F}}{k + 2k_{\rm F}}\right)^2 \right. \\ &\qquad + \frac{(2m^2 - k^2)\sqrt{k^2 + 4m^2}}{k} \\ &\qquad \left. \times \ln\left(\frac{2\mu^2(k^2 + 2m^2) - 2\mu k k_{\rm F}\sqrt{k^2 + 4m^2} - m^2(k^2 + 4m^2)}{2\mu^2(k^2 + 2m^2) + 2\mu k k_{\rm F}\sqrt{k^2 + 4m^2} - m^2(k^2 + 4m^2)}\right)\right] \end{aligned} \tag{6.71}$$

Here $k_{\rm F} = \sqrt{\mu^2 - m^2}$ is the Fermi momentum and $k = |\mathbf{k}|$. The vacuum part is derived in many books on QED, such as Berestetskii, Lifshitz, and Pitaevskii [4] and Quigg [5]. It is

$$\begin{aligned} F_{\rm vac}&(0, k) \\ &= -\frac{e^2}{4\pi^2}k^2\left[\frac{4m^2}{3}\frac{M^2 - k^2}{M^2 k^2} \right. \\ &\qquad + \frac{1}{3M}\left(1 - \frac{2m^2}{M^2}\right)\sqrt{M^2 + 4m^2}\ln\left(\frac{\sqrt{M^2 + 4m^2} - M}{\sqrt{M^2 + 4m^2} + M}\right) \\ &\qquad \left. - \frac{1}{3k}\left(1 - \frac{2m^2}{k^2}\right)\sqrt{k^2 + 4m^2}\ln\left(\frac{\sqrt{k^2 + 4m^2} - k}{\sqrt{k^2 + 4m^2} + k}\right)\right] \end{aligned} \tag{6.72}$$

where M is an arbitrary subtraction point such that $F_{\rm vac}(0, M) = 0$.

The integrand in (6.68) has poles at $k = \pm i m_{\rm el} \approx \pm i\sqrt{F(0, k \to 0)}$. The contribution from these poles gives a Debye screening function of the form (6.69). The integrand also has a pair of branch points at $k = 2k_{\rm F} \pm i\varepsilon$ and a mirror pair at $k = -2k_{\rm F} \pm i\varepsilon$. The branch cuts can be taken to be vertical lines going up from the points $k = \pm 2k_{\rm F} + i\varepsilon$ and vertical lines going down from the points $k = \pm 2k_{\rm F} - i\varepsilon$. The contribution to the potential between point charges from these branch cuts is tedious but straightforward to evaluate. The result for asymptotically large r is

$$V(r) = \frac{Q_1 Q_2 e^2}{4\pi^3} \frac{\mu}{(4+a)^2} \left\{ \frac{m^2}{\mu^2} \frac{\cos\, 2k_{\rm F} r}{(k_{\rm F} r)^3} - \frac{\sin\, 2k_{\rm F} r}{(k_{\rm F} r)^4} \right.$$
$$\left. \times \left[\frac{e^2}{\pi^2} \frac{m^4}{\mu^3 k_{\rm F}} \frac{1}{4+a} \left(\ln(4k_{\rm F} r) + \gamma_{\rm E} - \frac{3}{2} \right) - \frac{16}{4+a} \frac{m^2}{\mu^2} + \frac{m^4}{2\mu^4} - \frac{k_{\rm F}^2}{\mu^2} \right] \right\} \tag{6.73}$$

where $a = F(0, 2k_{\rm F})/k_{\rm F}^2$. The terms neglected in this expression are one order higher either in $1/r$ or in e^2. The contribution from the branch cuts dominates the Debye contribution at large r because the latter falls exponentially in r whereas the former falls as a power.

There are two especially interesting limits of this potential. Let us write $Q_i = Z_i e$. The nonrelativistic limit, $k_{\rm F} \ll m$, is

$$V(r) = \frac{Z_1 Z_2 e^2 \xi k_{\rm F}}{2\pi(4+\xi)^2} \frac{\cos(2k_{\rm F} r)}{(k_{\rm F} r)^3} \tag{6.74}$$

with

$$\xi = \frac{e^2}{2\pi^2} \frac{m}{k_{\rm F}}$$

This form of screening is usually referred to as Friedel oscillation in low-temperature physics (Fetter and Walecka [6]) and can be observed in the nuclear magnetic resonance lines in dilute alloys [7].

The relativistic limit, $k_{\rm F} \gg m$, is

$$V(r) = Z_1 Z_2 \frac{\bar{\alpha}^2}{4\pi} \frac{\sin\, 2k_{\rm F} r}{k_{\rm F}^3 r^4} \tag{6.75}$$

This involves the renormalization-group running coupling

$$\bar{\alpha} = \frac{\alpha}{1 - (2\alpha/3\pi)\, \ln(4\mu/eM)} = \frac{3\pi}{2\ln(e\Lambda_{\rm MOM}/4\mu)} \tag{6.76}$$

where $\Lambda_{\rm MOM}$ is the QED scale parameter. This is familiar from (5.80)–(5.82). The relativistic results were obtained by Sivak [8] and by Kapusta and Toimela [9]. There may be applications to the dense matter present in white dwarf and neutron stars.

Finally, consider what happens at small but nonzero temperature, $T \ll |\mu|$. The sharp Fermi surface is smeared over a thickness T in the energy. Consequently, the branch cuts do not extend to the real axis, and the branch points are shifted by an amount $2\pi\mu T i/k_{\rm F}$. Then the asymptotic formula for the potential must be multiplied by the factor $\exp(-2\pi\mu T r/k_{\rm F})$. When $T^2 > e^2 k_{\rm F}^3/4\pi^4\mu$ the contribution from the pole, $k \sim i m_{\rm el}$, begins to dominate the oscillating terms coming from the branch

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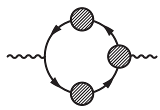
cuts. For a white dwarf star with $k_{\rm F} = 4m_e$ the crossover would be at 3×10^8 K or 30 keV.

6.5 Exact formula for screening length in QED

It is possible to derive an exact formula for the screening length of static electric fields in QED. This formula connects the screening length to the thermodynamic equation of state and so is a very interesting relation indeed.

An exact expression for the photon self-energy, known as the Schwinger–Dyson equation [10, 11], is

$$\Pi^{\mu\nu}(k) = e^2 T \sum_{n_p} \int \frac{d^3p}{(2\pi)^3} \, \mathrm{Tr}\left[\gamma^\mu \mathcal{G}(p) \Gamma^\nu(p, p-k) \mathcal{G}(p-k)\right]$$

$=$  (6.77)

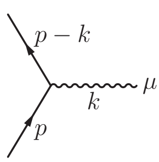
Here the blobs on the fermion lines represent the exact fermion propagator $\mathcal{G}$, and the blob at the vertex represents the exact photon–fermion vertex function Γ^μ. The latter depends in general on the incoming fermion momentum p and the outgoing fermion momentum $p-k$. To lowest order, the photon–fermion vertex function is the point (or contact) coupling appearing in the Lagrangian,

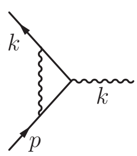
$$\Gamma_0^\mu = \gamma^\mu \tag{6.78}$$

Corrections due to interactions may be found order by order, by applying the formula

$$-e\Gamma = (\delta \ln Z / \delta \Gamma_0)_{\rm 1PI} \tag{6.79}$$

which may be derived in a way analogous to (3.35). For example, from (5.39), (5.62), and (5.77) we obtain

$-e\Gamma^\mu(p, p-k) =$ 

$+$ 

$+ \cdots$ (6.80)

It should be clear intuitively that a relation exists between the fermion propagator and the photon–fermion vertex, since the latter represents the propagation of a fermion while emitting a photon of momentum k. To see what this relation might be, consider the free-fermion inverse propagator

$$\mathcal{G}_0^{-1}(p) = \not{p} - m \tag{6.81}$$

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We notice that

$$\frac{\partial \mathcal{G}_0^{-1}}{\partial p_\mu} = \gamma^\mu = \Gamma_0^\mu \tag{6.82}$$

It turns out that the exact result is

$$\frac{\partial \mathcal{G}^{-1}}{\partial p_\mu} = \lim_{\delta_\mu \to 0} \Gamma^\mu(p, p - \delta_\mu) \tag{6.83}$$

where only the μ-component of the four-vector δ_μ is nonzero. Equation (6.83) is known as the differential form of Ward's identity. It relates the momentum derivative of the exact inverse fermion propagator to the exact photon–fermion vertex in the limit $k \to 0$.

The only change in the derivations of the Schwinger–Dyson equations and the Ward identity at $T > 0$ and $\mu \neq 0$ is the substitution of the frequency sums for energy integrals (the interested reader should consult Bjorken and Drell [3]; see also Fradkin [12], whose arguments we are following here).

At finite temperature and density, in the imaginary time formalism $p^0 = (2n_p + 1)\pi T i + \mu$. Thus from (6.83)

$$\frac{\partial \mathcal{G}^{-1}}{\partial \mu} = \Gamma^0(p, p) \tag{6.84}$$

The screening length follows from Π^{00} in the static infrared limit. Combining (6.77) and (6.84) yields

$$\begin{aligned} m_{\text{el}}^2 &= -\Pi^{00}(k_0 = 0, \mathbf{k} \to 0) \\ &= -e^2 T \sum_{n_p} \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}\left(\gamma^0 \mathcal{G}(p) \frac{\partial \mathcal{G}^{-1}}{\partial \mu}(p) \mathcal{G}(p)\right) \\ &= e^2 \frac{\partial}{\partial \mu} T \sum_{n_p} \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}\left[\gamma^0 \mathcal{G}(p)\right] \\ &= e^2 \left(\frac{\partial n}{\partial \mu}\right)_T = e^2 \frac{\partial^2 P(\mu, T)}{\partial \mu^2} \end{aligned} \tag{6.85}$$

The electric screening length is directly related to the equation of state.

To see how remarkable (6.85) is, notice that the static infrared limit of the photon propagator at one-loop order is determined by the pressure of a *noninteracting* fermion gas. To show the power of (6.85) we recall the formula for $P(\mu, T)$ for a massless electron–positron plasma from Exercise 2.7 and from (5.60), (5.67), and (5.68). Since the pressure is known to order e^3 when both T and μ are nonzero, the inverse screening length is known to order e^5. For $\mu = 0$,

$$m_{\text{el}}^2 = \left(\frac{e^2}{3} - \frac{e^4}{8\pi^2} + \frac{e^5}{4\sqrt{3}\pi^3} + \cdots \right) T^2 \tag{6.86}$$

This expression is phrased in terms of a fixed coupling constant e evaluated at a fixed scale. Let us denote that scale by M_0. At some other scale M the coupling constant changes to

$$e^2(M) = e^2(M_0) \left[1 + \frac{e^2(M_0)}{6\pi^2} \ln\left(\frac{M}{M_0}\right) \right] \tag{6.87}$$

according to the renormalization group. Then

$$m_{\text{el}}^2 = \left\{ \frac{e^2(M)}{3} - \frac{e^4(M)}{18\pi^2} \left[\ln\left(\frac{M}{M_0}\right) + \frac{9}{4} \right] + \frac{e^5(M)}{4\sqrt{3}\pi^3} + \cdots \right\} T^2 \tag{6.88}$$

The issue is how best to choose M and M_0 to minimize higher-order contributions. This may be resolved as follows.

Return to (6.68) and expand $F(0, \mathbf{k})$ in powers of $|\mathbf{k}|$, keeping terms up to and including $\mathbf{k}^2$. Including both the vacuum and finite-temperature parts, and using the above expression for $m_{\text{el}}^2(e)$, the integrand of (6.68) becomes

$$\frac{e^2}{m_{\text{el}}^2(e) + \{1 - (e^2/6\pi^2)\,[\ln(\pi T/M) + 4/3 - \gamma_{\text{E}}]\}\,\mathbf{k}^2}$$

If we use the electric screening mass to one-loop order only, this becomes

$$\frac{\bar{e}^2}{m_{\text{el}}^2(\bar{e}) + \mathbf{k}^2}$$

where

$$\bar{e}^2(T) = \frac{e^2}{1 - (e^2/6\pi^2)\,[\ln(\pi T/M) + 4/3 - \gamma_{\text{E}}]} = \frac{6\pi^2}{\ln\left(\mathrm{e}^{\gamma_{\text{E}} - 4/3}\Lambda/\pi T\right)} \tag{6.89}$$

The fixed coupling constant has been replaced by the renormalization-group running coupling with the absolute scale determined naturally. If we use the electric screening mass to order e^5 we get

$$m_{\text{el}}^2 = \left\{ \frac{e^2}{3} + \frac{e^4}{18\pi^2} \left[\ln\left(\frac{\pi T}{M}\right) - \gamma_{\text{E}} - \frac{11}{12} \right] + \frac{e^5}{4\sqrt{3}\pi^3} + \cdots \right\} T^2 \tag{6.90}$$

Expressing e in terms of $\bar{e}$ gives *exactly* the formula (6.85) with the fixed coupling replaced by the running coupling:

$$m_{\text{el}}^2 = \left(\frac{\bar{e}^2}{3} - \frac{\bar{e}^4}{8\pi^2} + \frac{\bar{e}^5}{4\sqrt{3}\pi^3} + \cdots \right) T^2 \tag{6.91}$$

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This has been verified in an explicit diagrammatic analysis by Blaizot, Iancu, and Parwani [13] (the constant following the logarithm in this work is different, on account of the different renormalization schemes).

The relation (6.85) can be understood very simply. Insert a charge Q at location $\mathbf{x}$ in an electron–positron plasma. If the plasma has a charge density $-en$ then there must be a net background charge density to ensure charge neutrality. Denote this background charge density by en_0, so that in equilibrium $n = n_0$. Owing to the insertion of the charge Q there will be a rearrangement of electrons and positrons in the plasma. The condition of local hydrostatic equilibrium requires a balance of forces:

$$-\nabla P = en\mathbf{E}_{\rm net} \tag{6.92}$$

Poisson's equation is

$$\nabla \cdot \mathbf{E}_{\rm net} = [Q\delta(\mathbf{x}) - e(n - n_0)] \tag{6.93}$$

where $\mathbf{E}_{\rm net}$ is the net electric field due to the external charge Q and the consequent rearrangement of the charged particles in the plasma. In equilibrium, T must be uniform but the charge chemical potential μ may depend on position. Thus we write $P = P(\mu, T)$, $T =$ constant, $\mu = \mu(\mathbf{x})$, and seek to determine $\mu(\mathbf{x})$. Let μ_0 denote the chemical potential in the absence of the charge, and let $\delta\mu(\mathbf{x})$ denote the difference after the introduction of the charge. Then $\nabla P = (\partial P/\partial\mu)\nabla\delta\mu$, and $n - n_0 = (\partial n/\partial\mu)\delta\mu$. Taking the divergence of $\mathbf{E}_{\rm net}$ in (6.92), identifying it with (6.93), and using the above information we arrive at the expressions

$$\begin{aligned} \left(\nabla^2 - m_{\rm el}^2\right)\delta\mu &= -eQ\delta(\mathbf{x}) \\ m_{\rm el}^2 &= e^2\,\frac{\partial^2 P}{\partial\mu^2} \end{aligned} \tag{6.94}$$

which have the solution

$$\delta\mu(r) = \frac{eQ}{4\pi r}\,\mathrm{e}^{-m_{\rm el}r} \tag{6.95}$$

This is the Thomas–Fermi approximation. Clearly (6.95) is only valid for large r, since the derivation assumes that $|\delta\mu/\mu| \ll 1$. At short distances, the momentum dependence of $F(0, \mathbf{k})$ in (6.68) cannot be neglected and the Thomas–Fermi result is modified. This result is also incorrect for a cold Fermi gas, as already detailed in Section 6.4.

6.6 Collective excitations

Instead of applying a static external field, let us hit the system with an impulsive perturbation. Without loss of generality, we may focus on a single Fourier component. Thus, for the scalar field discussed in Section 6.1,

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we take

$$J(\mathbf{x},t) = J_0(\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\delta(t)$$
$$\tilde{J}(\omega,\mathbf{q}) = (2\pi)^3 J_0(\mathbf{k})\delta(\mathbf{q}-\mathbf{k}) \tag{6.96}$$

This leads to the field response

$$\delta\langle\hat{\phi}(\mathbf{x},t)\rangle = J_0(\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\mathrm{e}^{-i\omega t}\, D^{\mathrm{R}}(\omega,\mathbf{k}) \tag{6.97}$$

The retarded Green function is analytic in the upper half-plane. Suppose that it has a simple pole located at $\omega = \omega(\mathbf{k}) - i\gamma(\mathbf{k})$ with $\gamma(\mathbf{k}) \geq 0$. Then

$$D^{\mathrm{R}}(\omega,\mathbf{k}) = \frac{R(\mathbf{k})}{\omega - \omega(\mathbf{k}) + i\gamma(\mathbf{k})} \tag{6.98}$$

where $R(\mathbf{k})$ is the residue. Evaluation of (6.97) leads to

$$\delta\langle\hat{\phi}(\mathbf{x},t)\rangle = -iJ_0(\mathbf{k})R(\mathbf{k})\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t)}\,\mathrm{e}^{-\gamma(\mathbf{k})t} \tag{6.99}$$

The field response is a traveling wave with dispersion relation $\omega(\mathbf{k})$ and damping constant $\gamma(\mathbf{k})$.

For a free field with mass m, $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2+m^2}$, $\gamma(\mathbf{k}) = 0$, and $R(\mathbf{k}) = \frac{1}{2}\omega(\mathbf{k})$. The amplitude of the wave is proportional to the residue and to the Fourier amplitude of the impulse.

6.7 Photon dispersion relation

Let us consider a QED plasma at such high temperature or density that the electron mass may be neglected. Based on the previous discussion, we would expect that the poles of the photon propagator would give the dispersion relations for traveling electromagnetic waves in the plasma. However, this requires careful consideration owing to the fact that the photon propagator is gauge dependent.

Transverse oscillations have a dispersion relation determined by

$$k_0^2 = \mathbf{k}^2 + G(k_0,\mathbf{k}) \tag{6.100}$$

in the temporal axial gauge (6.58), in the Coulomb gauge (6.59), and in the covariant gauges (5.46). We write $k_0 = \omega - i\gamma$ and assume weak damping, otherwise the oscillations would not propagate. The above equation can be decomposed into real and imaginary parts:

$$\omega^2 = \mathbf{k}^2 + \mathrm{Re}\,G(\omega,\mathbf{k})$$
$$\gamma = -\frac{\mathrm{Im}\,G(\omega,\mathbf{k})}{2\omega} \tag{6.101}$$

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Even at one-loop order, $G(k_0, \mathbf{k})$ is a complicated function. In general, the solutions can only be found numerically. In the limit of short or long wavelengths, however, analytical results may be found.

For short wavelengths we expect that the modification of the free-photon dispersion relation $\omega = |\mathbf{k}|$ by medium effects will be small. The reason is that if we probe the system at distances considerably less than the average interparticle spacing then medium effects should tend to zero. Thus we look for a solution when $\omega \approx |\mathbf{k}| \gg T, |\mu|$. From Exercise 5.7 we know that

$$G(\omega = |\mathbf{k}|) = \frac{1}{2}e^2\left(\frac{1}{3}T^2 + \frac{\mu^2}{\pi^2}\right) \equiv m_{\mathrm{P}}^2 \tag{6.102}$$

which is precisely $\frac{1}{2}m_{\mathrm{el}}^2 = \frac{1}{2}F(k_0 = 0, \mathbf{k} \to 0)$ to order e^2. The short-wavelength dispersion relation is then

$$\omega^2 = \mathbf{k}^2 + m_{\mathrm{P}}^2 + \cdots \tag{6.103}$$

Clearly this is a gauge invariant result. One may think of the high-momentum photons as having acquired a mass m_{P} due to plasma interactions.

For the long-wavelength transverse oscillations, we expect a substantial modification of the free-photon dispersion relation owing to many-body effects. The oscillatory electric and magnetic fields will cause any nearby electrons and positrons to be accelerated, giving the oscillation inertia. In fact, one might expect that it would take a finite amount of energy to excite an oscillation with vanishing momentum. To look for a solution to (6.100) and (6.101) we calculate G in the limit $|k^2| = |k_0^2 - \mathbf{k}^2| \ll T^2$. The functions F and G may be obtained from a combination of (5.46), (5.48), and (5.51). This is a straightforward calculation leading to [14]

$$G(k_0, \mathbf{k}) = m_{\mathrm{P}}^2 - \frac{1}{2}F(k_0, \mathbf{k}) \tag{6.104}$$

and

$$F(k_0, \mathbf{k}) = -2m_{\mathrm{P}}^2 \frac{k^2}{|\mathbf{k}|^2}\left[1 - \frac{k_0}{4|\mathbf{k}|}\ln\left(\frac{k_0 + |\mathbf{k}|}{k_0 - |\mathbf{k}|}\right)^2\right] \tag{6.105}$$

It must be emphasized that these expressions are valid not just for the case where $|k_0|$ and $|\mathbf{k}|$ are small compared with T but also near the light cone. In fact, note that $G(k^2 = 0) = m_{\mathrm{P}}^2$, the same as the limit obtained from the exact one-loop expression for G.

For small momenta we find that

$$G(|\mathbf{k}| \ll \omega < T, |\mu|) = \omega_{\mathrm{P}}^2\left(1 + \frac{\mathbf{k}^2}{5\omega^2} + \cdots\right) \tag{6.106}$$

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The plasma frequency $\omega_{\rm P}$ is related to the electric mass and the photon mass via $\omega_{\rm P}^2 = \frac{1}{3}m_{\rm el}^2 = \frac{2}{3}m_{\rm P}^2$ at order e^2 when the electron mass is set to zero. The long-wavelength dispersion relation for transverse excitations is

$$\omega^2 = \omega_{\rm P}^2 + \tfrac{6}{5}\mathbf{k}^2 + \cdots \tag{6.107}$$

Indeed, it does take a finite energy to excite one of these modes even at zero momentum.

Next we turn to longitudinal oscillations, or compressional charge-density waves. Without doing the full linear response analysis in each gauge, we would expect that the dispersion relation is determined by the poles of the following functions in the specified gauge:

temporal axial

$$\frac{1}{k^2 - F}\,\frac{k^2}{k_0^2}$$

Coulomb

$$\frac{1}{k^2 - F}\,\frac{k^2}{\mathbf{k}^2}$$

covariant

$$\frac{1}{k^2 - F} \tag{6.108}$$

Some of the subtleties involved in gauge invariance now arise.

Consider the limit of no interactions. Then $F = 0$, and the covariant gauges produce the spectrum $\omega = |\mathbf{k}|$, whereas in the temporal axial and Coulomb gauges there is no wave propagation. This could have been anticipated. Free electromagnetic radiation is transversely polarized. The temporal axial and Coulomb gauges are physical gauges in the sense that they have the correct number of polarization degrees of freedom, namely, two. The covariant gauges are unphysical in the same sense since they have four degrees of freedom. The extra two degrees of freedom are canceled by the ghosts in the partition function. There is nothing wrong in all this, but one must be careful to ask only physical questions of the theory. The situation is not altered when interactions are turned on at $T = \mu = 0$. Recall that $F = (k^2/\mathbf{k}^2)\Pi_{00}$. It turns out that Π_{00} is not singular enough at $k^2 = 0$ to cancel the factor of k^2. For example, to order e^2,

$$F_{\rm vac} = \frac{\alpha}{3\pi}k^2 \ln\left(\frac{-k^2}{M^2}\right) \tag{6.109}$$

The covariant gauges still have a singularity at $k^2 = 0$, a branch point due to pair production, while the other two gauges do not. The conclusion is that short-wavelength longitudinal excitations do not propagate.

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The spectrum of long-wavelength longitudinal excitations in the plasma is manifestly gauge invariant and is determined by

$$k_0^2 = \mathbf{k}^2 + F(k_0, \mathbf{k}) \tag{6.110}$$

or equivalently

$$\mathbf{k}^2 = \Pi_{00}(k_0, \mathbf{k}) \tag{6.111}$$

Decomposing into real and imaginary parts gives

$$\begin{aligned} \mathbf{k}^2 &= \operatorname{Re}\Pi_{00}(\omega, \mathbf{k}) \\ \gamma_{\mathrm{L}} &= \frac{\operatorname{Im}\Pi_{00}(\omega, \mathbf{k})}{\partial \operatorname{Re}\Pi_{00}(\omega, \mathbf{k})/\partial\omega} \end{aligned} \tag{6.112}$$

Expanding Π_{00} in powers of $\mathbf{k}^2/k_0^2$ leads to

$$\Pi_{00}(\omega, \mathbf{k}) = \omega_{\mathrm{P}}^2 \left(1 + \frac{3\mathbf{k}^2}{5\omega^2} + \cdots\right) \frac{\mathbf{k}^2}{\omega^2} \tag{6.113}$$

and finally to the dispersion relation

$$\omega^2 = \omega_{\mathrm{P}}^2 + \tfrac{3}{5}\mathbf{k}^2 + \cdots \tag{6.114}$$

The energy at zero momentum for longitudinal and transverse excitations is the same, which is no surprise since at zero momentum there is no distinction between longitudinal and transverse modes.

At arbitrary momentum, the dispersion relation cannot be obtained by analytic means for either mode. They must be found by numerical methods.

It is interesting that the damping constants as determined by the approximate expressions (6.104) and (6.105) are zero. However, if one returns to the exact one-loop expressions for F and G it turns out that

$$\gamma_{\mathrm{T}} = \gamma_{\mathrm{L}} = \frac{e^2}{24\pi}\omega_{\mathrm{P}} \tag{6.115}$$

at zero momentum. The origins of the various factors in this result are not difficult to find. The factor e^2 comes from the square of the photon–electron or photon–positron vertex and the factor ω_{P} comes from phase space.

Lastly, notice that the propagator in the covariant gauges has a term $\rho k^\mu k^\nu/k^2$. Clearly, no physical significance should be attached to this pole since the residue is proportional to the gauge parameter and in fact vanishes in the Landau gauge $\rho = 0$.

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6.8 Electron dispersion relation

The electron propagator is

$$\mathcal{G}(p) = \frac{1}{\not p - m_e + \Sigma(p)} \tag{6.116}$$

In the Feynman gauge the one-loop expression for the self-energy is

$$\Sigma(p) = e^2T^2 \sum_{n_k}\sum_{n_q} \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\,\frac{1}{k^2}\gamma^\mu \frac{1}{\not q - m_e}\gamma_\mu \beta \delta_{n_p, n_k+n_q}(2\pi)^3\delta(\mathbf{p}-\mathbf{k}-\mathbf{q}) \tag{6.117}$$

At very high temperature the electron mass may be neglected. The evaluation of this self-energy is rather tedious but straightforward. The vacuum contribution may be found in numerous textbooks. Here we shall focus on the matter contribution.

The leading contribution at order T^2 and μ^2 is [15]

$$\begin{aligned}
\Sigma^0_{\rm mat} &= -\frac{m_F^2}{8|\mathbf{p}|}\ln\left(\frac{p_0+|\mathbf{p}|}{p_0-|\mathbf{p}|}\right)^2 \\
\boldsymbol{\Sigma}_{\rm mat} &= \frac{m_F^2}{2|\mathbf{p}|^2}\mathbf{p}\left[1 - \frac{p_0}{4|\mathbf{p}|}\ln\left(\frac{p_0+|\mathbf{p}|}{p_0-|\mathbf{p}|}\right)^2\right]
\end{aligned} \tag{6.118}$$

where $m_F^2 = \frac{1}{2}(m_P^2 + \frac{1}{3}e^2T^2) = \frac{1}{4}e^2(T^2 + \mu^2/\pi^2)$. Equations (6.118) may be compared with the corresponding expressions for F and G for the photon self-energy. Although the electron self-energy is in general gauge dependent, the leading contributions (6.118) can be shown to be independent of the gauge. As with the photon self-energy, it must be emphasized that these expressions are valid not only for small electron energy and momentum but also near the light cone at any momentum.

The poles of the propagator are determined by

$$[p_0 + \Sigma^0_{\rm mat}(p_0,\mathbf{p})]^2 = [\mathbf{p} + \boldsymbol{\Sigma}_{\rm mat}(p_0,\mathbf{p})]^2 \tag{6.119}$$

There are two undamped solutions to this equation, referred to as $\omega_+(\mathbf{p})$ and $\omega_-(\mathbf{p})$. They can be expressed in parametric form as

$$\begin{aligned}
\omega_\pm^2(\mathbf{p}) &= z^2\mathbf{p}_\pm^2(z) \\
\mathbf{p}_\pm^2(z) &= \omega_F^2\left[\frac{\pm 1}{z\mp 1} \mp \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\right]
\end{aligned} \tag{6.120}$$

with $\omega_F^2 = \frac{1}{2}m_F^2$ and $z > 1$. The two solutions are shown in Figure 6.1.

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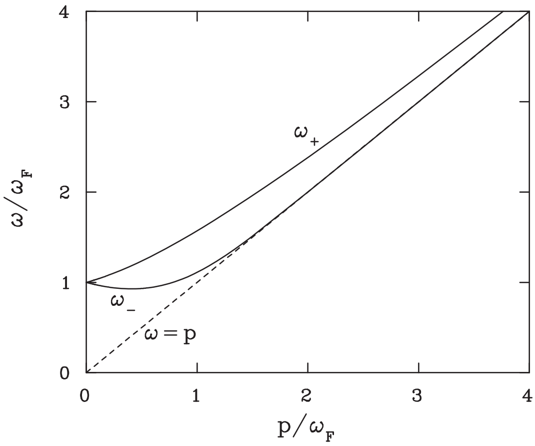

Fig. 6.1. The two branches ($\omega_\pm$) of the electron dispersion relation are shown. For comparison, the dispersion relation of a massless particle is also plotted (broken line).

At momenta that are high in comparison with T and μ, the solutions become

$$\begin{aligned} \omega_+^2 &= \mathbf{p}^2 + m_\mathrm{F}^2 + \cdots \\ \omega_-^2 &= \mathbf{p}^2 + 4\mathbf{p}^2 \exp\left(-\frac{4\mathbf{p}^2}{m_\mathrm{F}^2} - 1\right) + \cdots \end{aligned} \tag{6.121}$$

whereas at low momentum the solutions become

$$\omega_\pm = \omega_\mathrm{F} \pm \frac{1}{3}|\mathbf{p}| + \frac{1}{3}\frac{\mathbf{p}^2}{\omega_\mathrm{F}} + \cdots \tag{6.122}$$

The low-momentum spectra have an optical character. The high-momentum spectrum for ω_+ solution may be used to define a finite-temperature and finite-density fermion mass, just as the photon mass was defined at high momentum.

It is interesting to examine the behavior of the propagator in the vicinity of the poles. In the high-momentum limit,

$$\begin{aligned} \mathcal{G}(\omega \to \omega_+, p \gg m_\mathrm{F}) &\approx \frac{1}{2}\frac{\gamma^0 - \hat{\mathbf{p}}\cdot\boldsymbol{\gamma}}{\omega - \omega_+} \\ \mathcal{G}(\omega \to \omega_-, p \gg m_\mathrm{F}) &\approx \frac{2\mathbf{p}^2}{m_\mathrm{F}^2} \exp\left(-\frac{4\mathbf{p}^2}{m_\mathrm{F}^2} - 1\right) \frac{\gamma^0 + \hat{\mathbf{p}}\cdot\boldsymbol{\gamma}}{\omega - \omega_-} \end{aligned} \tag{6.123}$$

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and in the low-momentum limit

$$\mathcal{G}(\omega \to \omega_+, p \ll m_{\rm F}) \approx \frac{4}{3}\frac{\gamma^0 - \hat{\mathbf{p}}\cdot\boldsymbol{\gamma}}{\omega - \omega_+}$$
$$\mathcal{G}(\omega \to \omega_-, p \ll m_{\rm F}) \approx \frac{4}{3}\frac{\gamma^0 + \hat{\mathbf{p}}\cdot\boldsymbol{\gamma}}{\omega - \omega_-} \tag{6.124}$$

These display a number of features. The ω_+ solution has the same relation between chirality and helicity as free electrons whereas the ω_- solution has the opposite relation between chirality and helicity. This is true for all momenta, not just in the limits. It suggests that the ω_+ branch represents the modification of the dispersion relation of a real electron in the plasma, whereas the ω_- branch is a true collective excitation. Indeed, the residue of that branch vanishes as the momentum becomes large, which is the vacuum limit. The residue of the ω_+ branch in the high-momentum limit is the same as for free electrons. Finally, notice that the residues are the same as the momentum tends to zero since there is no distinction between different polarizations when the particle is at rest.

6.9 Kubo formulae for viscosities and conductivities

Many physical systems can be described using fluid dynamics. In the context of this book, examples of such systems are stars, the early universe and, to some extent, high-energy nuclear collisions. The state of the fluid can be described in terms of its temperature and chemical potentials, specified as functions of space and time, together with an equation of state. The dynamics of the fluid is described by equations of motion based on the energy–momentum tensor $T^{\mu\nu}(x)$. Here $x^\mu = (t, \mathbf{x})$. The local energy density is T^{00}, the local momentum density is T^{i0}, and the flux of these quantities in the direction j is $T^{\mu j}$. Local conservation of energy and momentum is expressed as

$$\partial_\nu T^{\mu\nu} = 0 \tag{6.125}$$

This conservation law is general and makes no assumption about local equilibrium or the types of interactions. Without loss of generality the energy–momentum tensor can always to taken to be symmetric. As a concrete example, the energy–momentum tensor for a set of N noninteracting particles labeled by index n is

$$T^{\mu\nu}(x) = \sum_{n=1}^{N} \frac{p_n^\mu p_n^\nu}{E_n}\delta(\mathbf{x} - \mathbf{x}_n(t)) \tag{6.126}$$

where $\mathbf{x}_n(t)$ is the trajectory of the nth particle.

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In a field theory with independent fields labeled ϕ_n and Lagrangian $\mathcal{L}$ the energy–momentum tensor is found in the usual way to be

$$T^{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial_\mu \phi_n} \partial^\nu \phi_n - g^{\mu\nu} \mathcal{L} \tag{6.127}$$

Specific examples include a self-interacting scalar field

$$T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \tag{6.128}$$

and the electromagnetic field

$$T^{\mu\nu} = F^\mu{}_\rho F^{\rho\nu} + \tfrac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{6.129}$$

To evaluate these in a classical field theory, the solutions to the field equations are inserted into these expressions. In a quantum theory the fields are operators and the expressions are therefore also operators. In a fluid, the expressions may be averaged over spacetime volumes that are large compared with typical thermal wavelengths and correlation lengths but small compared with distances and times over which local energy and momentum densities vary appreciably; this averaging process is referred to as coarse-graining.

Coarse-graining is easy to describe but usually difficult to implement. It can be done in numerical simulations, of course. In a hydrodynamic or perfect-fluid description, the assumption of local thermal equilibrium is made. Then the energy–momentum tensor is

$$T^{\mu\nu} = -P g^{\mu\nu} + w u^\mu u^\nu \tag{6.130}$$

where P is the local pressure, $w = \epsilon + P$ is the local enthalpy density, and $u^\mu = (\gamma, \gamma \mathbf{v})$ is the local flow velocity relative to some fixed reference frame. In a frame in which the fluid is locally at rest, $u^\mu = (1, 0, 0, 0)$, $T^{00} = \epsilon$, $T^{ij} = P\delta_{ij}$, and $T^{i0} = 0$. In general the trace of the energy–momentum tensor is $T^\mu{}_\mu = \epsilon - 3P$. For a noninteracting gas of massless particles, $\frac{1}{3}\epsilon = P$ and the trace vanishes. If there are conserved charges, such as baryon number or electric charge, there is an additional conservation law or equation of motion for each. For example, the baryon current is

$$J_B^\mu = n_B u^\mu \tag{6.131}$$

where $n_B = J_B^0$ is the local baryon density. The conservation law is

$$\partial_\mu J_B^\mu = 0 \tag{6.132}$$

Note that the baryon number flows with the same four-velocity as appeared in the energy–momentum tensor. The local pressure, energy, and baryon densities are related through the equation of state. Equivalently they can all be expressed in terms of T and μ_B.

When variations in temperature and chemical potential become appreciable over length scales that are not large compared with thermal wavelengths or correlation lengths then gradients in the thermodynamic variables must be taken into account. In a typical nonrelativistic fluid the massive particles carry the energy and momentum so that energy, momentum, and baryon number all flow together with only very minor departures associated with thermal conductivity. In a relativistic fluid, meaning one in which P is not much less than ϵ, or equivalently in which the temperature and chemical potential are not much less than the mass of the particles, the situation is more complicated. The energy and momentum may flow with a velocity different from that of the baryons if the system has gradients that are not negligibly small. The situation is then described in terms of (first-order) relativistic viscous-fluid dynamics. Dissipative contributions are added to the energy–momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= -Pg^{\mu\nu} + wu^\mu u^\nu + \Delta T^{\mu\nu} \\ J_B^\mu &= n_B u^\mu + \Delta J_B^\mu \end{aligned} \tag{6.133}$$

The dissipative terms are proportional to first-order derivatives of the flow velocity, temperature, and chemical potential. There are two common definitions of the flow velocity in relativistic dissipative fluid dynamics.

In the Eckart approach u^μ is the velocity of baryon number flow [16]. The dissipative terms must satisfy the conditions $\Delta J_B^\mu = 0$ and $u_\mu u_\nu \Delta T^{\mu\nu} = 0$, the latter following from the requirement that T^{00} be the energy density in the local (baryon) rest frame. The most general form of $\Delta T^{\mu\nu}$ is given by

$$\begin{aligned} \Delta T^{\mu\nu} = \eta(\Delta^\mu u^\nu + \Delta^\nu u^\mu) + \left(\tfrac{2}{3}\eta - \zeta\right) H^{\mu\nu}\partial_\rho u^\rho \\ - \chi(H^{\mu\alpha}u^\nu + H^{\nu\alpha}u^\mu)\, Q_\alpha \end{aligned} \tag{6.134}$$

Here

$$H^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu} \tag{6.135}$$

is a projection tensor normal to u^μ,

$$\Delta_\mu = \partial_\mu - u_\mu u^\beta \partial_\beta \tag{6.136}$$

is a derivative normal to u^μ, and

$$Q_\alpha = \partial_\alpha T - T u^\rho \partial_\rho u_\alpha \tag{6.137}$$

is the heat flow vector, whose nonrelativistic limit is $\mathbf{Q} = -\nabla T$. Furthermore, η is the shear viscosity, ζ is the bulk viscosity, and χ is the thermal conductivity. The entropy current is

$$s^\mu = su^\mu + \frac{1}{T} u_\nu \Delta T^{\mu\nu} \tag{6.138}$$

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and is defined is such a way that $u_\mu s^\mu = s$, the local entropy density. Its divergence is

$$\partial_\mu s^\mu = \frac{\eta}{2T}\left(\partial_i u^j + \partial_j u^i - \tfrac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right)^2 + \frac{\zeta}{T}\left(\nabla\cdot\mathbf{u}\right)^2 + \frac{\chi}{T^2}\left(\nabla T + T\dot{\mathbf{u}}\right)^2 \tag{6.139}$$

All three dissipation coefficients must be non-negative to ensure that entropy can never decrease.

In the Landau–Lifshitz approach, u^μ is the velocity of energy transport. The dissipative part of the energy–momentum tensor satisfies $u_\mu \Delta T^{\mu\nu} = 0$, and ΔJ_B^μ is not constrained to be zero. In this case the most general form of the energy–momentum tensor is

$$\Delta T^{\mu\nu} = \eta(\Delta^\mu u^\nu + \Delta^\nu u^\mu) + \left(\tfrac{2}{3}\eta - \zeta\right) H^{\mu\nu}\partial_\rho u^\rho \tag{6.140}$$

The baryon current is modified to

$$\Delta J_B^\mu = \chi\left(\frac{n_B T}{w}\right)^2 \Delta^\mu\left(\frac{\mu_B}{T}\right) \tag{6.141}$$

The three coefficients η, ζ, and χ are the same as in the Eckart approach. This can be proven in a variety of ways. For example, even though the entropy current in this approach is different, being

$$s^\mu = s u^\mu - \frac{\mu_B}{T}\Delta J_B^\mu \tag{6.142}$$

its divergence is the same. Physical, observable, results cannot depend on how one defines the frame of reference.

In the above approaches the dissipative coefficients are taken to be phenomenological constants or, rather, functions of temperature and chemical potential. However, it ought to be possible to derive them from the microscopic theory. In particular, it ought to be possible to derive them using linear response theory since departures from local thermal equilibrium are assumed to be small. Indeed this is so, and the resulting formulae are named after Kubo [17].

Consider the problem of pure baryon number diffusion in the absence of energy flow. The most direct approach to use in this case is that of Landau and Lifshitz: the vanishing of the energy flux implies that the flow velocity is zero. The equation of continuity for the baryon current, including dissipation, reduces to a diffusion equation for the baryon chemical potential:

$$\partial\mu_B/\partial t = D\,\nabla^2\mu_B \tag{6.143}$$

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Here

$$D \equiv \frac{\chi T}{dn_B/d\mu_B}\left(\frac{n_B}{w}\right)^2 \tag{6.144}$$

is the diffusion constant. A single Fourier mode $\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$ will relax towards equilibrium as $\exp(-Dk^2 t)$.

A nonuniform baryon distribution can be achieved by the imposition of an external force that is turned on and off, allowing the system to relax back towards equilibrium. It does not matter how this is done. For example, we could take the coupling Hamiltonian to be

$$H_{\text{ext}}(t) = \int d^3x\, \hat{J}^\mu_B(\mathbf{x},t) J^{\text{ext}}_\mu(\mathbf{x},t) \tag{6.145}$$

where J^{ext}_μ is an external perturbing current. The response of the baryon current is given in the usual way by

$$\delta\langle \hat{J}^\mu_B(\omega,\mathbf{k})\rangle = J^{\text{ext}}_\nu(\omega,\mathbf{k}) B^{\mu\nu}_{\text{R}}(\omega,\mathbf{k}) \tag{6.146}$$

where $B^{\mu\nu}_{\text{R}}(\omega,\mathbf{k})$ is the Fourier transform of the retarded current–current correlation or response function:

$$iB^{\mu\nu}_{\text{R}}(\mathbf{x},t;\mathbf{x}',t') = \left\langle \left[\hat{J}^\mu_B(\mathbf{x},t),\ \hat{J}^\nu_B(\mathbf{x}',t')\right]\right\rangle \theta(t-t') \tag{6.147}$$

Since baryon number is conserved the most general form of the response function is

$$B^{\mu\nu}_{\text{R}} = B_{\text{L}} P^{\mu\nu}_{\text{L}} + B_{\text{T}} P^{\mu\nu}_{\text{T}} \tag{6.148}$$

where B_{L} and B_{T} are longitudinal and transverse response functions. Without loss of generality it is convenient to parametrize the longitudinal response function, or equivalently the time–time component, as

$$B^{00}_{\text{R}}(\omega,\mathbf{k}) = \frac{\mathbf{k}^2}{k^2} B_{\text{L}}(\omega,\mathbf{k}) = \frac{i\mathbf{k}^2 D(\omega,\mathbf{k})}{\omega + i\mathbf{k}^2 D(\omega,\mathbf{k})} B^{00}_{\text{R}}(\omega=0,\,\mathbf{k}) \tag{6.149}$$

Here $D(\omega,\mathbf{k})$ is an unknown function. It is expected to be a smooth function of ω and $\mathbf{k}$, whereas the response function itself is expected to have singularities, usually poles. If we define $D \equiv D(\omega\to 0,\,\mathbf{k}\to 0)$, and if there is a slow perturbing variation in the baryon density, then the density will relax back towards equilibrium with a dispersion relation determined by the pole of the response function, namely, $\omega = -iD\mathbf{k}^2$. Therefore we may identify this D with the diffusion constant in the dissipative fluid dynamics calculation.

The diffusion constant can be extracted directly from the response function. First,

$$D = \lim_{\omega\to 0}\lim_{\mathbf{k}\to 0} \frac{i}{\omega}\frac{B_{\text{L}}(\omega,\mathbf{k})}{B_{\text{L}}(0,\,\mathbf{k})} \tag{6.150}$$

Now $B_{\rm L}(\omega = 0, \mathbf{k} \to 0) = -B_{\rm R}^{00}(\omega = 0, \mathbf{k} \to 0) = \partial^2 P/\partial\mu_B^2 = \partial n_B/\partial\mu_B$. (The reasoning is the same as for the electric screening mass.) Furthermore,

$$B_{\rm L}(\omega,\, |\mathbf{k}| \to 0) = \hat{k}^i \hat{k}^j\, B_{\rm R}^{ij}(\omega,\, |\mathbf{k}| \to 0) \tag{6.151}$$

where $\hat{k}^i = k^i/|\mathbf{k}|$ is a unit vector in the direction of $\mathbf{k}$. Putting all this together and using the rotational symmetry yields a linear response formula for the thermal conductivity:

$$\chi T = \frac{1}{3}\left(\frac{w}{n_B}\right)^2 \lim_{\omega\to 0}\frac{1}{\omega}\int d^4x\, \mathrm{e}^{i\omega t}\left\langle\left[\hat{J}_B^i(t,\mathbf{x}),\, \hat{J}_B^i(0,\mathbf{0})\right]\right\rangle\theta(t) \tag{6.152}$$

The factor $(w/n_B)^2$ arises in the conversion of baryon current to enthalpy current. Alternatively, (6.152) could be written in terms of the spectral densities for the longitudinal part of the baryon response function as

$$\begin{aligned}\chi T &= \frac{1}{3}\left(\frac{w}{n_B}\right)^2 \lim_{\omega\to 0}\frac{1}{\omega}\rho_{\rm L}^{\rm n}(\omega,\, |\mathbf{k}| = 0)\\ &= \frac{1}{3T}\left(\frac{w}{n_B}\right)^2 \lim_{\omega\to 0}\rho_{\rm L}^{+}(\omega,\, |\mathbf{k}| = 0)\end{aligned} \tag{6.153}$$

The latter equality follows from the relation $\rho^{\rm n} = (1 - \mathrm{e}^{-\beta\omega})\rho^{+}$, as discussed in Section 6.2.

There are Kubo-type linear-response expressions for the viscosities too. These may be derived in a way analogous to that for the thermal conductivity since $T^{\mu\nu}$ may be viewed as representing a set of four conserved currents. One obtains

$$\eta = \frac{1}{20}\lim_{\omega\to 0}\frac{1}{\omega}\int d^4x\, \mathrm{e}^{i\omega t}\left\langle\left[\mathcal{S}^{ij}(t,\mathbf{x}),\, \mathcal{S}^{ij}(0,\mathbf{0})\right]\right\rangle\theta(t) \tag{6.154}$$

$$\zeta = \frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\int d^4x\, \mathrm{e}^{i\omega t}\langle[\mathcal{P}(t,\mathbf{x}),\, \mathcal{P}(0,\mathbf{0})]\rangle\theta(t) \tag{6.155}$$

where $\mathcal{P} = -\frac{1}{3}T^i{}_i$ represents the trace of the momentum tensor (the pressure in equilibrium) and $\mathcal{S}^{ij} = T^{ij} - \delta^{ij}\mathcal{P}$ represents the traceless part. These follow from the dispersion relation for the transverse part of the momentum density,

$$\omega = -iD_{\mathcal{S}}\mathbf{k}^2 \tag{6.156}$$

where $D_{\mathcal{S}} = \eta/w$, and from the dispersion relation for pressure waves,

$$\omega^2 - v_{\mathcal{P}}^2\mathbf{k}^2 + iD_{\mathcal{P}}\omega\mathbf{k}^2 = 0 \tag{6.157}$$

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where $D_{\mathcal{P}} = (\frac{4}{3}\eta + \zeta)/w$ (when the thermal conductivity is neglected). In terms of the spectral densities we have

$$\eta = \frac{1}{20} \lim_{\omega \to 0} \frac{1}{\omega} \rho^{\rm n}_{\mathcal{S}S}(\omega, \, |\mathbf{k}| = 0) = \frac{1}{20T} \lim_{\omega \to 0} \rho^{+}_{\mathcal{S}S}(\omega, \, |\mathbf{k}| = 0) \quad (6.158)$$

$$\zeta = \frac{1}{2} \lim_{\omega \to 0} \frac{1}{\omega} \rho^{\rm n}_{\mathcal{P}P}(\omega, \, |\mathbf{k}| = 0) = \frac{1}{2T} \lim_{\omega \to 0} \rho^{+}_{\mathcal{P}P}(\omega, \, |\mathbf{k}| = 0) \quad (6.159)$$

It is worth noting that in all these formulae the relevant transport coefficient is proportional to a diffusion constant with dimensions of length. In a multicomponent fluid those particles or fields with the longest diffusion length tend to dominate the transport coefficient.

In a similar manner one may derive an expression for the electrical conductivity, which is the coefficient in Ohm's law $\mathbf{J}_{\rm EM} = \sigma_{\rm el}\mathbf{E}$:

$$\sigma_{\rm el} = \frac{1}{6} \lim_{\omega \to 0} \frac{1}{\omega} \int d^4x \, {\rm e}^{i\omega t} \left\langle \left[\hat{J}^i_{\rm EM}(t, \mathbf{x}), \, \hat{J}^i_{\rm EM}(0, \mathbf{0}) \right] \right\rangle \theta(t) \quad (6.160)$$

This may also be expressed in terms of the corresponding spectral density.

The viscosities in $\lambda\phi^4$ theory have been calculated by Jeon [18] and Jeon and Yaffe [19]. In the limit of weak coupling and high temperature, the shear viscosity is

$$\eta = 5.28 T^3/\lambda^2 \quad (6.161)$$

The parametric dependence of η on T and λ is straightforward. Recall that $\eta = wD_{\mathcal{S}}$. A diffusion constant may be estimated as $n\langle\sigma v\rangle$, where n is an average density, σ is a cross section, and v is the speed of the particles. For massless, or nearly massless, particles, $v \approx 1$, $n \propto T^3$, and $w \propto T^4$. The thermally averaged elastic cross section in $\lambda\phi^4$ theory is proportional to λ^2/T^2. Putting this all together yields the estimate $\eta \propto T^3/\lambda^2$, in agreement with the result quoted above. However, to calculate the overall coefficient is not easy. This may be seen immediately by the inverse dependence of η on λ. An infinite set of ladder diagrams corresponding to elastic scattering must be summed along with finite-temperature self-energy insertions. The calculation is ultimately reduced to a single integral equation that is solved numerically. The bulk viscosity for point particles with no internal degrees of freedom undergoing local interactions is generally much smaller than the shear viscosity. For the $\lambda\phi^4$ theory the bulk viscosity is nonzero at high temperature because of inelastic scatterings. When these are taken into account it is found that

$$\zeta = 0.002\,14 \, \lambda \, \ln^2(1.55\lambda) \, T^3 \quad (6.162)$$

The ratio of the two viscosities $\zeta/\eta = \lambda^3 \ln^2(1.55\lambda)/2470$. For $\lambda = 1/10$ the ratio is 1.4×10^{-6} and for $\lambda = 1$ it is 7.8×10^{-5}. The thermal and electrical conductivities have no meaning in this theory since there is no conserved charge.

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The shear viscosity, diffusion constant, and electrical conductivity have been evaluated at high temperature in gauge theories, to lowest order in the gauge coupling but to all orders in the logarithm of the coupling, by Arnold, Moore, and Yaffe [20]. Rather than applying the Kubo formulae directly they found it more expedient to do a numerical calculation based on the Boltzmann transport equation. For one flavor of lepton (electrons) the results are

$$D = \frac{0.596}{\alpha^2 \ln(1.46/\alpha)} \frac{1}{T} \tag{6.163}$$

$$\eta = \frac{2.39}{\alpha^2 \ln(5.99/\alpha)} T^3 \tag{6.164}$$

$$\sigma_{\text{el}} = \frac{2.50}{\alpha \ln(1.46/\alpha)} T \tag{6.165}$$

and for two flavors (electrons and muons) they are

$$D = \frac{0.392}{\alpha^2 \ln(1.08/\alpha)} \frac{1}{T} \tag{6.166}$$

$$\eta = \frac{1.53}{\alpha^2 \ln(2.33/\alpha)} T^3 \tag{6.167}$$

$$\sigma_{\text{el}} = \frac{3.29}{\alpha \ln(1.08/\alpha)} T \tag{6.168}$$

Here D refers to (conserved) lepton number diffusion. These QED expressions have an extra logarithmic factor arising from the screening of the long-range Coulomb force. The corresponding results for QCD will be discussed in later chapters.

6.10 Exercises

6.1 Find the linear response of the fermion number density to an applied neutral scalar field $\phi_{\text{ext}}(\mathbf{x}, t)$ for a Yukawa theory with interaction $\mathcal{L}_{\text{I}} = g\bar{\psi}\psi\phi$.

6.2 Repeat the analysis of Section 6.2 for a charged scalar field with a chemical potential.

6.3 Derive (6.47) for the interaction Lagrangian (5.36). You may choose whichever gauge you prefer.

6.4 Repeat the analysis leading to (6.70) but in the opposite limit, that of vanishing electron mass.

6.5 Derive the low-momentum expansion for $F(0, \mathbf{k})$ at finite temperature and chemical potential.

6.6 Derive the limiting form, (6.104) and (6.105), of the photon self-energy.

6.7 Is there an expression analogous to (6.120) for the photon dispersion relations?
6.8 Find the relationship between the flow velocities in the Eckart and the Landau–Lifshitz approaches.
6.9 Transport coefficients may be expressed in terms of differing correlation functions. As an example of this, express the thermal conductivity in terms of the density–density correlation function instead of the current–current one.
6.10 Derive the Kubo formula for the electrical conductivity.

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