§ 24. The perpendiculars from the vertices on the opposite sides of a triangle bisect the angles of the triangle formed by joining the feet of the perpendiculars. (Proof by means of the polar figure).

Let ABC be the polar triangle, L, M, N, the poles of the perpendiculars in the original triangle, *i.e.*, L is a point in BC such that LA is a quadrant, &c. Then DEF is the polar of the triangle formed by joining the feet of the perpendiculars in the original triangle; and it is required to show that L, M, N bisect the sides of DEF externally.

L, M, N are the poles of the perpendiculars of ABC;
∴ A, B, C are the middle points of the sides of DEF (§ 22).
∴ L, M, N bisect the sides externally.

Although not strictly within the scope of this paper, the following proof of the theorem of § 23 may be interesting.

Let ABCD (fig. 10) be the quadrilateral, AC and BD being quadrants. Then (AGCK) = − 1, and AC is a quadrant; ∴ GC = CK. Similarly, GB = BL.

Now, in the triangle LGK, B bisects LG internally, and A bisects GK externally; ∴ E bisects LK. And from triangle GLK F bisects LK externally; ∴ EF is a quadrant.

Note on the Condensation of a Special Continuant.

By Thomas Muir, M.A., F.R.S.E.

[*Held over from Third Meeting.*]

§ 1. The continuant referred to is that in which the elements of the main diagonal are all equal (to x, say), the elements of the one minor diagonal all equal (to b, say), and the elements of the other minor diagonal all equal (to c, say). It may be denoted by $F(b, x, c, n)$ when it is of the nth order. Professor Wolstenholme has recently given two elegant theorems regarding the condensation of $F(1, x, 1, n)$. I wish to establish the analogous theorems for $F(b, x, c, n)$.

§ 2. It may be necessary to premise that a determinant whose elements are all zeros, except those in the main diagonal and in the two diagonals drawn through the places (1, 3), (3, 1) parallel to the main diagonal, is expressible as the product of two continuants. Thus

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$$\begin{vmatrix} a_1 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & c_2 & 0 & 0 & 0 \\ b_1 & 0 & a_3 & 0 & c_3 & 0 & 0 \\ 0 & b_2 & 0 & a_4 & 0 & c_4 & 0 \\ 0 & 0 & b_3 & 0 & a_5 & 0 & c_5 \\ 0 & 0 & 0 & b_4 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & b_5 & 0 & a_7 \end{vmatrix} \text{ or } D_7,$$

$$= \begin{vmatrix} a_1 & c_1 & 0 & 0 \\ b_1 & a_3 & c_3 & 0 \\ 0 & b_3 & a_5 & c_5 \\ 0 & 0 & b_5 & a_7 \end{vmatrix} \quad \begin{vmatrix} a_2 & c_2 & 0 \\ b_2 & a_4 & c_4 \\ 0 & b_4 & a_6 \end{vmatrix},$$

$$= K\begin{pmatrix} & -b_1c_1 & & -b_3c_3 & & -b_5c_5 & \\ a_1 & & a_3 & & a_5 & & a_7 \end{pmatrix} K\begin{pmatrix} & -b_2c_2 & & -b_4c_4 & \\ a_2 & & a_4 & & a_6 \end{pmatrix} \qquad \text{....(I.)}$$

and similarly—

$$D_6 = \begin{vmatrix} a_1 & c_1 & 0 \\ b_1 & a_3 & c_3 \\ 0 & b_3 & a_5 \end{vmatrix} \cdot \begin{vmatrix} a_2 & c_2 & 0 \\ b_2 & a_4 & c_4 \\ 0 & b_4 & a_6 \end{vmatrix},$$

$$= K\begin{pmatrix} & -b_1c_1 & & -b_3c_3 & \\ a_1 & & a_3 & & a_5 \end{pmatrix} \cdot K\begin{pmatrix} & -b_2c_2 & & -b_4c_4 & \\ a_2 & & a_4 & & a_6 \end{pmatrix} \qquad \text{...(II.)}$$

and therefore, as we may observe in passing,

$$\frac{D_7}{D_6} = a_7 - \cfrac{b_5c_5}{a_5 - \cfrac{b_3c_3}{a_3 - \cfrac{b_1c_1}{a_1}}} \qquad \text{...(III.)}$$

§ 3. Also, we may note that since

$$K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & & -b^2 & \\ a+b & & a & & a & & a & & a+b \end{pmatrix} =$$

$$K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a & & a+b \end{pmatrix} + bK\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a+b \end{pmatrix}$$

and since the first term in the right-hand member equals

$$(a+b)K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a \end{pmatrix} - b^2K\begin{pmatrix} & -b^2 & & -b^2 & \\ a & & a & & a \end{pmatrix},$$

and the second term equals

$$b\cdot K\begin{pmatrix} & -b^2 & & -b^2 & & -b^2 & \\ a & & a & & a & & a \end{pmatrix} + b^2K\begin{pmatrix} & -b^2 & & -b^2 & \\ a & & a & & a \end{pmatrix},$$

we have the identity in continuants

$$K\begin{pmatrix} & -b^2 & -b^2 & -b^2 & -b^2 & \\ a+b & & a & a & a & a+b \end{pmatrix} = (a+2b)K\begin{pmatrix} & -b^2 & -b^2 & -b^2 & \\ a & & a & a & a \end{pmatrix} \quad \text{(IV.)}$$

§ 4. Now, taking the case of F (b, x, c, n) where n is odd, we have

$$F(b,x,c,7) = \begin{vmatrix} x & b & . & . & . & . & . \\ c & x & b & . & . & . & . \\ . & c & x & b & . & . & . \\ . & . & c & x & b & . & . \\ . & . & . & c & x & b & . \\ . & . & . & . & c & x & b \\ . & . & . & . & . & c & x \end{vmatrix} = \begin{vmatrix} x-c & . & . & . & . & . & . \\ -b & x-c & . & . & . & . & . \\ . & -b & x-c & . & . & . & . \\ . & . & -b & x-c & . & . & . \\ . & . & . & -b & x-c & . & . \\ . & . & . & . & -b & x-c & \\ . & . & . & . & . & -b & x \end{vmatrix}$$

and therefore by multiplication

$$\left[F(b,x,c,7)\right]^2 = \begin{vmatrix} x^2-bc & 0 & -b^2 & . & . & . & . \\ 0 & x^2-2bc & 0 & -b^2 & . & . & . \\ -c^2 & 0 & x^2-2bc & 0 & -b^2 & . & . \\ . & -c^2 & 0 & x^2-2bc & 0 & -b^2 & . \\ . & . & -c^2 & 0 & x^2-2bc & 0 & -b^2 \\ . & . & . & -c^2 & 0 & x^2-2bc & 0 \\ . & . & . & . & -c^2 & 0 & x^2-bc \end{vmatrix}$$

$$= \begin{vmatrix} x^2-bc & -b^2 & . & . \\ -c^2 & x^2-2bc & -b^2 & . \\ . & -c^2 & x^2-2bc & -b^2 \\ . & . & -c^2 & x^2-bc \end{vmatrix} \begin{vmatrix} x^2-2bc & -b^2 & . \\ -c^2 & x^2-2bc & -b^2 \\ . & -c^2 & x^2-2bc \end{vmatrix} \text{ by § 2.}$$

$$= x^2 \begin{vmatrix} x^2-2bc & -b^2 & . \\ -c^2 & x^2-2bc & -b \\ . & -c^2 & x^2-2bc \end{vmatrix}^2 \text{ by § 3.}$$

and consequently we have

$$F(b, x, c, 7) = xF(b^2, x^2-2bc, c^2, 3),$$

the general theorem evidently being

$$F(b, x, c, 2n+1) = xF(b^2, x^2-2bc, c^2, n) \qquad \text{....(V.)}$$

In exactly the same way we find the complementary theorem

$$F(b, x, c, 2n) = F(b^2, x^2-2bc, c^2, n) + bcF(b^2, x^2-2bc, c^2, n-1) \text{...(VI.)}$$

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