A MARTINGALE CONVERGENCE THEOREM IN VECTOR LATTICES

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1. Introduction. The martingale convergence theorem was first proved by Doob **(3)** who considered a sequence of real-valued random variables. Since various collections of real-valued random variables can be regarded as vector lattices, it seems of interest to prove the martingale convergence theorem in an arbitrary vector lattice. In doing so we use the concept of order convergence that is related to convergence almost everywhere, the type of convergence used in Doob's theorem.

Unfortunately, our general theorem does not contain Doob's theorem as a special case. However, it does enable us to give a new proof of his theorem; this is done in §5. Recall that Doob's proof uses an ingenious lemma concerning the number of up-crossings of sample functions; the proof given here does not use this fact.

The reader will find the basic facts concerning martingales in the books of Doob **(3)** and Loève **(5)**. The author is indebted to Professor Doob for his helpful comments.

2. Basic definitions. As usual, a *vector lattice* is a linear space over the reals which is partially ordered in such a way that each pair of elements has a least upper bound (supremum) and a greatest lower bound (infimum). In this paper we shall also assume that the vector lattices under consideration are countably complete, i.e., each bounded countable set has a supremum and an infimum. Examples of such spaces are plentiful, but the one of greatest interest here is the space of all real-valued bounded random variables defined on some probability measure space.

If X is a countably complete vector lattice, then it is possible to define a notion of limit for bounded sequences. Thus let $\{x_n\}$ be a bounded sequence of elements from X (the sequence is said to be bounded if there exists $u \in X$ such that $-u \leqslant x_n \leqslant u$ for all n). Define

$$\limsup x_n = \inf_n \sup_k \{x_k : k \geqslant n\}$$

and

$$\liminf x_n = \sup_n \inf_k \{x_k : k \geqslant n\}.$$

We always have $\liminf x_n \leqslant \limsup x_n$; if equality holds, then we write $\lim x_n = \liminf x_n = \limsup x_n$.

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As usual, we define $x^+ = \sup\{x, 0\}$ and $|x| = x^+ + (-x)^+$. The following lemma will be used in the proof of the convergence theorem.

LEMMA 1. *Let* $\{x_n\}$ *be a sequence of elements from* X *which is bounded below. If* $\inf\{x_n\} = y$, *then*

$$\inf\{(x_n - z)^+\} = (y - z)^+ \qquad \text{for any } z \in X.$$

The proof of this lemma and many other details concerning vector lattices may be found in Birkhoff (**1**; see also **4** and **6**).

DEFINITION 1. *A positive projection* P *on the vector lattice* X *is a positive linear mapping of* X *into itself such that* $P(P(x)) = P(x)$ *for all* $x \in X$.

DEFINITION 2. *A martingale is a double sequence* $\{x_n, P_n\}$, *where* $x_n \in X$ *and* P_n *is a positive projection, such that* $P_n(x_k) = x_n$ *for* $n \leqslant k$ *and* $P_n P_k = P_k P_n = P_n$ *for* $n \leqslant k$. *If* $P_n(x_n) = x_n \leqslant P_n(x_k)$ *for* $n \leqslant k$, *then the above double sequence is called a submartingale.* Thus, a martingale is a special case of a submartingale. *A submartingale* $\{x_n, P_n\}$ *is said to be bounded if the sequence* $\{x_n\}$ *is bounded.*

LEMMA 2. *Let* $\{P_n\}$ *be a sequence of positive projections on the vector lattice* X *such that* $P_n P_k = P_k P_n = P_n$ *for* $n \leqslant k$. *Then if* $\{y_n\}$ *is any sequence of elements from* X *such that* $y_1 \leqslant y_2 \leqslant \ldots$ *and if we define* $x_n = P_n(y_n)$, *then the double sequence* $\{x_n, P_n\}$ *is a submartingale. In particular, if* $y_1 = y_2 = \ldots$, *then* $\{x_n, P_n\}$ *is a martingale.*

LEMMA 3. *If* $\{x_n, P_n\}$ *is a martingale and* α *is any real number, then* $\{\alpha x_n, P_n\}$ *is a martingale.*

LEMMA 4. *If* $\{x_n, P_n\}$ *and* $\{y_n, P_n\}$ *are (sub)martingales, then* $\{x_n + y_n, P_n\}$ *is a (sub)martingale.*

LEMMA 5. *If* $\{y_n, P_n\}$ *is a submartingale and if* $x_n = P_n(y_n^+)$, *then* $\{x_n, P_n\}$ *is a submartingale.*

Proof of Lemma 5. If $n \leqslant k$, we have

$$y_n \leqslant P_n(y_k) \leqslant P_n(y_k^+) = P_n P_k(y_k^+) = P_n(x_k).$$

Therefore, since $0 \leqslant P_n(x_k)$, we have $y_n^+ = \sup\{y_n, 0\} \leqslant P_n(x_k)$. Hence, $x_n = P_n(y_n^+) \leqslant P_n(x_k)$.

LEMMA 6. *If* $\{z_n, P_n\}$ *is a submartingale, then it is possible to write* $z_n = x_n + y_n$, *where* $\{x_n, P_n\}$ *is a martingale and* $0 = y_1 \leqslant y_2 \leqslant \ldots$.

Proof. Define $y_1 = 0$ and $x_1 = z_1$. Then by induction define

$$x_{n+1} = x_n + z_{n+1} - P_n(z_{n+1}).$$

Therefore,

$$x_{n+1} = \sum_{i=1}^{n} (x_{i+1} - x_i) + x_1$$
$$= \sum_{i=1}^{n} (z_{i+1} - z_i) + z_1 - \sum_{i=1}^{n} [P_i(z_{i+1}) - z_i] = z_{n+1} - y_{n+1}$$

where

$$y_{n+1} = \sum_{i=1}^{n} [P_i(z_{i+1}) - z_i].$$

Since $z_i \leqslant P_i(z_{i+1})$, we see that $0 = y_1 \leqslant y_2 \leqslant \ldots$. It is easy to show that $\{x_n, P_n\}$ is a martingale.

Lemmas 2–6 may be found in **(3** or **5)** but in a slightly different form in certain cases. For example, Lemma 5 is slightly more general than the corresponding assertion in **(3** or **5)** and therefore its proof is given here. Lemma 6 is due to Doob, but its proof is given here because some of the computations involved will appear later.

The definition of a martingale given above includes that given by Doob. We shall give some examples to show that it includes more.

Example 1. Let X be the collection of all real-valued functions defined on some non-empty set Ω. If X is partially ordered pointwise, then X is a countably complete vector lattice. Now let $M_1 \subset M_2 \subset \ldots$ be a sequence of subsets of Ω. Each subset M_n determines a positive projection P_n defined on X as follows: for each $x \in X$, $P_n(x) = y$, where $y(t) = x(t)$ for $t \in M_n$ and $y(t) = 0$ for $t \in \Omega - M_n$. It is easy to show that $P_n P_k = P_k P_n = P_n$ for $n \leqslant k$. Hence, if we take any $z \in X$ and define $x_n = P_n(z)$, then $\{x_n, P_n\}$ is a martingale by Lemma 2.

Example 2. Let X be as in Example 1. Now let $\phi = M_0 \subset M_1 \subset M_2 \subset \ldots$ be a sequence of subsets of Ω and t_n a sequence of points from Ω such that $t_n \in M_n - M_{n-1}$ (note that this requires that Ω be infinite). Each pair M_n, t_n determines a positive projection P_n defined on X as follows: for each $x \in X$, $P_n(x) = y$, where $y(t) = x(t)$ for $t \in M_n$ and $y(t) = x(t_n)$ for $t \in \Omega - M_n$. It is easy to show that $P_n P_k = P_k P_n = P_n$ for $n \leqslant k$. Hence, if we take any $z \in X$ and define $x_n = P_n(z)$, then $\{x_n, P_n\}$ is a martingale.

Example 3. Let X be the collection of all real-valued functions defined on the closed interval $[-1, 1]$. If X is partially ordered pointwise, then X is a countably complete vector lattice. Now let $\{-1, 1\} \subset M_1 \subset M_2 \subset \ldots$ be a sequence of finite subsets of Ω. We shall write

$$M_n = \{t_n^0, t_n^1, \ldots, t_n^n\}, \qquad \text{where } -1 = t_n^0 < t_n^1 < \ldots < t_n^n = 1.$$

Each set M_n determines a positive projection P_n defined on X as follows: for each $x \in X$, $P_n(x) = y$, where

$$y(t) = x(t_n^{k-1}) + \frac{x(t_n^k) - x(t_n^{k-1})}{t_n^k - t_n^{k-1}} (t - t_n^{k-1})$$

$$\text{for } t_n^{k-1} \leqslant t \leqslant t_n^k \text{ and } k = 1, \ldots, n.$$

It is easy to show that $P_n P_k = P_k P_n = P_n$ for $n \leqslant k$. Hence, if we take any $z \in X$ and define $x_n = P_n(z)$, then $\{x_n, P_n\}$ is a martingale.

In particular, let us define $M_n = \{-1, 1/n, 1/(n-1), \ldots, 1\}$ and then define $z \in X$ as follows: $z(t) = 0$ unless $t = 1/j$, where j is a positive even integer, in which case $z(t) = 1$. If we define $x_n = P_n(z)$, then $\{x_n, P_n\}$ is a martingale. We note that $\liminf x_n(0) = 0$ and $\limsup x_n(0) = 1$; hence, $\lim x_n$ does not exist even though the sequence $\{x_n\}$ is bounded.

The above examples will be referred to later. For the moment, though, they illustrate that the martingale concept extends beyond its original confines in probability (i.e., measure) theory.

3. The martingale convergence theorem in a vector lattice. From Example 3 of the previous section we see that it is not possible to prove a theorem to the effect that every bounded martingale $\{x_n, P_n\}$ converges. Additional conditions must be imposed on the positive projections P_n. The following three conditions are selected because they can easily be verified in the case when each P_n is a conditional expectation operator and X is a vector lattice of real-valued, bounded (almost everywhere) random variables defined on some probability measure space.

C_1: *If* $x \in X$ *and* $P_n(x) = x$, *then* $P_n(x^+) = x^+$.
C_2: *If* $z_1 \geqslant z_2 \geqslant \ldots \geqslant 0$ *and* $\inf z_n = 0$, *then*

$$\inf_k \sup_n P_n(z_k) = 0.$$

We are implicitly assuming here that if $z_1 \geqslant 0$, then the sequence $\{P_n(z_1)\}$ is bounded.

C_3: *If* $\{z_n, P_n\}$ *is any bounded submartingale such that* $z_n \geqslant 0$ *for all* n *and* $\liminf z_n = 0$, *then* $z_n = 0$ *for all* n.

Referring to the three examples given in §2, we see that C_1 is satisfied in Examples 1 and 2 but not in 3. C_2 is satisfied in Example 1 but not in 3; it is satisfied in Example 2 only if

$$\bigcup_{n=1}^{\infty} M_n = \Omega.$$

C_3 is satisfied in all three examples.

The three conditions given above may not be necessary to prove that every bounded martingale converges; they are only sufficient. Furthermore, they may not be independent conditions, but the author is unable to establish this.

In the proof of the main theorem we shall need the following lemma.

LEMMA 7. *Let $P_1, P_2, \ldots$ be a sequence of positive projections on the vector lattice X such that $P_n P_k = P_k P_n = P_n$ for $n \leqslant k$ and such that conditions C_1 and C_2 are satisfied. Then if X_0 is the collection of all $x \in X$ such that $x = \lim P_n(x)$, then X_0 is a countably complete vector sublattice of X* (this means that supremums and infimums computed in X_0 are the same as if they were computed in X).

Proof. Since $0 \in X_0$, X_0 is non-empty. It is easy to show that X_0 is a linear subspace of X. We shall now show that if $\{y_n\}$ is a sequence of elements from X_0 and $\lim y_n = y$, then $y \in X_0$. We note that

$$|P_n(y) - y| \leqslant |y_k - y| + |P_n(y_k) - y_k| + |P_n(y_k - y)|.$$

Since each $y_k \in X_0$, we have

$$\limsup_n |P_n(y_k) - y_k| = 0 \qquad \text{for each } k.$$

If we define $z_k = \sup_n |P_n(y_k - y)|$, then

$$\limsup_n |P_n(y) - y| \leqslant |y_k - y| + z_k \qquad \text{for all } k.$$

But by using condition C_2, we obtain $\lim z_k = 0$. Hence, $\lim P_n(y) = y$, which means that $y \in X_0$.

We shall now show that if $y \in X_0$, then $y^+ \in X_0$. Thus, we are assuming that $\lim P_n(y) = y$. Hence, $\lim P_n(y)^+ = y^+$. Using the fact that $P_n P_k = P_k P_n = P_n$ for $n \leqslant k$, we see that $P_n(x) \in X_0$ for all n and all $x \in X$. Therefore, by using condition C_1 and the fact that $P_n(P_n(y)) = P_n(y)$, we see that $P_n(y)^+ \in X_0$ for all n. Hence, by the results of the previous paragraph we have

$$\lim P_n(y)^+ = y^+ \in X_0.$$

The results of the previous two paragraphs can be used in a standard fashion to show that X_0 is a countably complete vector sublattice of X.

We are now ready to prove the main theorem.

THEOREM. *If $\{x_n, P_n\}$ is a bounded submartingale and conditions C_1, C_2, and C_3 are satisfied, then* $\lim x_n = y$ *exists. Furthermore, $x_n \leqslant P_n(y)$ for all n. If $\{x_n, P_n\}$ is actually a bounded martingale, then $x_n = P_n(y)$ for all n.*

Proof. Define $y_n = \inf\{x_k : k \geqslant n\}$ and then $y = \sup y_n = \lim y_n$. Note that $y_n \leqslant x_n$ for all n. If we define $u_n = P_n(y)$, then $\{u_n, P_n\}$ is, by condition C_2, a bounded martingale.

Recalling the definition of the subspace X_0 and its properties, we see that since $x_n \in X_0$, we must have $y_n \in X_0$ and $y \in X_0$. Hence, $\lim u_n = y$. To prove the theorem we need only show that $x_n \leqslant u_n$ for all n.

We now note that $\{(x_n - u_n)^+, P_n\}$ is a bounded non-negative submartingale; to prove this, we combine Lemmas 3, 4, and 5 and also condition C_1.

We shall now make use of condition C_3. Define $w_n = \inf\{u_k: k \geqslant n\}$. Thus, $\lim w_n = \lim u_n = y$. Now if $n \leqslant k$, then $w_n \leqslant w_k \leqslant u_k$. Therefore, when $n \leqslant k$ we have

$$x_k - u_k \leqslant x_k - w_k \leqslant x_k - w_n.$$

If we define $a_n = \inf\{(x_k - u_k)^+ : k \geqslant n\}$, then

$$\lim a_n = \lim\inf (x_n - u_n)^+.$$

However, by Lemma 1,

$$a_n \leqslant \inf\{(x_k - w_n)^+ : k \geqslant n\} = (y_n - w_n)^+.$$

Since $\lim y_n = \lim w_n = y$, we have $\lim a_n \leqslant \lim (y_n - w_n)^+ = 0$. Therefore,

$$\lim a_n = \lim\inf (x_n - u_n)^+ = 0.$$

Using condition C_3, we see that $(x_n - u_n)^+ = 0$ and, hence, $x_n \leqslant u_n = P_n(y)$ for all n.

If $\{x_n, P_n\}$ is actually a bounded martingale, then the above results can be applied to the bounded submartingales $\{x_n, P_n\}$ and $\{-x_n, P_n\}$. Therefore, $\lim x_n = y$ and $\lim -x_n = -y$ and $x_n \leqslant P_n(y)$ and $-x_n \leqslant P_n(-y)$ for all n. Hence, $x_n = P_n(y)$ for all n. This completes the proof of the theorem.

4. Proof of Doob's theorem (special case). In this section, X will denote the vector lattice of all real-valued, bounded (almost everywhere) random variables defined on a probability measure space $(\Omega, \mathfrak{F}, \mu)$. Let $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$ be a sequence of Borel subfields of $\mathfrak{F}$. The conditional expectations $E\{\cdot|\mathfrak{F}_n\}$ are the positive projections P_n in the special case under consideration. We may easily verify that they have all the properties of positive projections that were assumed in §2.

In order to prove Doob's theorem in the special case of a submartingale $\{x_n, P_n\}$, where the sequence $\{x_n\}$ of random variables is bounded above and below by constant random variables, we need only verify that the conditions C_1, C_2, and C_3 are satisfied and then apply the main theorem proved in §3. It should be noted here that in the main theorem the limit was obtained in the sense of order convergence and, therefore, in applying the main theorem in this special case the limit is computed in the sense of convergence almost everywhere. This is because order convergence in the vector lattice X under consideration in this section implies convergence almost everywhere.

Condition C_1 is easily verified.

To verify condition C_2, we select any sequence $\{z_n\}$ of random variables from X such that $0 \leqslant \ldots \leqslant z_2 \leqslant z_1 \leqslant u$, where u is a constant random variable and $\inf z_n = 0$. Define $y_{nk} = E\{z_k|\mathfrak{F}_n\}$; hence, $0 \leqslant \ldots \leqslant y_{n2} \leqslant y_{n1} \leqslant u$ for all n. Note that y_{nk} is measurable $\mathfrak{F}_n$ for all k and n.

We must now prove that $\lim_k y_{nk} = 0$ uniformly in n for almost all $\omega \in \Omega$. For any $\epsilon > 0$ define $F_{nk} = \{\omega : y_{nk}(\omega) \geqslant \epsilon\}$. Define

$$f_k(A) = \int_A z_k \, d\mu \qquad \text{for any } A \in \mathfrak{F}.$$

We now note the following facts. If $A \subset B$, then $f_k(A) \leqslant f_k(B)$. The sequence $\{f_n(\Omega)\}$ converges monotonically to zero. If $A \in \mathfrak{F}_n$, then

$$f_k(A) = \int_A y_{nk}\, d\mu.$$

For all k and n, we have $F_{nk} \in \mathfrak{F}_n$ and $F_{n1} \supset F_{n2} \supset \ldots$.

We now define

$$H_k = \bigcup_{n=1}^{\infty} F_{nk};$$

hence, $H_1 \supset H_2 \supset \ldots$. Then define

$$S_{tk} = \bigcap_{n=1}^{t} (\Omega - F_{nk}).$$

We now put $T_{1k} = F_{1k}$ and then by induction $T_{t+1,k} = S_{tk} \cap F_{t+1,k}$. We note that if $t \neq j$, then $T_{tk} \cap T_{jk} = \emptyset$. Also note that

$$H_k = \bigcup_{j=1}^{\infty} T_{jk}$$

and $T_{nk} \in \mathfrak{F}_n$ for all k and n. Now

$$f_k(H_k) = \sum_{j=1}^{\infty} f_k(T_{jk}) = \sum_{j=1}^{\infty} \int_{T_{jk}} y_{jk}\, d\mu \geqslant \sum_{j=1}^{\infty} \epsilon\cdot\mu(T_{jk}) = \epsilon\cdot\mu(H_k).$$

Since $f_k(\Omega) \geqslant f_k(H_k) \geqslant \epsilon\mu(H_k)$ for all k and all $\epsilon > 0$, we see that $\lim \mu(H_k) = 0$. Since $H_1 \supset H_2 \supset \ldots$, we have

$$\mu\left(\bigcap_{j=k}^{\infty} H_j\right) = 0 \qquad \text{for all } k.$$

By checking the definition of H_k and F_{nk}, we see that this verifies condition C_2.

Condition C_3 can be verified by using (**3**, Chapter 7, Theorem 2.1) to show that $\int_\Omega z_n\, d\mu = 0$ for all n. Since we assume in C_3 that $z_n \geqslant 0$ for all n, this shows that $z_n = 0$ for all n.

5. Proof of Doob's theorem (general case). It is possible to prove Doob's theorem (**3**, Chapter 7, Theorem 4.1s) by using the result of the previous section and the following lemmas. The reader should note that in what follows the random variables under consideration need not be bounded by constants. It should be pointed out that in (**3** and **5**) the term "semimartingale" is used instead of submartingale. In (**3**) Doob uses the notation $\{z_n, \mathfrak{F}_n, n \geqslant 1\}$ for a submartingale. Our notation $\{z_n, P_n\}$ has the same meaning since we are assuming here that $P_n = E\{\cdot|\mathfrak{F}_n\}$ for all n.

LEMMA 8. *If a submartingale $\{z_n, P_n\}$ is bounded above by a constant, then it converges almost everywhere.*

Proof. The upper bound on the sequence $\{z_n\}$ means that $\sup_n E\{z_n^+\} < \infty$. Therefore, for any $\epsilon > 0$ there exists a real number $\lambda < 0$ such that Prob $\{z_n \geqslant \lambda$ for all $n\} \geqslant 1 - \epsilon$; this follows from **(3**, Chapter 7, Theorem 3.2**)**. If we define $y_n = \max\{z_n, \lambda\}$, then $\{y_n, P_n\}$ is a submartingale bounded above and below by constants; hence, it converges almost everywhere to the same values as the sequence $\{z_n\}$ on the set $\{z_n \geqslant \lambda$ for all $n\}$. Therefore, $\lim z_n$ exists almost everywhere.

LEMMA 9. *If a submartingale $\{z_n, P_n\}$ is bounded below by a constant and $\sup_n E\{z_n^+\} < \infty$, then it converges almost everywhere.*

Proof. From Lemma 6 we see that we can write $z_n = x_n + y_n$, where $\{x_n, P_n\}$ is a martingale and $0 = y_1 \leqslant y_2 \leqslant \ldots$ and

$$y_{n+1} = \sum_{i=1}^{n} [P_i(z_{i+1}) - z_i].$$

Therefore,

$$E\{y_{n+1}\} = E\{z_{n+1}\} - E\{z_1\} \leqslant \sup_n E\{z_n^+\} - E\{z_1\}.$$

Hence, $\lim y_n = u$ exists with $E\{u\} < \infty$.

Since $u \geqslant y_n$ and $P_n(y_n) = y_n$, we have $u_n = P_n(u) \geqslant y_n$ for all n. Now $\{u_n, P_n\}$ is a martingale bounded below by 0. Also $\{y_n - u_n, P_n\}$ is a submartingale bounded above by 0; by Lemma 8 it therefore converges almost everywhere.

Now $z_n = x_n + y_n = (x_n + u_n) + (y_n - u_n) \leqslant x_n + u_n$ for all n. By assumption the sequence $\{z_n\}$ is bounded below by a constant; hence, $\{-(x_n + u_n), P_n\}$ is a martingale bounded above by a constant and, therefore, by Lemma 8 it converges almost everywhere. Hence,

$$\lim (x_n + u_n) + \lim (y_n - u_n) = \lim z_n$$

exists almost everywhere.

DOOB'S THEOREM. *If $\{z_n, P_n\}$ is a submartingale with $\sup_n E\{z_n^+\} < \infty$, then it converges almost everywhere.*

Proof. Using the definition of x_n and y_n as given in Lemma 9, we see that $z_n = x_n + y_n \geqslant x_n$ and, hence, $z_n^+ \geqslant x_n^+$ for all n. Therefore, $\sup_n E\{x_n^+\} < \infty$. Since $\{x_n, P_n\}$ is a martingale, $\{x_n^+, P_n\}$ is a submartingale bounded below by 0; by Lemma 9 it therefore converges almost everywhere. By similar reasoning we can show that the submartingale $\{(-x_n)^+, P_n\}$ converges almost everywhere. In Lemma 9 we showed that $\lim y_n$ exists almost everywhere. Therefore,

$$\lim x_n^+ - \lim (-x_n)^+ + \lim y_n = \lim x_n + \lim y_n = \lim z_n$$

exists almost everywhere.

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