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OPTIMAL SHAPE DESIGN FOR A NOZZLE PROBLEM

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Abstract

In this paper, a gradient method is developed for the optimal shape design in a nozzle problem described by variational inequalities. It is known that this method can be used for the optimal shape design for systems described by partial differential equations (Pironneau [6]); it is used here for differential inequalities by taking limits of the expression resulting from an approximations scheme. The computations are done by the finite element method; the gradient of the criteria as a function of the coordinates nodes is computed, and the performance criterion is then minimised by the gradient method.

1. Introduction

The optimal shape problem can be solved for systems described by differential equations (Pironneau [6]). The purpose of this paper is to develop an optimal shape design for a nozzle problem described by a variational inequality.

Let Ω be a given domain and D be any fixed domain which is contained in ω; $\partial\Omega$ is the boundary of the domain Ω. The velocity $u(x)$ at a point x in a nonviscous incompressible potential flow (such as for air or water at moderate speed) may be approximated by

$$u(x) = \nabla\varphi(x) \qquad x \in \Omega, \tag{1.1}$$

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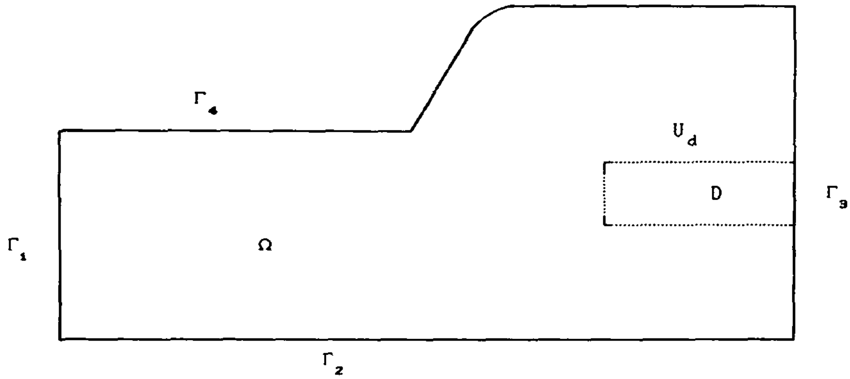

FIGURE 1. Physical set-up of the problem with domain Ω, subdomain D (i.e., $D \subset \Omega$), boundary of the domain $\partial\Omega = \bigcup_{i=1}^{4} \Gamma_i$ and velocity near the exit U_d.

where φ satisfies a second-order partial differential equation on Ω,

$$-\nabla^2\varphi = f \quad \text{in} \quad \Omega. \tag{1.2}$$

Then the flow in a nozzle Ω (the region occupied by the fluid) with a prescribed pressure drop $\varphi_{\Gamma_1} - \varphi_{\Gamma_3}$ is obtained by solving (1.2) with boundary conditions

$$\partial\varphi/\partial n|_{\Gamma_2\cup\Gamma_4} = 0, \qquad \varphi|_{\Gamma_1\cup\Gamma_3} = 0, \tag{1.3}$$

where $\partial\Omega = \cup\Gamma_i$, $i = 1, 2, 3, 4$. The physical set-up is depicted schematically in Figure 1. Consider the Sobolev space

$$H_0^1(\Omega) = \{\phi \mid \phi \in H^1(\Omega), \quad \phi = 0, \quad \text{on } \partial\Omega\},$$

where $H^1(\Omega)$ is the set of square integrable functions with square integrable first derivatives. Define the inner products $(.,.)$ and $a(.,.)$ on $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, by

$$(f, g) = \int_\Omega fg\, dx \qquad \forall f, g \in L^2(\Omega),$$

and

$$a(\varphi, \phi) = \int_\Omega \nabla\varphi \cdot \nabla\phi\, dx \qquad \forall \varphi, \phi \in H_0^1(\Omega), \tag{1.4}$$

with the associated norms being denoted by $|f|^2 = (f, f)$ and $||\varphi||^2 = a(\varphi, \varphi)$.

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The problem we want to consider consists of finding the solution φ so that

$$\varphi \in H_0^1(\Omega),$$

and

$$a(\varphi, \phi - \varphi) \geq (f, \phi - \varphi) \qquad \forall \phi \in H_0^1(\Omega). \tag{1.5}$$

We note that the bilinear form $a(.,.)$ in (1.5) is elliptic, i.e.,

$$a(\phi, \phi) \geq \alpha||\phi||^2 \quad \alpha > 0, \quad \forall \phi \in H_0^1(\Omega). \tag{1.6}$$

In our problem, we are interested in designing a nozzle that gives a prescribed velocity U_d under the exit, say in some given domain D which is a subset of the domain Ω. To obtain an approximate design we shall solve the following optimisation problem:

$$\min_{\Omega \in \theta} E(\Omega) = \int_D |\nabla\varphi(\Omega) - U_d|^2 dx, \tag{1.7}$$

where $\theta = \{\Omega : \Omega \supset D, \ ; \ \Gamma_1, \Gamma_2, \Gamma_3 \text{ are fixed, } \Gamma_4 \text{ is any curve}\}$.

The optimal shape problem can be solved for the systems described by differential equations (Angrand [1]). So to solve the problem for the systems described by a differential inequality we shall introduce the first penalised equation,

$$A\varphi_\varepsilon + (1/\varepsilon)\varphi_\varepsilon^- = f, \quad \varphi_\varepsilon \in H_0^1(\Omega), \tag{1.8}$$

where $V^- = -\sup(-V, 0)$, and $A : V = H_0^1 \to V'$ is a linear continuous and symmetric operator satisfying the coercivity condition, i.e. $(A\phi, \phi) = a(\phi, \phi) \geq \alpha||\phi||^2$, for all $\phi \in V$, $\alpha > 0$, and $A = -\nabla.\nabla$, whose solution φ_ε tends to the solution of (1.5) when $\varepsilon \to 0$. For the existence and uniqueness of a solution of this equation, see Lions [5].

2. Discretisation and optimisation

We briefly review the method of finite elements. To illustrate the method, let (1.8) be discretised by triangulation elements of degree m. In variational form, (1.8) becomes: seek $\varphi_\varepsilon \in H_0^1(\Omega)$ so that

$$\int_\Omega (\nabla\varphi_\varepsilon.\nabla\omega + F(\varphi_\varepsilon)\omega - f\omega)dx = 0, \qquad \forall \omega \in H_0^1(\Omega), \tag{2.1}$$

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where $F(\varphi_\varepsilon) = (1/\varepsilon)\varphi_\varepsilon^-$.

Let τ_h be a triangulation of Ω, and T_k is called the triangle, $\cup T_k = \Omega_h \subset \Omega$. The parameter h is the size of the largest side or edge, and we assume that we have a family of triangulations of τ_h. Let P^m be the space of polynomials of degree m on Ω_h, and denote by

$$H_h^m(\Omega_h) = \{\omega_h \in C^0(\Omega_h) : \omega_h|_{T_k} \in P^m \quad \forall\, T_k \in \tau_h\}, \tag{2.2}$$

the space of continuous piecewise polynomial functions on Ω_h (Pironneau [6]).

It is well known (see Ciarlet [3]) that $H_h^m(\Omega_h)$ is of finite dimension; then

$$\int_{\Omega_h} (\nabla\varphi_{h,\varepsilon} \cdot \nabla\omega_h + F(\varphi_{h,\varepsilon})\omega_h - f\Omega_h)dx = 0 \tag{2.3}$$

reduces to the solution of a linear symmetric positive definite system plus the numerical computation of some integrals. More precisely, if $\{\omega^i\}_i^N$ is a basis for $H_h^m(\Omega_h)$, (2.3) is equivalent to (Pironneau [6])

$$\widehat{A}\varphi = F, \tag{2.4}$$

where $\widehat{A}_{ij} = \int_{\Omega_h} (\nabla\omega^i \cdot \nabla\omega^j + F(\varphi_{h,\varepsilon})\omega^j)dx; \quad F_i = \int_{\Omega_h} f\omega^i dx$

$$\varphi_h = \sum_{i=1}^{N} \varphi_i \omega^i\,. \tag{2.5}$$

The $\{\omega^i\}$ are polynomials of degree $\leq m$ on T_k, so $\widehat{A}_{ij}$ can be computed exactly. In the case $m = 1$, if $\{q^j\}_i^N$ denote the vertices of τ_h, then $\{\omega^i\}$ are uniquely determined by

$$\omega^i(q^j) = \delta_{ij} \quad \forall\, i, j = 1, \dots, N.$$

In the case $m = 2$, if $\{q^{jk}\}$ denote the middles of the sides of vertices $\{q^j, q^k\}$, then $\{\omega^i\}$ is uniquely determined by

$$\omega^i(q^j) = \delta_{ij} \quad \forall\, i, j = \{1, \dots, N'\} \cup (\{1, \dots, N'\} \times \{1, \dots, N'\}).$$

It is possible to consider our optimisation problem in this new setting. The optimal shape will be found by successive approximation starting with an initial guess Ω_h^0, and the algorithm is then developed by means of a gradient method. We note that the problem has been discretised, so that the shape Ω_h is defined by the coordinates of the nodes. The expression for the cost function E is now

$$E(\Omega_h) = \int_{D_h} (\nabla\varphi_{h,\varepsilon} - U_{d,h})^2 dx, \tag{2.6}$$

where $\varphi_{h,\varepsilon}$ is the solution of the differential equation (2.3) on Ω_h and $U_{d,h}$ and D_h are the approximations of U_d and D respectively. The following theorem has been adapted from Pironneau [6] to compute the gradient of the cost function E at Ω_h. For the proof of this theorem see Butt [2].

THEOREM 1. *If E is given by* (2.6) *and $\varphi_{h,\varepsilon}$ by* (2.3), *then*

$$\begin{aligned}\partial E/\partial q_l^k \;=\; &\int_{\Omega_h} \partial/\partial x_l \left\{\omega^k(\nabla\varphi_{h,\varepsilon} - U_{d,h})^2\right\} dx\\ &+\int_{\Omega_h} \left\{(\nabla\varphi_{h,\varepsilon}\cdot\nabla\omega^k)\partial P_{h,\varepsilon}/\partial x_l - (\nabla\varphi_{h,\varepsilon}\cdot\nabla P_{h,\varepsilon})\partial\omega^k/\partial x_l\right\} dx\\ &+\int_{\Omega_h} \left\{\partial/\partial x_l(f P_{h,\varepsilon}\omega^k) - (f\omega^k\partial P_{h,\varepsilon}/\partial x_l)\right\} dx\\ &+\int_{\Omega_h} \left\{(F(\varphi_{h,\varepsilon})\nabla\omega^k\partial P_{h,\varepsilon}/\partial x_l) - \partial/\partial x_l(\omega^k F(\varphi_{h,\varepsilon})P_{h,\varepsilon})\right\} dx,\\ &\qquad l = 1, 2 \text{ and } k = 1,\dots,n, \quad q^k \in D_h,\end{aligned} \tag{2.7}$$

where

$$P_{h,\varepsilon} \in H_h^1(\Omega_h)$$

is the solution of

$$\begin{aligned}&\int_{\Omega_h} (\nabla P_{h,\varepsilon}\cdot\nabla\omega_h + F'(\varphi_{h,\varepsilon})P_{h,\varepsilon}\omega_h)dx\\ &= 2\int_{D_h} (\nabla\varphi_{h,\varepsilon}) - U_{d,h})\nabla\omega_h dx, \qquad \omega_h \in H_0^1(\Omega_h)\end{aligned} \tag{2.8}$$

with $F'(\varphi_{h,\varepsilon}) = (1/\varepsilon)\, d/d\varphi\ (\varphi_{h,\varepsilon}^-)$, *and* (2.8) *is equivalent to the second penalised equation,*

$$AP_{h,\varepsilon} + F'(\varphi_{h,\varepsilon})P_{h,\varepsilon} = -2\nabla(\nabla\varphi_{h,\varepsilon} - U_{d,h}) = f_1\,. \tag{2.9}$$

We note that the function $\varphi_{h,\varepsilon} \to \varphi_{h,\varepsilon}^-$ is not differentiable at $\varphi_{h,\varepsilon} = 0$; we have *defined* $F'(0) = 0$. This choice turns out to be unimportant because $f_{h,\varepsilon} > 0$, on Ω_h, with exception of a (zero-measure) subset of $\partial\Omega_h$. For more details see Butt [2], where an approximation scheme is introduced for proving this.

3. Optimal shape design for a variational inequality

Now we come to the implementation of the main idea of our treatment, that is, to take the limit as ε tends to zero of these quantities. First we shall find the value of the limit of the cost function E, as ε tends to zero. Since we know (Lions [5]) that

$$\varphi_{h,\varepsilon} \to \varphi_h \quad \text{in} \quad H_h^1(\Omega_h) \quad \text{weakly as} \quad \varepsilon \to 0, \tag{3.1}$$

and also,

$$\varphi_{h,\varepsilon} \to \varphi_h \quad \text{in} \quad L^2(\Omega_h) \quad \text{strongly as} \quad \varepsilon \to 0,$$

by taking the limit (as $\varepsilon \to 0$), on both sides of (2.6), we obtain:

$$\lim_{\varepsilon \to 0} E(\Omega_h) = \lim_{\varepsilon \to 0} \int_{D_h} (\nabla\varphi_{h,\varepsilon} - U_{d,h})^2 \, dx.$$

Since $\varphi_{h,\varepsilon} \to \varphi_h$ in $L^2(\Omega_h)$ strongly, so (Glowinski *et al.* [4])

$$\lim_{\varepsilon \to 0} \int_{D_h} (\nabla\varphi_{h,\varepsilon} - U_{d,h})^2 dx \to \int_{D_h} (\nabla\varphi_h - U_{d,h})^2 dx;$$

since the functional $\varphi_h \to \int_{D_h} (\nabla\varphi_h - U_{d,h})^2 dx$ is continuous in $L^2(\Omega_h)$; so the above equation becomes:

$$E(\Omega_h) = \int_{D_h} (\nabla\varphi_h - U_{d,h})^2 dx, \tag{3.2}$$

which is the required value of the cost function E as ε tends to zero. Now we shall find the value of the gradient of the cost function (2.7) as ε tends to zero:

$$\begin{aligned}
\lim_{\varepsilon \to 0} \partial E/\partial q_l^k = \lim_{\varepsilon \to 0} \Bigg[& \int_{\Omega_h} \partial/\partial x_l \Big\{\omega^k (\nabla\varphi_{h,\varepsilon} - U_{d,h})^2\Big\} dx \\
& + \int_{\Omega_h} \Big\{(\nabla\varphi_{h,\varepsilon} \cdot \nabla\omega^k)\partial P_{h,\varepsilon}/\partial x_l - (\nabla\varphi_{h,\varepsilon} \cdot \nabla P_{h,\varepsilon})\partial\omega^k/\partial x_l\Big\} dx \\
& + \int_{\Omega_h} \Big\{\partial/\partial x_l (f P_{h,\varepsilon}\omega^k) - (f\omega^k \partial P_{h,\varepsilon}/\partial x_l)\Big\} dx \\
& + \int_{\Omega_h} \Big\{(F(\varphi_{h,\varepsilon})\nabla\omega^k (\partial P_{h,\varepsilon}/\partial x_l) - \partial/\partial x_l (\omega^k F(\varphi_{h,\varepsilon}) P_{h,\varepsilon})\Big\} dx \Bigg].
\end{aligned} \tag{3.3}$$

Now we need to find the limit of the vector $P_{h,\varepsilon}$ as ε tends to zero; in the Appendix we prove the following theorem which shows that this limit, P_h, is itself the solution of a variational inequality.

THEOREM 2. *As* $\varepsilon \to 0$, $P_{h,\varepsilon} \to P_h$ *in* $H_0^1(\Omega_h)$, P_h *being the solution of the variational inequality*

$$a(P_h, \omega_h - P_h) \geq (f_1, \omega_h - P_h), \quad \forall \omega_h \in H_0^1(\Omega_h) \tag{3.4}$$

where f_1 *is defined by* (2.9).

The proof of Theorem 2 has been explained in detail in the appendix. Now we shall compute the gradient of the cost function when ε tends to zero, by using (3.3).

Since we know (Glowinski *et al.* [4]) that

$$\lim_{\varepsilon \to 0} \int_{\Omega_h} (\nabla P_{h,\varepsilon} \cdot \nabla \varphi_{h,\varepsilon}) dx = \int_{\Omega_h} (\nabla P_h \cdot \nabla \varphi_h) dx,$$

and

$$\partial/\partial x_1(\varphi_{h,\varepsilon}) \to \partial/\partial x_1(\varphi_h) \quad \text{in} \quad L^2(\Omega_h) \quad \text{weakly, as} \quad \varepsilon \to 0,$$

then $\varphi_{h,\varepsilon} \to \varphi_h$ in $L^2(\Omega_h)$ strongly as $\varepsilon \to 0$; so (3.3) gives rise to

$$\begin{aligned} \partial E/\partial q_l^k = & \int_{\Omega_h} (\nabla\varphi_h - U_{d,h})^2 \partial/\partial x_l(\omega^k) dx \\ & + \int_{\Omega_h} \Big\{ (\nabla\varphi_h \cdot \nabla\partial/\partial x_l(P_h))\omega^k \\ & \quad + (\nabla\varphi_h \cdot \nabla\partial/\partial x_l(\omega^k)) P_h - (\nabla\varphi_h \cdot \nabla\partial/\partial x_l(P_h))\nabla\omega^k \Big\} dx \\ & + \int_{\Omega_h} \Big\{ 2(\nabla\varphi_h \cdot \nabla\omega^k)\partial/\partial x_l(P_h) \\ & \quad + (\nabla P_h \cdot \nabla\omega^k)\partial/\partial x_l(\varphi_h) - f\omega^k \partial/\partial x_l(P_h) \Big\} dx \\ & - \int_{\Omega_h} \Big\{ (\nabla\varphi_h \cdot \nabla\nabla\omega^k)\partial/\partial x_l(P_h) \\ & \quad - (\nabla P_h \cdot \nabla\partial/\partial x_l(\varphi_h))\omega^k - f\nabla\omega^k \partial/\partial x_l(P_h) \Big\} dx, \\ & \qquad l = 1, 2, \quad \text{and} \quad k = 1, \dots, n \end{aligned} \tag{3.5}$$

where φ_h is the solution of (1.5) and P_h is the solution of (3.4).

We define then an algorithm to solve the optimal shape problem for the systems described by a differential inequality, i.e. when ε tends to zero, and in this algorithm it has been found necessary to use a second-order approximation, that is, $m = 2$, which made us able to compute (3.5).

ALGORITHM

1. Choose Ω_h^0, i.e., $\{q^{k,0}\}$.
2. Compute $\varphi_h^{m'}$ (with $m = 2$).
3. Compute $P_h^{m'}$.

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4. Compute $G_l^k = -\partial E/\partial q_l^k$, $l = 1, 2$, and $k = 1, \dots, n$, $q^k \notin D_h$.
5. Let $q^{k,m'}(\rho) = q^{k,m'} + \rho G_l^k$. Compute $\rho^{m'}$, an approximation of arg $\min\limits_{0<\rho<\rho_{\max}} E(\{q^{k,m'}(\rho)\})$, where E is given by (3.2).
6. Set $q^{k,m'+1} = q^{k,m'}(\rho)$.
7. Perform a terminal check, if necessary go on with the same procedure in $q^{k,m'+1}$, i.e. go back to 1.

4. Description of the program and algorithms used

The implementation of an algorithm ($m = 2$) will be described here. The optimum design program is composed of the following modules.

MODULE 1. A module for solving the direct problem (or state problem). Find $\varphi_h \in H_h^2(\Omega_h)$ such that

$$\int_{\Omega_h} (\nabla\varphi_h \cdot \nabla\omega_h - f\omega_h)dx \geq 0 \quad \forall\, \omega_h \in H_h^2(\Omega_h) \tag{4.1}$$

or, find $\varphi_h \in H_h^2(\Omega_h)$ such that

$$I(\varphi_h) \leq I(\omega_h) \quad \forall\, \omega_h \in H_h^2(\Omega_h), \tag{4.2}$$

where $I(\varphi_h)$ is defined as follows:

$$I(\varphi_h) = 1/2\int_{\Omega_h} |\nabla\varphi_h|^2 dx - \int_{\Omega_h} f\varphi_h dx, \tag{4.3}$$

minimised over the convex set $K_1 = \{\psi_h \in H^2(\Omega_h),\ \psi_h \geq 0 \text{ a.e. in } \Omega_h\}$, where φ_h is the solution of (4.1). The method used for the minimisation of this functional will be explained briefly. The function $I(\varphi_h)$ may be written $I(\varphi_1, \dots, \varphi_{N(h)})$ to emphasise the dependence of φ_h on the coefficients in (2.5). The problem (4.2) is solved by the relaxation method, with

$$\varphi_h^0 = (\varphi_1^0, \dots, \varphi_{Nh}^0) \quad \text{given in } H_h^2(\Omega_h),$$

with φ_h^n known, then φ_h^{n+1} is determined coordinate by coordinate, further iterations in the algorithm being given by

$$\varphi^{n+1} = \varphi^n + \omega(\varphi^{n+1/2} - \varphi^n).$$

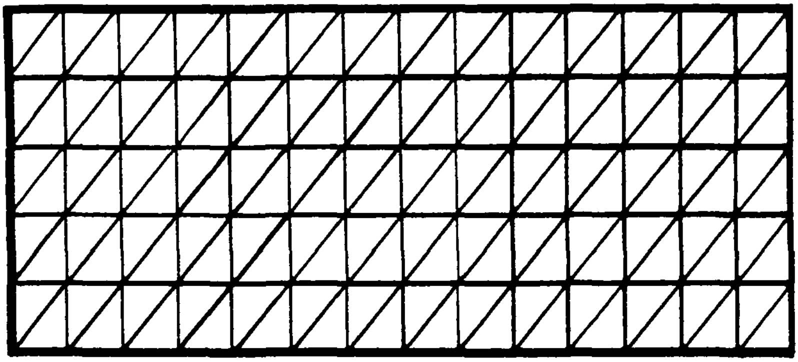

FIGURE 2. Indicates the initial shape with performance criterion $E(\Omega_h^0) = 0.16452$ after iteration zero.The total number of nodes is 90 and the total number of triangles is 140 for the domain Ω. For the subdomain D the total number of nodes is 36 and the total number of triangles is 50.

Here ω is the relaxation parameter, $0 < \omega < 2$. The process is stopped when

$$\sum_{i=1}^{Nh} \left|\varphi_i^{n+1} - \varphi_i^n\right| \Big/ \sum_{i=1}^{Nh} \left|\varphi_i^{n+1}\right| \leq \varepsilon_r.$$

(In our computational experiments we took $\varepsilon_r = 10^{-5}$.)

MODULE 2. A module for solving the adjoint-state problem, whose solution is needed to compute the descent direction (the vector G). The adjoint state $P_h \in H_h^2(\Omega_h)$ given by the solution of the following variational inequality,

$$\int_{\Omega_h} (\nabla P_h \cdot \nabla \omega_h) dx - \int_{D_h} (f_1 \omega_h) dx \geq 0 \quad \forall \omega_h \in H_h^2(\Omega_h). \tag{4.4}$$

In the Appendix, we show that this variational inequality has a solution which minimises the following functional:

$$I(P_h) = \frac{1}{2} \int_{\Omega_h} |\nabla P_h|^2 dx - \int_{D_h} f_1 P_h dx, \quad P_h \in H_h^2(\Omega_h) \tag{4.5}$$

over the convex set K_1 and P_h is the solution of (4.4), and f_1 is defined by (2.9). For this problem, we use the same optimisation method used in the case of the state problem.

MODULE 3. A module for the computation of the descent direction, i.e. the gradient of the cost function E when we know the solution φ_h of the state

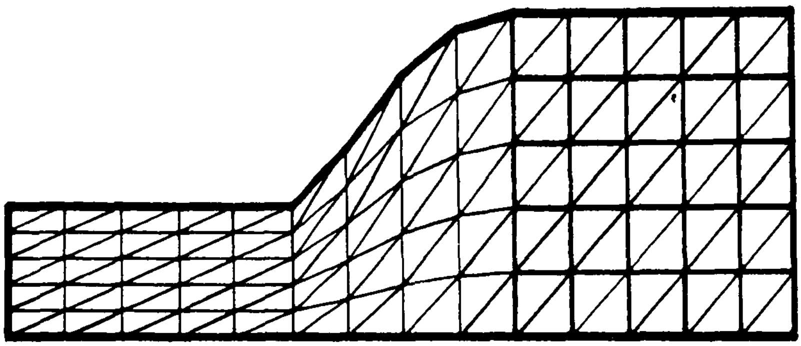

FIGURE 3. Indicates the new shape after 15 iterations with new performance criterion $E(\Omega_h^{15}) = 0.010921$. The total number of nodes is 90 and the total number of triangles is 140 for the domain Ω. For the subdomain D the total number of nodes is 36 and the total number of triangles is 50.

problem and the solution P_h of the adjoint state problem. In the formula we must account for the variability of the criterion domain.

MODULE 4. A module minimising the criterion functional when we know a descent direction. We used the gradient method with optimal choice of step length ρ and eventually projection.

MODULE 5. A drawing module for the plotting of the results related to a given geometry. This is convenient for quickly analysing computational results.

The finite element method was used to solve (1.5), (3.4), and (3.5) with $f = 0$ and $U_{d,h} = 0.1$. The triangulation is composed of 90 nodes and 140 triangles for the domain Ω_h and 36 nodes and 50 triangles for the subdomain D_h. The initial shape of the problem is shown in Figure 2 and we can also see in Figure 2 the subdomain D_h where the criterion E

$$E(\Omega_h) = \int_{D_h} (\nabla\varphi_h - U_{d,h})^2 dx$$

is evaluated. The starting value of the criterion is $E(\Omega_h^0) = 0.16452$ with $U_{d,h} = 0.1$ given, at iteration zero. The new shape of the problem is shown in Figure 3 after 15 iterations with criterion $E(\Omega_h^{15}) = 0.010921$, and Figure 4 shows the final shape of the problem with criterion $E(\Omega_h^{27}) = 0.000415$ after 24 iterations. Figure 5 shows the relation between the performance criterion E and the number of iterations.

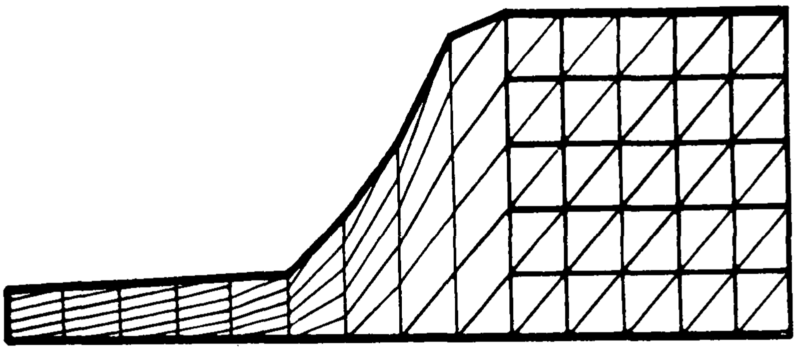

FIGURE 4. Indicates the final shape of the problem after 27 iterations with performance criterion $E(\Omega_h^{27}) = 0.000415$. The total number of nodes is 90 and the total number of triangles is 140 for the domain Ω. For the subdomain D the total number of nodes is 36 and the total number of triangles is 50.

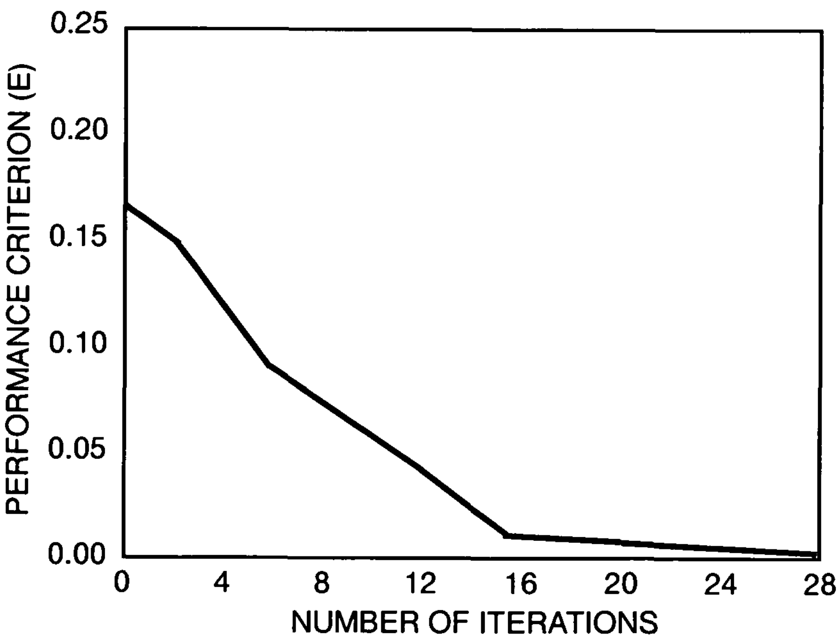

FIGURE 5: Indicates the relation between performance criterion and number of iterations.

5. Conclusions

We have developed a method for the optimal shape design for the nozzle problem. The work has been helped by the fact that our system is governed by a variational inequality, with all its strong properties, which make the approximation and computation of solutions and optimal shapes that much simpler. The main theoretical result – Theorem 2 in Section 3 – shows that the vector which eventually defines the search direction for a minimum, is itself the solution of an associated variational inequality. The practical results consist of the development of a computationally-complex method for the determination of the optimal shapes, which can be adapted to other problems of current interest.

6. Appendix

The main purpose of this appendix is to sketch the proof of Theorem 2. We can prove that the functions $P_{h,\varepsilon}$ are non-negative on Ω_h. Before proving Theorem 2, we prove the following lemma.

LEMMA 1. *Let $a(P_{h,\varepsilon}, \phi_{h,\varepsilon})$ be a bilinear, continuous form on $H_h^2(\Omega_h) \times H_h^2(\Omega_h)$ such that*

$$a(P_{h,\varepsilon}, P_{h,\varepsilon}) \geq 0 \quad \forall\, P_{h,\varepsilon} \in H_h^2(\Omega_h). \tag{1}$$

Then the function $\phi_{h,\varepsilon} \to a(\phi_{h,\varepsilon}, \phi_{h,\varepsilon})$ is lower-semicontinuous with respect to the weak topology.

PROOF. From the bilinearity, we have for all $P_{h,\varepsilon} \in H_h^2(\Omega_h)$, $\phi_{h,\varepsilon} \in H_h^2(\Omega_h)$

$$\begin{aligned} a(\phi_{h,\varepsilon}, \phi_{h,\varepsilon}) \;&=\; a(P_{h,\varepsilon}, P_{h,\varepsilon}) + \Big[a(P_{h,\varepsilon}, \phi_{h,\varepsilon} - P_{h,\varepsilon}) \\ &\quad + a(\phi_{h,\varepsilon} - P_{h,\varepsilon}, P_{h,\varepsilon}) + a(P_{h,\varepsilon} - \phi_{h,\varepsilon}, P_{h,\varepsilon} - \phi_{h,\varepsilon})\Big]. \end{aligned} \tag{2}$$

Now we use the condition of ellipticity, i.e.

$$a(P_{h,\varepsilon}, P_{h,\varepsilon}) \geq 0,$$

which implies that

$$a(\phi_{h,\varepsilon}, \phi_{h,\varepsilon}) \geq a(P_{h,\varepsilon}, P_{h,\varepsilon}) + \Big[a(P_{h,\varepsilon}, \phi_{h,\varepsilon} - P_{h,\varepsilon}) + a(\phi_{h,\varepsilon} - P_{h,\varepsilon}, P_{h,\varepsilon})\Big].$$

Now, let $\phi_{h,\varepsilon} \to P_h$ in $H_h^2(\Omega_h)$ weakly; from the continuity of "a", and the fact that

$$a(P_h, \phi_{h,\varepsilon} - P_h) \to 0 \quad \text{and} \quad a(\phi_{h,\varepsilon} - P_h, P_h) \to 0,$$

we have

$$\liminf_{\phi_{h,\varepsilon} \to P_h} a(\phi_{h,\varepsilon}, \phi_{h,\varepsilon}) \geq a(P_h, P_h). \tag{3}$$

Hence the map $\phi_{h,\varepsilon} \to a(\phi_{h,\varepsilon}, \phi_{h,\varepsilon})$ is weakly lower-semicontinuous.

In connection with the behaviour of the subsequence $P_{h,\varepsilon}$ as $\varepsilon \to 0$, we have

THEOREM 2. *As* $\varepsilon \to 0$, $P_{h,\varepsilon} \to P_h$ *in* $H_h^2(\Omega_h)$, P_h *being the solution of the variational inequality*

$$a(P_h, \omega_h - P_h) \geq (f_1, \omega_h - P_h) \quad \forall \omega_h \in H_h^2(\Omega_h), \tag{4}$$

where f_1 *is defined by* (2.9).

PROOF. Consider the second penalised equation

$$AP_{h,\varepsilon} + (1/\varepsilon)(d/d\varphi(\varphi_{h,\varepsilon}^-))P_{h,\varepsilon} = f_1 , \tag{5}$$

or, in variational form,

$$\int_{\Omega_h} \Big((\nabla P_{h,\varepsilon} \cdot \nabla \omega_h) + (1/\varepsilon)((d/d\varphi(\varphi_{h,\varepsilon}^-))P_{h,\varepsilon}, \omega_h)\Big)dx = \int_{D_h} (f_1, \omega_h)dx. \tag{6}$$

With $P_{h,\varepsilon} = \omega_h$, we have

$$\int_{\Omega_h} \Big((\nabla P_{h,\varepsilon} \cdot \nabla P_{h,\varepsilon}) + (F'(\varphi_{h,\varepsilon})P_{h,\varepsilon}, P_{h,\varepsilon})\Big)dx = \int_{D_h} (f_1, P_{h,\varepsilon})dx, \tag{7}$$

where $F'(\varphi_{h,\varepsilon}) = (1/\varepsilon)(d/d\varphi(\varphi_{h,\varepsilon}^-)) \geq 0$, and

$$\int_{\Omega_h} (d/d\varphi(\varphi_{h,\varepsilon}^-))P_{h,\varepsilon}^2 dx \geq 0.$$

Then, by (1.6) and (7), we have

$$0 \leq \alpha||P_{h,\varepsilon}||^2 \leq \int_{\Omega_h} \Big((\nabla P_{h,\varepsilon} \cdot \nabla P_{h,\varepsilon}) + F'(\varphi_{h,\varepsilon})P_{h,\varepsilon}^2\Big)dx = \int_{D_h} (f_1, P_{h,\varepsilon})dx,$$

or,

$$\begin{aligned} \alpha||P_{h,\varepsilon}||^2 &\leq \int_{D_h} (f_1, P_{h,\varepsilon})dx \leq ||f_1||\ ||P_{h,\varepsilon}||, \\ \alpha||P_{h,\varepsilon}||^2 &\leq C^1||P_{h,\varepsilon}|| \quad (C^1 = ||f_1||), \\ ||P_{h,\varepsilon}|| &\leq C_{11}, \quad (C_{11} = c^1/\alpha = \text{ constant, independent of } \varepsilon). \end{aligned} \tag{8}$$

A subsequence, also denoted by $P_{h,\varepsilon}$, can then be extracted from the sequence $P_{h,\varepsilon}$, such that

$$P_{h,\varepsilon} \to P_h \quad \text{weakly in} \quad H_h^2 .$$

Since we have assumed that $P_{h,\varepsilon} \geq 0$ on Ω_h, $P_h \geq 0$ on Ω_h. By writing (7) in the following form:

$$\begin{aligned} a(P_{h,\varepsilon}, \omega_h - P_{h,\varepsilon}) & \\ -(f_1, \omega_h - P_{h,\varepsilon}) &= -\Big(F'(\varphi_{h,\varepsilon})P_{h,\varepsilon}, \omega_h - P_{h,\varepsilon}\Big) \\ &= \Big(F'(\varphi_{h,\varepsilon})P_{h,\varepsilon}, P_{h,\varepsilon}\Big) - \Big(F'(\varphi_{h,\varepsilon})P_{h,\varepsilon}, \omega_h\Big) \\ &= 1/\varepsilon\Big[(\widehat{H}P_{h,\varepsilon}, P_{h,\varepsilon}) - (\widehat{H}P_{h,\varepsilon}, \omega_h)\Big], \end{aligned} \tag{9}$$

where $\widehat{H} \equiv d/d\varphi\ (\varphi_{h,\varepsilon}^-)$. Consider now (9) only for those $\omega_h = W_h \in B \subset H_h^2(\Omega_h)$, with B the subset of the convex set K_1 composed of the basis elements for $H_h^2(\Omega_h)$. Now we shall prove that the right-hand side of (9) is positive, that is,

$$(\widehat{H}P_{h,\varepsilon}, P_{h,\varepsilon}) > (\widehat{H}P_{h,\varepsilon}, W_h), \quad \text{provided } h \text{ is sufficiently small.} \tag{10}$$

Since $\widehat{H}$ is positive operator, $(\widehat{H}P_{h,\varepsilon}, P_{h,\varepsilon}) > 0$; we can assume that $(\widehat{H}P_{h,\varepsilon}, W_h) \geq 0$; otherwise (9) is automatically true. We can make the right-hand side of the inequality (10) as small as possible; note that $P_{h,\varepsilon}$ does not depend much on h (from (4)), but that the support of W_h can be made as small as possible by taking h small enough, the maximum value of W_h is of course 1. Therefore from (10) we can see that, under these conditions,

$$\begin{aligned} a(P_{h,\varepsilon}, W_h - P_{h,\varepsilon}) &- (f_1, W_h - P_{h,\varepsilon}) \\ &= 1/\varepsilon\Big[(\widehat{H}P_{h,\varepsilon}, P_{h,\varepsilon}) - (\widehat{H}P_{h,\varepsilon}, W_h)\Big] \geq 0, \\ &\qquad \text{for } W_h \in B \subset H_h^2(\Omega_h). \end{aligned} \tag{11}$$

Hence (11) can be written as

$$a(P_{h,\varepsilon}, W_h - P_{h,\varepsilon}) - (f_1, W_h - P_{h,\varepsilon}) \geq 0, \qquad W_h \in B \subset H_h^2(\Omega_h)$$

or

$$a(P_{h,\varepsilon}, W_h) - (f_1, W_h - P_{h,\varepsilon}) \geq a(P_{h,\varepsilon}, P_{h,\varepsilon}). \tag{12}$$

Letting $\varepsilon \to 0$ in (12), we obtain

$$a(P_h, W_h) - (f_1, W_h - P_h) \geq \liminf_{\varepsilon \to 0} a(P_{h,\varepsilon}, P_{h,\varepsilon}).$$

By applying the lemma, we obtain now

$$\liminf_{\varepsilon \to 0} a(P_{h,\varepsilon}, P_{h,\varepsilon}) \geq a(P_h, P_h) \geq 0,$$

which implies that

$$a(P_h, W_h - P_h) \geq (f_1, W_h - P_h), \quad W_h \in B \subset H_h^2(\Omega_h). \tag{13}$$

Now we shall show that (13) holds for all $\omega_h \in H_h^2(\Omega_h)$. Indeed,

$$\omega_h = \sum_i \alpha_i \omega_i, \quad \alpha_i \geq 0, \quad \omega_i \in B,$$

so that

$$\begin{aligned} a(P_h, \omega_h - P_h) = a\Big(P_h, \Big(\sum_i \alpha_i \omega_i\Big) - P_h\Big) &= \sum_i \alpha_i a(P_h, \omega_i - P_h) \\ &\geq \sum_i \alpha_i (f_1, \omega_i - P_h), \end{aligned}$$

since the α_i's are positive, and (13) is valid for all ω_i's. Thus

$$a(P_h, \omega_h - P_h) \geq (f_1, \omega_h - P_h); \quad \omega_h \in H_h^2(\Omega_h), \tag{14}$$

which shows that P_h is a solution of the inequality (4).

Since $P_h \geq 0$, that is $P_h \in H_h^2(\Omega_h)$, (14) is a variational inequality; the unique solution (in K_1) of (14) minimises

$$I(P_h) = 1/2 \int_{\Omega_h} |\nabla P_h|^2 dx - \int_{D_h} (f_1, P_h) dx, \tag{15}$$

on the convex set K_1.

Thus, we can estimate P_h by actually performing the minimisation (15). In the case of Theorem 2, we can see that the functional I is the limit as ε tends to zero of

$$I_\varepsilon(P_{h,\varepsilon}) = 1/2 \int_{\Omega_h} \Big(|\nabla P_{h,\varepsilon}|^2 + F'(\varphi_{h,\varepsilon}) P_{h,\varepsilon}^2\Big) dx - \int_{D_h} f_1 P_{h,\varepsilon} dx,$$

(for more detail see Butt [2]); this fact in effect can be proved by Theorem 2.

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