SMALL ISOMORPHISMS BETWEEN GROUP ALGEBRAS

by G. V. WOOD

(Received 27 June, 1989)

If G_1 and G_2 are locally compact groups and the algebras $L^1(G_1)$ and $L^1(G_2)$ are isometrically isomorphic, then G_1 and G_2 are isomorphic (Wendel, 1952, [**8**]). There is evidence that the following generalization of Wendel's result is true.

If T is an algebra isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \sqrt{2}$, then G_1 and G_2 are isomorphic.

This was proved for abelian groups and for connected groups in [**1**], but in the general case, it is still unproved. Some partial results have been obtained. That G_1 and G_2 are isomorphic when $\|T\| < 1{\cdot}246$ was proved in [**1**]. This was improved to the condition $\|T\| < (1+\sqrt{3})/2$ in [**8**], and the number $(1+\sqrt{3})/2$ has some special significance, as we shall see later.

In this paper, we prove the conjecture for a large class of non-abelian groups when T is a $*$-isomorphism. We also show that, for groups outside this class, the existence of a $*$-isomorphism between their group algebras with norm $<\sqrt{2}$ means that the groups are "nearly" isomorphic. (See Propositions 14, 15, and 16). Corresponding results are also true for the algebra $M(G)$ and for $C(G)$ when G is compact.

It was shown in [**8**] that the problem reduces to the discrete case. Let G_1 and G_2 be discrete groups and let T be an algebra isomorphism of $l^1(G_1)$ and $l^1(G_2)$ with $\|T\| < \sqrt{2}$. Then there exists a map t of G_1 into G_2 defined by the equation $Tx = at(x) + f$, where $|a| > 1/\sqrt{2}$. (See [**1**, Proposition 2.1].)

If $\|T\| < (1+\sqrt{3})/2$, then t is a group isomorphism. (This was proved for abelian groups in [**1**, Theorem 2.6], and in the general case in [**8**, Theorem 2.2].) For $\|T\| \geqslant (1+\sqrt{3})/2$, t need not be a isomorphism.

EXAMPLE. Let G be a cyclic group of order 6 with generator x. Define $Tx = -x/2 + i\sqrt{3}x^4/2$, and extend to an algebra isomorphism of $\mathbb{C}G$ onto $\mathbb{C}G$. Then $\|T\| = (1+\sqrt{3})/2$, yet $t(x) = x^4$.

Even though t need not be an isomorphism, it is always true that $t(x^{-1}) = t(x)^{-1}$. (See Lemma 2.1 in [**8**].)

We now assume that T is a $*$-map. If $Tx = \sum a_i y_i$, then $Tx^{-1} = \sum \bar{a}_i y_i^{-1}$. It follows that T is an isometry for the l^2 norm. Comparing the coefficient of the identity in $(Tx)(Tx^{-1})$ gives $\sum |a_i|^2 = 1$. It is this property that makes the case of $*$-isomorphisms more tractible than the general case. This fact, together with the $\sqrt{2}$ bound on the norm gives inequalities for the coefficients independent of the group structure.

LEMMA 1 ([**6**, Lemma 1]). *If* $(a_i) \in l^1$ *with* $\sum |a_i| = K < \sqrt{2}$, $\sum |a_i|^2 = 1$, *and* $|a_1| \geqslant |a_2| \geqslant |a_3|, \ldots,$ *then*

(a) $|a_2| \geqslant (1 - |a_1|^2)/(K - |a_1|)$,

(b) $|a_2| \geqslant (K - |a_1|)/2 + \sqrt{((1-|a_1|^2)/2 - (K-|a_1|)^2/4)}$,

whenever the expression under the square root sign is positive; i.e. when $|a_1| \leqslant K/3 + (2/3)\sqrt{((3-K^2)/2)}$,

(c) $|a_3| \leqslant K/3 - \sqrt{((3-K^2)/2)}/3$.

As in [**1**], we consider the two cases—whether or not $t(x^2) = t(x)^2$.

LEMMA 2. *If* $t(x^2)=t(x)^2$ *and* $Tx=at(x)+f$, *with* $|a|>1/\sqrt{2}$, *then* $|a|>0{\cdot}907$.

Proof. Let $Tx=at(x)+bw+f$, where $|b|$ is greater than all the coefficients in f. We consider two cases:

(1) If w commutes with $t(x)$, then the coefficient of $t(x)w$ in Tx^2 has modulus $2|ab|-\|f\|_2^2=(|a|+|b|)^2-1$. Since this is not the largest coefficient in Tx^2, $(|a|+|b|)^2-1<1/\sqrt{2}$ i.e.

$$|a|+|b|<\sqrt{(1+1/\sqrt{2})}<1{\cdot}307.$$

(2) If w does not commute with $t(x)$, then the coefficient of $t(x)w$ in Tx^2 has modulus $>|ab|-(|a|+|b|)(\sqrt{2}-|a|-|b|)$. The same is true for the coefficients of $wt(x)$. Since one of these is neither the first nor second largest coefficient in Tx^2, by Lemma 1(c), we have

$$|ab|-(|a|+|b|)(\sqrt{2}-|a|-|b|)<1/(3\sqrt{2}).$$

Now $|a|^2+|b|^2\leqslant 1$ and so

$$(|a|^2+|b|^2)/2+|ab|-(|a|+|b|)(\sqrt{2}-|a|-|b|)<1/2+1/(3\sqrt{2}).$$

Putting $|a|+|b|=A$, we have

$$A^2/2-A(\sqrt{2}-A)<1/2+1/(3\sqrt{2});$$

i.e.

$$3A^2/2-\sqrt{2}A<1/2+1/(3\sqrt{2}),$$

i.e.

$$A^2-(2\sqrt{2})A/3<1/3+\sqrt{2}/9,$$

or

$$(A-\sqrt{2}/3)^2<(5+\sqrt{2})/9.$$

Hence $A<\sqrt{2}/3+\sqrt{(5+\sqrt{2})}/3<1{\cdot}316$. Thus in both cases, we have $A<1{\cdot}316$, and using Lemma 1(b), $|a|>0{\cdot}907$, as required.

If we have this condition for all x in G_1, then t is a homomorphism.

THEOREM 3. *If* $t(x^2)=t(x)^2$, *for all* x *in* G_1, *then* t *is a homomorphism.*

Proof. Let $Tx=at(x)+f$ and $Ty=bt(y)+g$. By Lemma 2, $|a|>0{\cdot}907$ and $|b|>0{\cdot}907$. Hence the coefficient of $t(x)t(y)$ in Txy has modulus greater than

$$|ab|-\|f\|_2\,\|g\|_2=|ab|-\sqrt{(1-|a|^2)(1-|b|^2)}.$$

But this is greater than $(0{\cdot}907)^2-(1-(0{\cdot}907)^2)>0{\cdot}65$. Now the largest coefficient in Txy has modulus $>0{\cdot}907$ by Lemma 2. Since $(0{\cdot}64)^2+(0{\cdot}907)^2>1$, $t(xy)=t(x)t(y)$. Since this is true for all x and y, t is a homomorphism.

We now turn to the case when $t(x^2)\neq t(x)^2$.

THEOREM 4. *If* $Tx=at(x)+f$ *and* $t(x^2)\neq t(x)^2$, *then* $u=t(x^2)t(x)^{-2}$ *has order* 2, *commutes with* $t(x)$, *and we have* $Tx=at(x)+but(x)+g$ *with* $|a|+|b|>1{\cdot}29$.

Proof. Let $u=t(x^2)t(x)^{-2}$. Then $ut(x)^2$ has the largest coefficient in Tx^2. Let $Tx=at(x)+but(x)+g$. If u does not commute with $t(x)$, the coefficient of $ut(x)^2$ in Tx^2 has modulus at most

$$|ab|+(|a|+|b|)(\sqrt{2}-|a|-|b|).$$

Since $1/\sqrt{2} < |a| \leq 1$,

$$|ab| + (|a| + |b|)(\sqrt{2} - |a| - |b|) < (1/\sqrt{2})\,|b| + (1/\sqrt{2} + |b|)(1/\sqrt{2} - |b|)$$
$$= (1/\sqrt{2})\,|b| + 1/2 - |b|^2$$
$$= 5/8 - (|b| - 1/(2\sqrt{2}))^2$$
$$< 5/8.$$

This contradicts the fact that $ut(x)^2$ has the largest coefficient in Tx^2. The rest of the proof now follows as in the abelian case (Lemma 2.4 of **[1]**). However, there is a minor error in that part of the proof that shows u has order 2. This is rectified as follows. In estimating the coefficient of u^{-1}, the inequality should be

$$|a\bar{b}| \leq (|a| + |b|)\,\|f\| + \|f\|^2,$$

which implies that $|b| < 0{\cdot}195$, but this still gives a contradiction to $|b| > 0{\cdot}37$.

Corollary 5. *Under the hypothesis of Theorem* 4, *if* $Tx^2 = a_1ut(x)^2 + b_1t(x)^2 + f_1$, *then either* $|a| + |b| \geq 1{\cdot}36$ *or* $|a_1| + |b_1| \geq 1{\cdot}36$.

Proof. Since $|a| + |b| > 1{\cdot}29$, it is clear that $|b|$ is the second largest coefficient in Tx. Now if $|a| + |b| < 1{\cdot}36$, then, by Lemma 1(b), $|a| > 0{\cdot}87$ and $|b| < 0{\cdot}49$, and so $|a_1| \leq 2\,|ab| + \|f\|_2^2 = 2\,|ab| + (1 - |a|^2 - |b|^2) = 1 - (|a| - |b|)^2 < 1 - (0{\cdot}38)^2 < 0{\cdot}86$. Now

$$|b_1| \geq |a|^2 - |b|^2 - \|f\|_2^2 = 2\,|a|^2 - 1 > 0{\cdot}51,$$

so is certainly the second largest coefficient in Tx^2. Thus, by Lemma 1(b) again, $|a_1| + |b_1| \geq 1{\cdot}36$.

We now show that only one element u of order 2 can arise in this way.

Lemma 6. *The set* $[t(x^2)t(x)^{-2} : x \text{ in } G_1]$ *contains at most one non-trivial element.*

Proof. Suppose that $u = t(x^2)t(x)^{-2}$, $v = t(y^2)t(y)^{-2}$, where $u \neq v$, and both have order 2. Then, by Corollary 5, we may assume that $Tx = a_1t(x) + b_1t(x)u + f_1$, and $Ty = a_2t(y) + b_2t(y)v + f_2$ with $|a_1| + |b_1| \geq 1{\cdot}36$ and $|a_2| + |b_2| \geq 1{\cdot}36$. Now

$$Txy = a_1a_2t(x)t(y) + a_1b_2t(x)t(y)v + a_2b_1t(x)ut(y)$$
$$+ b_1b_2t(x)ut(y)v + (a_1t(x) + b_1t(x)u)^*f_2$$
$$+ f_1^*(a_2t(y) + b_2t(y)v) + f_1^*f_2.$$

Since $t(x)t(y)$, $t(x)t(y)v$, $t(x)ut(y)$ and $t(x)ut(y)v$ are all distinct, we have

$$\|Txy\| \geq |a_1a_2| + |a_1b_2| + |a_2b_2| + |b_1b_2| - (|a_1| + |b_1|)\,\|f_2\|$$
$$- (|a_2| + |b_2|)\,\|f_1\| - \|f_1\|\,\|f_2\|$$
$$\geq (1{\cdot}36)^2 - 2(1{\cdot}36)(\sqrt{2} - 1{\cdot}36) - (\sqrt{2} - 1{\cdot}36)^2$$
$$= (1{\cdot}36)^2 - (\sqrt{2} - 1{\cdot}36)(\sqrt{2} + 1{\cdot}36)$$
$$= 2(1{\cdot}36)^2 - 2$$
$$> 1{\cdot}69.$$

This contradicts $|T| < \sqrt{2}$ and so $u = v$.

We would now like to quotient out by the subgroup $[e, u]$, but to do this we must first prove that the subgroup is normal—i.e. that u commutes with all elements of G_2. We know already that u commutes with $t(x)$ whenever $t(x^2) \neq t(x)^2$. We will show that u commutes with $t(x)$ for all x, and then show that t is onto. We need a refinement of Lemma 6.

LEMMA 7. *Let* $[e, u] = [t(x^2)t(x)^{-2}: x \text{ in } G_1]$. *For* y *in* G_1, *let* $Ty = at(y) + bt(y)v + f$, *where* $|a| > 1/\sqrt{2}$ *and* b *is the second largest coefficient. If* $v \neq u$ *or if* $v = u$ *but does not commute with* $t(y)$, *then* $|a| + |b| < 1{\cdot}256$ *and* $|a| > 0{\cdot}933$.

Proof. Suppose $v \neq u$. As in Lemma 6, choose x in G_1 such that $Tx = a_1 t(x) + b_1 t(x)u + f_1$ with $|a_1| + |b_1| > 1{\cdot}36$. If $|a| + |b| = A$, the final inequality of Lemma 6 becomes

$$\begin{aligned}\|Txy\| &\geq A(1{\cdot}36) - A(\sqrt{2} - 1{\cdot}36) - 1{\cdot}36(\sqrt{2} - A)\\ &\quad - (\sqrt{2} - A)(\sqrt{2} - 1{\cdot}36)\\ &= 2A(1{\cdot}36) - 2\end{aligned}$$

Since $\|T\| < \sqrt{2}$, we have $A < (2 + \sqrt{2})/2{\cdot}72 < 1{\cdot}256$. By Lemma 1(b), $|a| > 0{\cdot}933$ as required. Now if $v = u$, but does not commute with $t(y)$, then the second largest coefficient in Ty cannot be both $t(y)u$ and $ut(y)$. Thus the same argument shows that in this case we also have $|a| + |b| < 1{\cdot}256$ and $|a| > 0{\cdot}933$.

THEOREM 8. $t(x)u = ut(x)$, *for all* x *in* G_1.

Proof. Suppose that there exists x in G_1 such that $t(x)u \neq ut(x)$ and let $Tx = at(x) + f$ with $|a| > 1/\sqrt{2}$. By Lemma 7, $|a| > 0{\cdot}933$. Choose y in G_1 such that $t(y^2) \neq t(y)^2$, and let $Ty = a_1 t(y) + f_1$ with $|a_1| > 1/\sqrt{2}$. We shall prove that $t(xy) = t(x)t(y)$. Let $Txy = a_2 t(xy) + f_2$ with $|a_2| > 1/\sqrt{2}$. Now the coefficient of $t(x)t(y)$ in Txy has modulus greater than

$$|aa_1| - \|f\|_2\, \|f_1\|_2 \geq (1/\sqrt{2})(|a_1| - \sqrt{(1 - |a_1|^2)}) > 0{\cdot}4.$$

Now if $t(xy) \neq t(x)t(y)$, then $|a_2| \leq \sqrt{(1 - (0{\cdot}4)^2)} < 0{\cdot}92$. By Lemma 1(c), the coefficient of $t(x)t(y)$ must be the second largest in Txy and so, by Lemma 7, $t(x)t(y) = t(xy)u = ut(xy)$. But this is a contradiction since u commutes with $t(y)$, but not with $t(x)$. Hence $t(xy) = t(x)t(y)$. Applying the same argument with xy in place of x, we obtain $t(xy^2) = t(xy)t(y)$, but with y^2 in place of y, we get $t(xy^2) = t(x)t(y^2)$. It follows that $t(y^2) = t(y)^2$, which is a contradiction. This completes the proof that $t(x)u = ut(x)$.

We now prove that t is onto. We know that t maps the identity e_1 of G_1 into the identity e_2 of G_2, but here is a stronger result.

LEMMA 9. *If* $x \neq e_1$ *and* $Tx = ce_2 + f$, *then* $|c| < 1/(2\sqrt{2} + 1) < 0{\cdot}262$.

Proof. If $Tx = ce_2 + f$, then $T(x - ce_1) = f$ and $T(x - ce_1)^n = f^n$. Now $\|f\| \leq K - |c|$, and so $\|f\|^n \leq (K - |c|)^n$. On the other hand, $\|(x - ce_1)^n\| \geq (1 + |c| + |c|^2)^{n/2}$. To see this, there exists a character ϕ on the group generated by x such that $|\phi(x) - c| \geq (1 + |c| + |c|^2)^{1/2}$. Thus

$$\|T(x - ce_1)^n\| / \|(x - ce_1)^n\| \leq (K - |c|)^n / (1 + |c| + |c|^2)^{n/2}.$$

https://doi.org/10.1017/S0017089500007990 Published online by Cambridge University Press

Since T has a continuous inverse, this cannot tend to zero. Therefore $(K-|c|)^2 \geq 1+|c|+|c|^2$ or $|c|\,(1+2K) \leq K^2-1$. Hence

$$|c| \leq (K^2-1)/(2K+1) < 1/(2\sqrt{2}+1) < 0{\cdot}262.$$

THEOREM 10. *t is one-to-one and onto.*

Proof. Suppose that t is not one-to-one. If x, y in G_1 satisfy $t(x)=t(y)=z$, then we have

$$Tx = a_1z + f_1 \qquad (|a_1| > 1/\sqrt{2}),$$
$$Ty = a_2z + f_2 \qquad (|a_2| > 1/\sqrt{2}).$$

We consider first the case when x and y commute. Then

$$T(a_2x - a_1y) = a_2f_1 - a_1f_2,$$
$$T(a_2x - a_1y)^n = (a_2f_1 - a_1f_2)^n.$$

Now

$$\|a_2f_1 - a_1f_2\| \leq |a_2|\,(K-|a_1|) + |a_1|\,(K-|a_2|)$$
$$\leq K\,(|a_1|+|a_2|) - 2\,|a_1|\,|a_2| < 1$$

since $|a_1| > 1/\sqrt{2}$, $|a_2| > 1/\sqrt{2}$ and $K < \sqrt{2}$. Also

$$\|(a_2x - a_1y)^n\| = \|(a_2e_1 - a_1x^{-1}y)^n\|$$
$$\geq |a_2|^n\,(1+|a_1/a_2|+|a_1/a_2|^2)^{n/2},$$

as in Lemma 9. Thus

$$|(a_2x-a_1y)^n| \geq (|a_1|^2+|a_1a_2|+|a_2|^2)^{n/2} \geq 1.$$

This again contradicts the boundedness of T^{-1}.

If x and y do not commute, we have that $Ty^{-1} = \bar{a}_2z^{-1} + f_2^*$. Thus $Txy^{-1} = (a_1z+f_1)^*(\bar{a}_2z^{-1}+f_2^*)$.

By Lemma 9, the coefficient of e_2 in Txy^{-1} has modulus less than $1/(2\sqrt{2}+1)$. We have $|a_1a_2| - \|f_1\|\,\|f_2\| \leq 1/(2\sqrt{2}+1)$. Therefore

$$|a_1a_2| - (\sqrt{2}-|a_1|)(\sqrt{2}-|a_2|) \leq 1/(2\sqrt{2}+1),$$
$$|a_1|+|a_2| \leq (4\sqrt{2}+3)/(4+\sqrt{2}) < 1{\cdot}6.$$

It follows from Lemma 1(b), that $|a_1| < 0{\cdot}9$ and $|a_2| < 0{\cdot}9$, and so, by Lemma 2, $t(x^2) \neq t(x)^2$ and $t(y^2) \neq t(y)^2$. By Theorem 4, we have $Tx = a_1z + b_1zu + g_1$, with $|a_1|+|b_1| > 1{\cdot}29$, and $Ty = a_2z + b_2zu + g_2$, with $|a_2|+|b_2| > 1{\cdot}29$. Now

$$Txy^{-1} = (a_1\bar{a}_2 + b_1\bar{b}_2)e + (a_1\bar{b}_2 + \bar{a}_2b_1)u + h,$$

where

$$\|h\| \leq |g_1|\,(|a_1|+|b_2|) + |g_2|\,(|a_1|+|b_2|) + \|g_1\|\,\|g_2\|$$
$$\leq K(|g_1|+|g_2|) - \|g_1\|\,\|g_2\|$$
$$< \sqrt{2}(\sqrt{2}-1{\cdot}29) - (\sqrt{2}-1{\cdot}29)(\sqrt{2}-1{\cdot}29)$$
$$= 1{\cdot}29(\sqrt{2}-1{\cdot}29) < 0{\cdot}17.$$

Thus $t(xy^{-1})=u$ and has coefficient with modulus greater than 0·933. This follows from Lemma 7 if the second coefficient is not that of e_2, and if it is, it is necessarily less than 0·262. Now, by Lemma 1(b), the biggest coefficient has modulus greater than 0·94. Similarly $t(y^{-1}x)=u$ and has coefficient with modulus greater than 0·933. Now $xy^{-1}\neq y^{-1}x$, since x and y do not commute. Repeating the argument with xy^{-1} and $y^{-1}x$ in place of x and y, we obtain a contradiction, since the sum of coefficients is less than 1·6. This completes the proof that t is one-to-one.

To show that t is onto, let $K=t(G_1)$ and P the linear projection of $l_1(G_1)$ onto $l_1(K)$. Define $Sx=a(x)t(x)$ where $Tx=a(x)t(x)+f$ and $|a(x)|>1/\sqrt{2}$. Extend S linearly to a map from $l_1(G_1)$ to $l_1(K)$. Then $\|T-S\|/1/\sqrt{2}$, so that S is invertible with $\|S^{-1}\|\leqslant\sqrt{2}$. $PS=S$ and so

$$\|PTS^{-1}-I\|=\|P(T-S)S^{-1}\|<(1/\sqrt{2})\sqrt{2}=1.$$

Thus PTS^{-1} is invertible. In particular P is invertible and $K=G_2$.

REMARK. It seems likely that $|(x-cy)^n|\geqslant 1$ for all n, even when x and y do not commute. If this were true, the above proof would be considerably shortened.

We have proved the main theorem.

THEOREM 11. *Let G_1 and G_2 be groups and T a $*$-isomorphism of $l_1(G_1)$ onto $l_1(G_2)$ with $|T|<\sqrt{2}$. Then either G_1 and G_2 are isomorphic, or there exist elements v in G_1 and u in G_2 both of order 2 and a map $t:G_1$ to G_2, such that*

(i) *t is a bijection preserving inverses,*

(ii) *$t(v)=u$, and $t:G_1$ onto $G_2/[e,u]$ is a homomorphism.*

Using the techniques for abelian groups contained in [**1**], we can obtain the following result.

THEOREM 12. *Under the hypothesis of Theorem 11, if u does not belong to the commutator subgroup of G_2, then G_1 and G_2 are isomorphic.*

Proof. If I_2 is the identity character on G_2, then $I_2\circ T$ is a character on G_1. By multiplying T by the inverse of this character, we may assume that $I_2\circ T=I_1$.

If u does not belong to the commutator subgroup of G_2, there exists a character ψ with $\psi(u)=-1$. Then the composition $\psi\circ T$ is a character on G_1, and since $t(xy)=t(x)t(y)$ or $t(x)t(y)u$, we have $\psi(t(xy))=\pm\psi(t(x))\psi(t(y))$. Thus $(\psi\circ t)^2$ is also a character on G_1. Define $\varphi=(\psi\circ t)^{-1}(\psi\circ T)$. Then φ^2 is a character on G_1. We show that φ^2 has odd order.

If φ^2 does not have odd order, there exists x in G_1 such that $\varphi^2(x)$ is arbitrarily close to -1. Thus, given $\varepsilon>0$, there exists x in G_1 such that $|\varphi(x)+i|<\varepsilon$. If

$$Tx=at(x)+bt(x)u+\sum c_iy_i,$$

then

$$\varphi(x)=a-b+\sum c_i\varphi(t(x))^{-1}\varphi(y_i).$$

Thus we have $a+b+\sum c_i=1$ (since $I_2\circ T=I_1$), and

$$|a-b+\sum c_i\varphi(t(x)^{-1}y_i)+i|<\varepsilon.$$

Substituting for a, we obtain

$$|1+i-2b+\sum c_i(\varphi(t(x)^{-1}y_i)-1)|<\varepsilon.$$

In particular,

$$|b|>|1+i|/2-\sum |c_i|-\varepsilon/2.$$

Since $a>1/\sqrt{2}$, $|a|+|b|+\sum |c_i|>\sqrt{2}-\varepsilon/2$, which is a contradiction.

Thus φ^2 has odd order, n say. Let $\theta=\varphi^{n+1}$, another character on G_1, with $\theta(x)=\pm\varphi(x)$. Define $s:G_1$ to G_2 by

$$\begin{aligned} s(x)&=t(x) \quad \text{if} \quad \theta(x)=\varphi(x),\\ s(x)&=t(x)u \quad \text{if} \quad \theta(x)=-\varphi(x). \end{aligned}$$

Then s is a homomorphism since φ is, and since t is injective and onto, s is also.

This gives us the main theorem.

THEOREM 13. *If T is a $*$-isomorphism of $l_1(G_1)$ onto $l_1(G_2)$ with $\|T\|<\sqrt{2}$, and if G_1 (or G_2) does not contain a central element of order 2 in the commutator subgroup, then G_1 and G_2 are isomorphic.*

If G_1 has a central element of order 2 in the commutator subgroup, then the map t in Theorem 11 has the following two additional properties.

PROPOSITION 14. *t maps the centre of G_1 into the centre of G_2.*

Proof. If x is in the centre of G_1, it is in the centre of $l_1(G_1)$, and hence Tx is in the centre of $l_1(G_2)$. Therefore if $Tx=at(x)+f$, with $|a|>1/\sqrt{2}$, then for each y in G_2,

$$Tx=y^{-1}(Tx)y=ay^{-1}t(x)y+y^{-1}fy.$$

No coefficient in $y^{-1}fy$ can have modulus greater than $1/\sqrt{2}$, and so $y^{-1}t(x)y=t(x)$ and $t(x)$ belongs to the centre of G_2.

In fact, it can be shown, using similar techniques to those in [**1**, Theorem 3.4], that on the centre Z_1 of G_1 either T has the form $Tx=\psi(x)t(x)$, with t an isomorphism and ψ a character on Z_1, or the form

$$Tx=\psi(x)[((1+\theta(x))/2)s(x)+((1-\theta(x))/2)s(x)u],$$

where ψ, θ are characters on Z_1 with θ of odd order, and s is an isomorphism.

PROPOSITION 15. *t maps the commutator subgroup of G_1 into the commutator subgroup of G_2.*

Proof. $t(xyx^{-1}y^{-1})$ is either $t(x)t(y)t(x)^{-1}t(y)^{-1}$ or $t(x)t(y)t(x)^{-1}t(y)^{-1}u$, both of which are in the commutator subgroup of G_2.

Under these circumstances, we also have the following result.

PROPOSITION 16. *G_1 and G_2 have the same number of elements of each order.*

Proof. Let $\tilde{x}$ be the image of x under the quotient map $G_1\to G_1/[e,v]$. If $\tilde{x}$ has odd order n, then one of x and xv has order n, the other $2n$. $\widetilde{t(x)}$ also has order n, and so, of the elements $t(x)$ and $t(x)u$, one will have order n, the other $2n$.

We next consider elements of order 2. We need to show that

$$x^2 = e_1 \Leftrightarrow t(x)^2 = e_2 \qquad (*)$$

If the implication in either direction is false, then $t(x^2) \neq t(x)^2$. But, if $Tx = at(x) + f$, the coefficient of $t(x)^2$ in Tx^2 has modulus $>|a| - \|f\|^2 > 0$, which gives a contradiction.

Now suppose that x and $t(x)$ have order $2n$. For the result to be false, one of two things must happen.

(a) x and xv have order $2n$ (i.e. $x^{2n} \neq v$) and $t(x)$ and $t(x)u$ have order $4n$ (i.e. $t(x)^{2n} = u$). But then $(x^n)^2 = e_1$, yet $(t(x^n))^2 = u$, which contradicts (*),

(b) x and xv have order $4n$ ($x^{2n} = v$) and $t(x)$ and $t(x)u$ have order $2^n(t(x)^{2n} \neq u)$. Then $(x^n)^2 = v$, but $(t(x)^n)^2 = e_2$. This also contradicts (*).

Whether these conditions in themselves mean that the groups are isomorphic is not clear. Using the book **[4]** it is possible, though very tedious, to confirm that no counterexample exists with groups of order up to 32.

The corresponding results for locally compact groups follows easily from the discrete case. (See **[1]**, **[3]**, and **[8]** for the details.)

THEOREM 17. *Let T be a $*$-isomorphism of $L^1(G_1)$ onto $L^1(G_2)$, $[M(G_1)$ onto $M(G_2)]$ satisfying $\|T\| < \sqrt{2}$. If G_1 (or G_2) does not contain a central element of order 2 in the commutator subgroup, then G_1 and G_2 are isomorphic.*

THEOREM 18. *Let G_1 and G_2 be compact groups without central elements of order 2 in the commutator subgroup. If T is a $*$-isomorphism of $C(G_1)$ onto $C(G_2)$, $[L^\infty(G_1)$ onto $L^\infty(G_2)]$, satisfying $\|T\| < \sqrt{2}$, then G_1 and G_2 are isomorphic.*

REFERENCES

1. N. J. Kalton and G. V. Wood, Homomorphisms of group algebras with norm less than $\sqrt{2}$, *Pacific J. Math.* **62** (1976), 439–460.

2. W. Rudin, *Fourier analysis on groups* (Interscience, 1960).

3. R. S. Strichartz, Isomorphism of group algebras, *Proc. Amer. Math. Soc.* **17** (1966), 858–862.

4. A. D. Thomas and G. V. Wood, *Group Tables* (Shiva, 1981).

5. J. G. Wendel, Left centralizers and isomorphisms of group algebras, *Pacific J. Math.* **2** (1952), 251–261.

6. G. V. Wood, Distance between group algebras, *Proc. Roy. Irish Acad. Sect. A* **76** (1976), 339–342.

7. G. V. Wood, Measures with bounded powers on locally compact groups, *Trans. Amer. Math. Soc.* **268** (1981), 187–209.

8. G. V. Wood, Isomorphisms of group algebras, *Bull. London Math. Soc.* **15** (1983), 247–252.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY COLLEGE OF SWANSEA
SWANSEA SA2 8PP
WALES