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A TOPOLOGICAL CHARACTERIZATION OF CONJUGATE NETS

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1. Introduction. One aspect of topological analysis that authors, such as G. T. Whyburn and Marston Morse, have pointed to ([**16**; **6**] for instance) as being fundamental in the development of function theory is the topological study of the level sets of analytic and harmonic functions or of their topological analogues, light open maps and pseudo-harmonic functions. The first step in this direction seems to have been made by H. Whitney [**14**] when he studied families of curves, given abstractly using a condition of regularity. In the plane, Kaplan [**9**] showed that Whitney's condition is equivalent to the condition that the family is locally homeomorphic to parallel lines and that this condition topologically characterizes the level sets of harmonic functions without critical points. Boothby [**2**; **3**] a few years later included critical points. He showed that a family of branched curves filling the plane and locally structured like the level curves of Re z^n characterized topologically the level curves of harmonic functions. M. Morse and J. Jenkins [**7**] shortly after generalized Boothby's work by omitting a non-recurrence hypothesis and considered the problem on an open Riemann surface. They subsequently considered [**6**; **7**] the question of the existence of a second family of branched curves such that locally both families were topologically like the level lines of Re z^n and Im z^n. When a second such family did exist they called the resulting pair of families a *conjugate net*. On simply connected surfaces, they showed that for a given branched curve family there always existed a second such family and that the concept of conjugate net topologically characterizes the level sets of the real and imaginary parts of analytic functions. In a different vein, G. T. Whyburn gave point set characterizations of the type of continuity required for decompositions into compact [**15**] as well as non-compact [**16**] elements caused by analytic functions.

The purpose of this paper is to formulate as a point set topological concept the idea of a conjugate net. In [**13**] the definition of conjugate net was extended by considering families of locally connected generalized continua (locally compact connected sets of more than one point) locally structured by z^n (see definition, § 4). Here necessary and sufficient conditions for two families of

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generalized continua to form a conjugate net are found (Theorems 4.1 and 4.2).

This concept of conjugate net includes the level sets of the real and imaginary parts of light open maps and as such in [**13**] was used to generalize a theorem of Stoïlow. Here theorem 4.1 approaches another theorem of Stoïlow which states that a light open map from one 2-manifold to another is locally topologically equivalent to z^n. The absence of semi-closed sub-collections (see § 2) and bounding unions parallels openess and the discrete intersection condition parallels lightness.

The topological properties required for one family of generalized continua to behave like the level lines of an harmonic function must be stricter. This problem will be taken up in a subsequent paper.

Notation. Capital letters will be used to denote spaces as well as their subsets and small letters will denote points of the space. Script letters, such as $\mathscr{A}$, $\mathscr{B}$, $\mathscr{G}$, $\mathscr{H}$ will denote families of subsets of a space. If $\mathscr{G}$ is a family of subsets of the space X and S is a subset of X then $\mathscr{G}_S$ will denote the family $\{$components of $G \cap S | G \in \mathscr{G}\}$. If the family $\mathscr{G}$ consists of mutually disjoint subsets of X and $p \in \bigcup \mathscr{G}$ then the element of $\mathscr{G}$ containing p will be denoted by G_p.

An *arc* (*open arc*) is the image under a continuous $1 - 1$ function of a closed (open) interval. Arcs will be denoted by lower case Greek letters, such as α, β, γ. An arc joining two points a and b will also be denoted by ab, or by axb to stress an intermediate point x on ab.

In a metric space X, $N(p, r)$ will be used to denote the open ball about the point p of radius r and $V(S, r)$ will denote the open neighborhood about the subset S of X of radius r. In a 2-manifold M, neighborhoods homeomorphic to the open unit disk in the plane will be referred to as *disks in* M. By a *closed disk* or a *closed Jordan domain*, it will be meant the homeomorph of the closed unit disk in the plane.

If S is a subset of a space X, Int S, Ext S, Cl S, βS, $\mathscr{C} S$ will denote respectively the interior, exterior, closure, boundary and complement of S. Finally, $\mathbf{Z}$ and $\mathbf{R}$ will denote the integers and the real numbers respectively.

2. Admissible families. This section establishes minimal requirements for the continuity of the families of generalized continua.

Since conjugate nets need not be generated globally by one function and since at best the elements are the components of the fibers of the real or imaginary parts of a complex valued function, we may be dealing with families of generalized continua which are neither lower semicontinuous (LSC) nor upper semicontinuous (USC) (see [**15**] or [**16**]) either in the open set sense or limit sense (for example the components of the fibres of $f(x, y) = x$ defined on $\mathbf{R}^2 - 0$). However, locally they are *USC* in the weaker limit sense but still need not be continuous. This suggests the following definition: a collection $\mathscr{G}$ of sets in a space X is *locally USC* if each point x of X has a neighborhood N in which $\mathscr{G}_N$ is *USC* in the limit sense. In locally compact spaces, it is easily seen that compact neighborhoods can be chosen in which $\mathscr{G}_N$ is *USC* in both the limit and open set sense [**5**, 3–35].

Although locally it is not required that limits occupy a complete element, we want to avoid sequences collapsing to points, that is the following behavior: a collection of disjoint sets $\mathscr{G}$ in a space X is *semi-closed* if each set of $\mathscr{G}$ is closed and any convergent sequence of sets of $\mathscr{G}$ whose limit intersects $X - \bigcup\mathscr{G}$ converges to a single point of $X - \bigcup\mathscr{G}$ [**15**, VII, 5].

Thus on a 2-manifold (locally euclidean, connected, Hausdorff space) M a family $\mathscr{G}$ of disjoint closed generalized continua which fills M will be termed *admissible* if it is locally *USC* and contains no semi-closed subcollection. This second condition gives the following useful fact.

LEMMA 2.1. *Let $\mathscr{G}$ be an admissible family on a 2-manifold M. Then for each point $p \in M$ there is a disk D about p such that $G \cap \mathscr{C}D \neq \emptyset$ for each $G \in \mathscr{G}$.*

3. Planar nets. Because conjugate nets are essentially characterized locally, this section considers the problem in the setting of the plane.

A component of a level curve of the real part of an analytic function f in the plane will meet one of the imaginary part in at most one point; otherwise, the union of two such components will bound a relatively compact domain D such that $f(\beta D)$ does not separate the plane and so $\beta(f(D)) \not\subset f(\beta D)$, contrary to the behavior for analytic maps. This behavior is reflected in the following definition.

Definition. Two admissible families of closed generalized continua $\mathscr{A}$ and $\mathscr{B}$ in the plane form a *planar net*, denoted by $[\mathscr{A}, \mathscr{B}]$, if each point p is the intersection of an element of $\mathscr{A}$ and an element of $\mathscr{B}$. The elements of $\mathscr{A}$ and $\mathscr{B}$ as well as of $\mathscr{A}_S$ and $\mathscr{B}_S$ for any subset S of the plane will be called *fibers*.

That the intersections in this definition must be no larger than single points is forced on us by the work of R. D. Anderson [**1**] who showed the existence of a continuous decomposition of the plane by pseudo-arcs such that the decomposition space is a plane and the induced map is open. The above condition also assures us that each family fills the plane. The main theorem of this section is the following "spoke theorem" for planar nets.

THEOREM 3.1. *Let $[\mathscr{A}, \mathscr{B}]$ be a planar net and let p be a point in the plane. There exists an integer $n \geqq 2$, a neighborhood N of p and an homeomorphism h of N onto the unit disk E in the z-plane about* 0, *such that* $h(0) = 0$ *and h carries $\mathscr{A}_N$ and $\mathscr{B}_N$ onto the level curves of* $\operatorname{Re} z^{n/2}$ *and* $\operatorname{Im} z^{n/2}$ *respectively.*

The first step towards the proof of this theorem is showing that these families have the following properties in common with the level sets of open real valued maps in the plane: thinness, local connectedness and absence of end points. That the fibers are of dimension one is an immediate result of the definition. The other two properties are less trivial to prove. However before proving these properties for planar nets, I will show them to be sufficient in producing another well known behavior of the above level sets: the absence of simple closed curves.

LEMMA 3.1. *A locally connected continuum C which contains at most one end point must contain a simple closed curve.*

Proof. The Non-Cut Point Existence Theorem [**15**, III, 6.1] says that C has at least two non-cut points. Thus there is a non-cut point p in C which is not an end point. But any such point of a continuum must be on a simple closed curve of C [**10**, II, 30].

THEOREM 3.2. *Let $\mathscr{A}$ be a family of closed disjoint locally connected generalized continua which fills the plane. If no member of $\mathscr{A}$ contains an open set or an end point, then no member of $\mathscr{A}$ contains a simple closed curve.*

Proof. Assume that the fiber A of $\mathscr{A}$ contains a simple closed curve. Then there exists a bounded domain D in $\mathscr{C}A$. Each fiber of the subfamily $\mathscr{A}' = \{A_p \in \mathscr{A} | p \in D\}$ is a locally connected continuum without end points and so by Lemma 3.1, contains a simple closed curve. Thus each fiber of $\mathscr{A}'$ has a bounded complementary domain. Define on $\mathscr{A}'$ the following partial ordering: for $A_1, A_2 \in \mathscr{A}, A_1 \leqq A_2$ if $A_1 = A_2$ or there is a bounded domain D_1 in $\mathscr{C}A_1$ such that $A_2 \subset D_1$.

It is now easy to show that chains have upper bounds but that there are no maximal elements, thus contradicting Zorn's lemma.

Although admissible families of thin sets are not necessarily free of simple closed curves globally, they are locally.

LEMMA 3.2. *Let $\mathscr{A}$ be an admissible family in the plane (or a 2-manifold) whose sets contain no open set. For any point p let D be a disk given by lemma 2.1. Then no element of $\mathscr{A}_D$ contains a simple closed curve.*

Proof. If there is an element of $\mathscr{A}_D$ containing a simple closed curve then there is a fiber A of $\mathscr{A}$ which has a complementary domain D' such that $D' \subset D$ and $\beta D' \subset A$. But then for any point q of D', the fiber A_q of $\mathscr{A}$ meets $\mathscr{C}D$ Ext D' and so must meet A, contrary to disjointness of $\mathscr{A}$.

The fibers are locally connected. Because a locally connected metrizable generalized continuum is arcwise connected [**15**, II, 5.2], considerable structural strength will be added to a planar net if its fibers are shown to be locally connected.

THEOREM 3.3. *Each fiber of a planar net $[\mathscr{A}, \mathscr{B}]$ is locally connected.*

Proof. An outline of the proof is given here. Assume there is a fiber A of $\mathscr{A}$ which is not locally connected at the point p of A. There is a closed disk D about p such that (i) $\mathscr{A}_D$ and $\mathscr{B}_D$ are *USC*, (ii) Lemma 2.1 holds, and (iii) there exists a sequence of distinct components C_i of $A \cap D$ converging to a continuum C of $A \cap D$ containing p and such that $C \cap (\bigcup_{i=1}^{\infty} C_i) = \emptyset$ (see [**15**, I, 12.1] or [**5**, 3, 3–12]).

Since $C_i \cap \beta D \neq \emptyset$ for all i, there is a cluster point a on βD of $\bigcup_{i=1}^{\infty} (C_i \cap \beta D)$ which is necessarily a point of C. Then there is a sequence $\{a_n\}$ of points of $\bigcup_{i=1}^{\infty} (C_i \cap \beta D)$ such that $\{a_n\}$ is strictly monotone converging to a and if $a_n \in C_{i_n}$ there is no point of C_{i_n} between a and a_n. Let L be the right bisector of the segment $[a, p]$ and let ϵ be a number such that $0 < \epsilon < \operatorname{diam}(D)/4$ (see Figure 1). There is an $n(\epsilon)$ large enough so that $N(a, \epsilon) \cap C_{i_{n(\epsilon)}} \neq \emptyset$. Since L

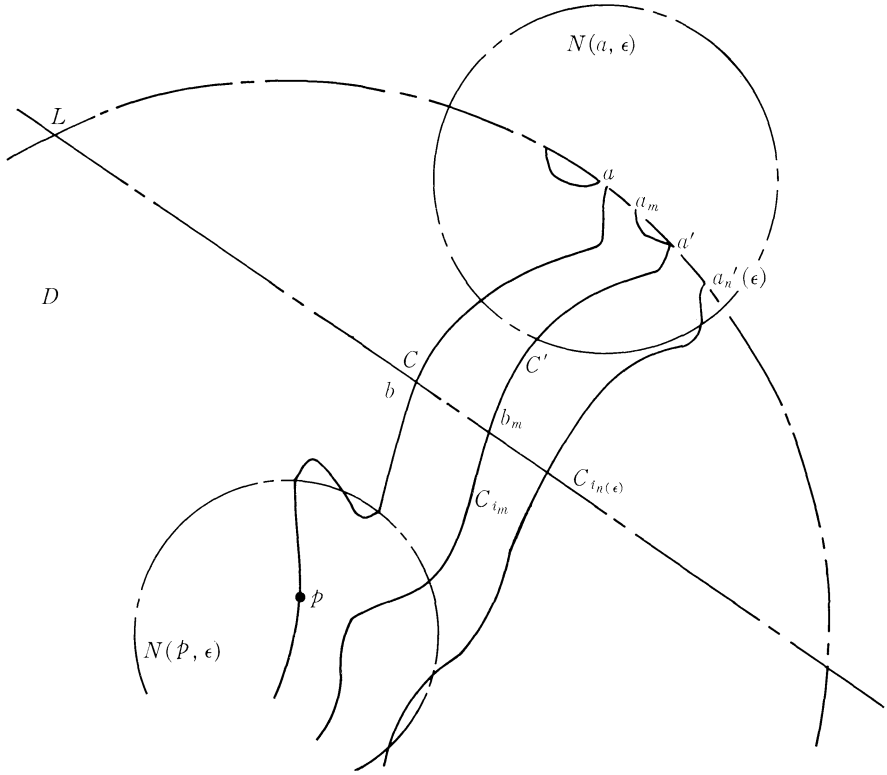

Figure 1

separates a from p then $C_{i_{n(\epsilon)}} \cap L \neq \emptyset$. Since the sets

$$X(\epsilon) = (\mathrm{Cl}\, N(a, \epsilon) \cap \beta D) \cup C_{i_{n(\epsilon)}} \cup \mathrm{Cl}\, N(p, \epsilon),$$
$$Y(\epsilon) = (\mathrm{Cl}\, N(a, \epsilon) \cap \beta D) \cup C \cup \mathrm{Cl}\, N(p, \epsilon)$$

are two continua such that $X(\epsilon) \cap Y(\epsilon) = (\mathrm{Cl}\, N(a, \epsilon) \cap \beta D) \cup \mathrm{Cl}\, N(p, \epsilon)$ is not connected, then [**10**, IV, 20] $X(\epsilon) \cup Y(\epsilon)$ separates the plane.

For each $m > n(\epsilon)$, $C_{i_m} \cap L \neq \emptyset$ by converging, since $C \cap L \neq \emptyset$. By the ordering of $\{a_n\}$ a point $b_m \in C_{i_m} \cap L$ can be found in a bounded complementary domain of $X(\epsilon) \cup Y(\epsilon)$, so that $\{b_m\}$ converges to a point $b \in C \cap L$.

The fibers B_{b_m} of $\mathscr{B}_D$ must meet βD (Lemma 2.1) and must do so by running through $N(p, \epsilon)$ or $N(a, \epsilon)$ infinitely often. Thus p or $a \in \lim\sup B_{b_m}$. But $b \in \lim\inf B_{b_m} \cap B_b$ so that $\lim\sup B_{b_m} \subset B_b$ by *USC*. Hence $\{p, b\}$ or $\{a, b\} \subset B_b \cap A$ contrary to the definition of planar net.

Remark 3.1. If C is a locally connected generalized continuum in the plane and D a closed disk then, using [**15**, I, 12.1, 12.3], it is not hard to see that the components of $C \cap D$ are also locally connected. Thus for a planar net $[\mathscr{A}, \mathscr{B}]$ the fibers of $[\mathscr{A}_D, \mathscr{B}_D]$ are also locally connected.

The fibers have no end points. A point p of a set K is an *end point* of K if there exists arbitrarily small neighborhoods of p with boundaries consisting of a single point. When K is a closed locally connected generalized continuum in the plane, which contains no simple closed curve, these neighborhoods may be given by Jordan domains D containing p such that $\beta D \cap K$ is a singleton and $D - K$ is connected [**4**, I, 2.1].

The non existence of end points depends on a phenomenon for planar nets which is analogous to the following behavior of the level curves of a light open map f of the plane to itself: if A is a level curve of $\operatorname{Re} f$ containing an arc α and if B is a level curve of $\operatorname{Im} f$ meeting A in a point p of α, then there is an arc β of B which crosses α ($\alpha \cap \beta = \{p\}$ and for some simple closed curve J containing an open arc of β containing p, no one of the complementary domains of J contains an open subarc from each component of $\alpha - p$, each incident on p; see [**10**]).

An analogue of this situation for planar nets depends on a theorem of Rutt-Roberts in the following form (see [**10**, IV, 112; **11**; or **12**]).

THEOREM (Rutt-Roberts). *Let D be a closed Jordan domain in the plane with points $a, b \in \beta D$. Let $\mathscr{H}$ be a collection of disjoint continua in D each meeting βD such that $R = \bigcup \mathscr{H}$ is a compact set not containing a or b and no element H of $\mathscr{H}$ separates a from b in D. Then there is an arc ab in $D - R$ spanning βD (i.e. $(ab - \{a, b\}) \subset \operatorname{Int} D$).*

THEOREM 3.4. *Let D be a closed Jordan domain chosen within a disk satisfying Lemma* 2.1 *and in which $\mathscr{A}_D$ and $\mathscr{B}_D$ are USC. Suppose that β is an arc in βD contained in some set B of $\mathscr{B}$. Then, if the fiber A of $\mathscr{A}$ separates β, it separates D by an arc aa' in $A \cap D$ spanning βD where $a = A \cap B$. A similar statement with $\mathscr{A}$ and $\mathscr{B}$ interchanged holds.*

Proof. Since $\beta \subset B$, we may suppose $A \cap \beta = a$ and a separates β. Assume A does not separate D. The component A_0 of $A \cap D$ containing a (possibly only $\{a\}$) does not meet $\beta D - \beta$ [**15**, VI, 3.5]. Let x and y be the end points of β on βD and let $b \in \beta D - \beta$. Choose $\epsilon > 0$ such that the neighborhood $V(A_0, \epsilon)$ of A_0 in D excludes the arc xby of βD (see Figure 2). By *USC* of $\mathscr{A}_D$ let $U \subset V(A_0, \epsilon)$ be an open set in D containing A_0 and such that each fiber A' of $\mathscr{A}_D$ meeting U is contained in $V(A_0, \epsilon)$. Choose $\delta > 0$ such that $\operatorname{Cl} V(A_0, \delta) \subset U \subset V(A_0, \epsilon)$. Let $W = \mathscr{C}(\operatorname{Cl} V(A_0, \delta)) \cup V(A_0, \delta/4)$. Define $\mathscr{G} =$

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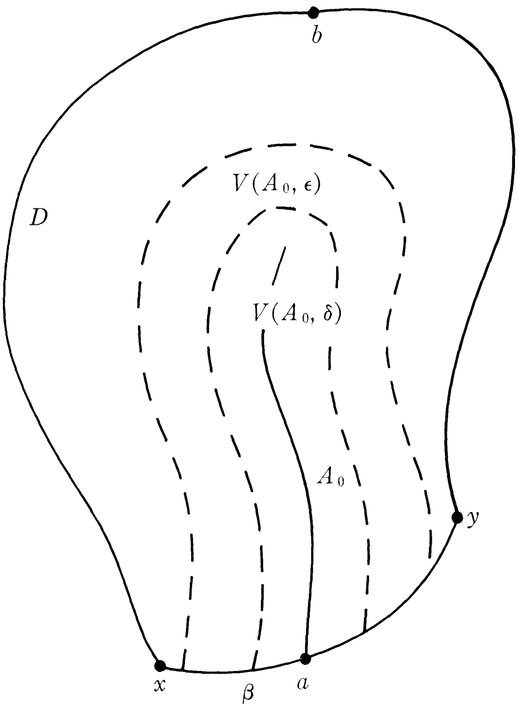

Figure 2

$\{A \in \mathscr{A}_D | A \subset W\}$. Then $\bigcup \mathscr{G}$ is open in D [**5**, 3–32]. Letting $\mathscr{H} = \mathscr{A}_D - \mathscr{G}$, then $\bigcup \mathscr{H}$ is closed.

No A' of $\mathscr{H}$ meets arc xby since each A' must meet Cl $V(A_0, \delta) \subset U$ and so by USC is in $V(A_0, \epsilon)$ which excluded arc xby. On the other hand A' must meet βD since the fiber of $\mathscr{A}$ containing it meets $\mathscr{C}D$. Therefore A' meets β but only in one point since $\beta \subset B$. Thus no A' of $\mathscr{H}$ separates a from b in D and so a, b, D and $\mathscr{H}$ satisfy the Rutt-Roberts theorem, which gives an arc α joining a and b in $D - \bigcup \mathscr{H}$. Since $a \in V(A_0, \delta/2)$ and $b \in \mathscr{C}V(A_0, \delta/2)$ then α meets $\beta V(A_0, \delta/2)$. Let $s \in \alpha \cap \beta V(A_0, \delta/2)$ and A_s be the fiber of $\mathscr{A}_D$ containing s. Thus A_s not being completely in $\mathscr{C}$ Cl $V(A_0, \delta)$ or $V(A_0, \delta/4)$, is an element of $\mathscr{H}$. But then $s \in \bigcup \mathscr{H}$, contradicting the fact that $s \in \alpha \subset D - \bigcup \mathscr{H}$. Thus A_0 must separate D by meeting $\beta D - \beta$ and the theorem follows.

THEOREM 3.5. *No fiber of* $[\mathscr{A}, \mathscr{B}]$ *has an end point.*

Proof. Assume p is an end point of some fiber A of $\mathscr{A}$. Let D' be a disk about p such that Lemma 2.1 holds and $\mathscr{A}_{D'}$ and $\mathscr{B}_{D'}$ are USC and let A_p be the fiber of $\mathscr{A}_{D'}$ containing p. Since A_p is a closed (in D') locally connected generalized continuum containing no simple closed curve (Lemma 3.2), but having an end point p, there is a Jordan domain $D \subset D'$ containing p such that $\beta D \cap A_p = q$

and $D - A_p$ is connected. Let qp be the arc joining q to p in $A_p \cap \mathrm{Cl}\, D$ and B_p the fiber of $\mathscr{B}$ containing p. Since $B_p \cap \mathscr{C}D' \neq \emptyset$ then B_p meets βD. Let the arc pr in B_p have only r in βD (see Figure 3). Thus the arc $qp \cup pr$ spans βD and so [**15**, VI, 3–5] separates D into two domains D_1 and D_2.

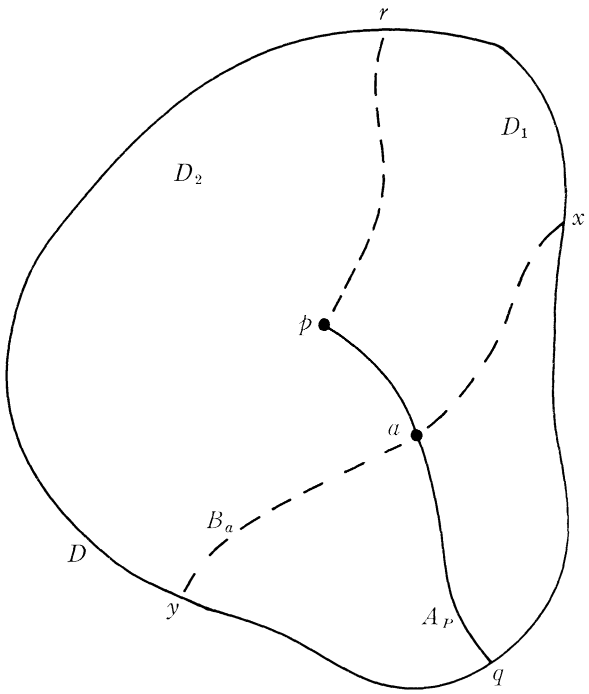

Figure 3

Now consider $a \in qp - \{p, q\}$ and the fiber B_a of $\mathscr{B}$. By Theorem 3.4 applied to D_1, $qp \subset \beta D_1$ and B_a, B_a contains an arc xa spanning βD_1; in fact $x \in \beta D$ since $B_a \cap (qp \cup pr) = a$. Apply Theorem 3.4 once again to D_2, $qp \subset \beta D_2$ and B_a to obtain an arc ay in B_a joining a to a point $y \in \beta D - \beta D_1 \subset \beta D_2$. The arc xay spans βD, thereby separating it into two domains D_3 and D_4. Assume $p \in D_3$. Since xay crosses qp (use βD_1 as the simple closed curve) then qp crosses xay [**10**, IV, 32] so that $q \in \beta D_4 - xay$. Now apply Theorem 3.4 once more to D_3, $xay \subset \beta D_3$ and A_p, to obtain an arc az in A_p joining a to $z \in \beta D_3 - xay = \beta D - xqy$. But then $qa \cup az$ is an arc in A_p spanning βD, separating D and so giving a contradiction to the choice of D.

Remark 3.2. It now follows from Theorem 3.2 that the fibers of a planar net contain no simple closed curves and furthermore that Lemma 2.1 is valid for any relatively compact set.

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Singular points of a planar net. The *order of a point p in a set X* is said to be less than or equal to the integer $n > 0$, if for any neighborhood V of p there exists a neighborhood U of p such that $U \subset V$ and βU contains $\leqq n$ points. The order of p in X is equal to n if the order is $\leqq n$ but not $\leqq n - 1$. If no such n exists the order will be said to be infinite. If p is a point in the fiber A of $\mathscr{A}$, the order of p in A will be denoted by $O_A(p)$. Similarly $O_B(p)$ will denote the order of p in the fiber B of $\mathscr{B}$. It is now shown that for any point p, $2 \leqq O_{A_p}(p) = O_{B_p}(p) < \infty$ and that in fact they are equal to 2 except for an isolated set. Because no fiber has an end point the lower bound of 2 is immediate.

LEMMA 3.3. *Let D be a disk in the plane and C a dendrite (a locally connected continuum with no simple closed curves) in* Cl D *with its end points in βD. Let C' be a subcontinuum or a point such that $C' \subset C - \beta D$. Then there are only a finite number of components in $C - C'$.*

This lemma follows easily from local connectedness.

THEOREM 3.6. *Let A be a fiber of $\mathscr{A}$. Then $O_A(p) = 2$ for all but a countable number of points of A. Furthermore, we have $2 \leqq O_A(p) < \infty$ for all $p \in A$.*

Proof. Every point of A is a cut point for if p is a non cut point which is not an end point (Theorem 3.5), then p must lie on a simple closed curve [**10**, II, 30] contrary to Remark 3.2. Thus the cut-point order theorem [**15**, III, 3.2] proves the first statement. For the second statement, let $p \in A$ and D be a disk about p. Apply Lemma 3.3 to the component C of $A \cap$ Cl D containing p and with $C' = \{p\}$. Thus the number of components of $C - p$ is finite and equal to $O_A(p)$ by [**15**, V, 1.3(2)].

THEOREM 3.7. *Each fiber of a planar net $[\mathscr{A}, \mathscr{B}]$ is locally a finite tree.*

For a fiber $A \in \mathscr{A}$, it can be shown that the countable set $S = \{p \in A | O_A(p) > 2\}$ does not cluster by standard arguments using local connectivity, Lemma 3.3 and Theorem 3.6.

THEOREM 3.8. *Let A be a fiber of $\mathscr{A}$ and B a fiber of $\mathscr{B}$ such that $p = A \cap B$. Then $O_A(p) = O_B(p)$.*

Proof. Let D be a closed disk about p which excludes any point in A and B of order > 2 (except possibly p) and where $\mathscr{A}_D$ and $\mathscr{B}_D$ are *USC*. Let $O_B(p) = n$. Let C be the component of $A \cap D$ containing p and C' be that of $B \cap D$ containing p. Then C' consists of n arcs with n complementary domains whose boundaries contain an arc of B containing p. Apply Theorem 3.4 to each domain to obtain n arcs in C joining p to βD. Therefore $C - p$ has at least n components so that $O_B(p) \leqq O_A(p)$. The reverse inequality is obtained by symmetric arguments on A and B.

Definition. Let $[\mathscr{A}, \mathscr{B}]$ be a planar net. For the point p in the plane let $A \in \mathscr{A}$ and $B \in \mathscr{B}$ be the fibers such that $p = A \cap B$. *The order of p in* $[\mathscr{A}, \mathscr{B}]$ is

$O_A(p) = O_B(p)$ and will be denoted by $O(p)$. The *singular set* of $[\mathscr{A}, \mathscr{B}]$ is the set $S = \{p|O(p) > 2\}$ and a point p of S is called a *singular point* of $[\mathscr{A}, \mathscr{B}]$.

THEOREM 3.9. *The set S has no cluster points.*

Proof. Assume p is a cluster point of S. Let $\{p_n\}$ be a sequence of points of S converging to p. Let D be a closed disk about p chosen such that $\mathscr{A}_D$ and $\mathscr{B}_D$ are USC and D contains no singular point of A_p except possibly p. Let A, A_n be the fibers of $\mathscr{A}_D$ containing p and p_n respectively, for each n. The components of $D - A$ incident with p are finite in number so assume $\{p_n\}$ is contained in one of them. Let C be the arc in A bounding this component in D with end points $a, b \in \beta D$. Given $\epsilon > 0$, by USC of $\mathscr{A}_D$ there is a $\delta > 0$ such that each A_n meeting $V(C, \delta)$ is contained in $V(C, \epsilon)$. Since there is an integer N such that for all $n > N$, $p_n \in N(p, \delta)$, $A_n \subset V(C, \epsilon)$ for all $n > N$. Let B_n and B be the fibers of $\mathscr{B}_D$ containing p_n and p respectively (see Figure 4). For each $n > N$, since $O(p_n) > 2$ there is a component, D_n, of $D - A_n$ such that $D_n \subset V(C, \epsilon)$ so that $\beta D_n \cap \beta D \subset N(a, \epsilon)$ or $N(b, \epsilon)$. By Theorem 3.4, since B_n separates the boundary arc $\beta D_n \cap A_n$ of D_n, B_n contains an arc which spans the boundary of D_n. Hence B_n must meet $\beta D_n \cap \beta D \subset N(a, \epsilon)$ or $N(b, \epsilon)$. Therefore the B_n's are infinitely often near a or b, say a. Thus $a \in \limsup B_n$. But $p \in B \cap$

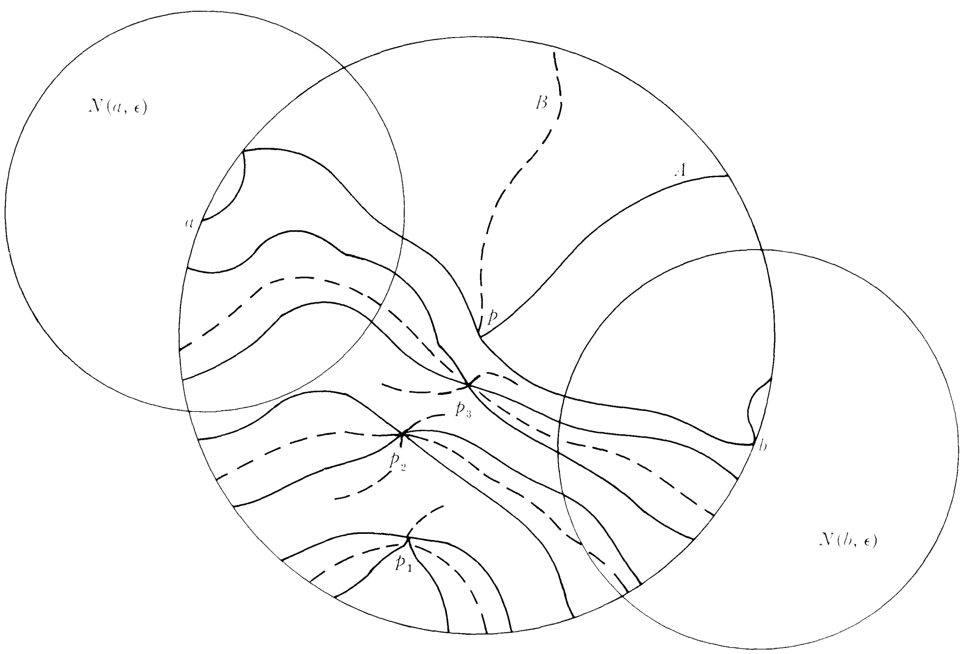

Figure 4

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$\liminf B_n$ so that $\limsup B_n \subset B$ and hence $a \in B$. This contradicts $p = A \cap B$ and establishes the theorem.

The proof of Theorem 3.1. Let D be a closed disk about p such that D contains no singular point of $[\mathscr{A}, \mathscr{B}]$ except possibly p for which $O(p) = n \geqq 2$ and $\mathscr{A}_D$ and $\mathscr{B}_D$ are *USC*. Let A and B be the fibers of $\mathscr{A}_D$ and $\mathscr{B}_D$ respectively such that $A \cap B = p$. Let $O(p) = n \geqq 2$ so that $A - p$ and $B - p$ each have n components $\{\alpha_i : i \in \mathbf{Z}_n\}$ and $\{\beta_i : i \in \mathbf{Z}_n\}$ respectively lettered counter clockwise from a fixed component α_0 such that for each $i \in \mathbf{Z}_n$, β_i is between α_i and α_{i+1}. For each $i \in \mathbf{Z}_n$, let the component of $D - A$ containing β_i be denoted by D_i and $a_i \in \alpha_i \cap \beta D$ and $b_i \in \beta_i \cap \beta D$ be the first points of α_i and β_i respectively on βD in the order starting from p (see Figure 5).

Consider the following construction. For each $i \in \mathbf{Z}_n$, let p_i be a point on the open arc pa_i of α_i and let B_i be the fiber of $\mathscr{B}_D$ containing p_i. By choice of D, B_i is an arc and by Theorem 3.4 B_i separates D_{i-1} and D_i. Hence B_i meets βD in the open arcs b_{i-1}, a_i and $a_i b_i$ and separates a_i from B. Let $\epsilon > 0$ be chosen so small that for each $i \in \mathbf{Z}_n$, $V(\gamma_i, \epsilon) \cap B_i = \emptyset$ where γ_i is the arc $\mathrm{Cl}(\alpha_i - pa_i)$. Then since γ_i is connected so is $V(\gamma_i, \epsilon)$ and hence B_i separates $V(\gamma_i, \epsilon)$ from B. Let $\delta > 0$ be chosen so small that $V(A, \delta) \cap \beta D \subset \bigcup_{i \in \mathbf{Z}_n} V(\gamma_i, \epsilon)$ and choose $\delta' > 0$ so that $V(A, \delta') \subset V(A, \delta)$ satisfies *USC* for $V(A, \delta)$. For each $i \in \mathbf{Z}_n$, let $q_i \in (\beta_i - p) \cap V(A, \delta')$ and let A_i be the fiber of $\mathscr{A}_D$ containing q_i. Then $A_i \subset V(A, \delta)$. Let pb_i be the arc joining p to b_i in β_i. Then $D_i - pb_i$ is the union of two disjoint domains each of which is separated by A_i (Theorem 3.4). Thus A_i must meet $a_i b_i$ and $b_i a_{i+1}$ within $V(\gamma_i, \epsilon)$ and $V(\gamma_{i+1}, \epsilon)$ respectively. Since $q_i \in B$, q_i is separated from $V(\gamma_i, \epsilon)$ by B_i and from $V(\gamma_{i+1}, \epsilon)$ by B_{i+1}. Hence $A_i \cap B_i \neq \emptyset \neq A_i \cap B_{i+1}$. Let $r_i = A_i \cap B_i$ and $s_i = A_i \cap B_{i+1}$. Finally denote the closure of the domain in D_i bounded by the arcs p_i, p_{i+1} on A, $p_{i+1} s_i$ on B_{i+1}, $s_i r_i$ on A_i and $r_i p_i$ on B_i by R_i.

Now I construct a homeomorphism $h_i : R_i \to [-1, 1] \times [0, (-1)^i]$ such that $\mathscr{A}_{R_i}$ and $\mathscr{B}_{R_i}$ are mapped onto {lines y = constant} and {lines x = constant} respectively. First, for each $i \in \mathbf{Z}_n$, the arcs $p_i p_{i+1} \subset \alpha_i \cup p \cup \alpha_{i+1}$ and pq_i in R_i can be mapped homeomorphically into $\mathbf{R}^2$ by $f_i : p_i p_{i+1} \to [-1, 1] \times 0$ such that $f_i(p) = (0, 0)$ and $f_i(p_i) = ((-1)^i, 0)$ and by $g_i : pq_i \to 0 \times [0, (-1)^i]$ such that $g_i(p) = (0, 0)$ and $g_i(q_i) = (0, (-1)^i)$. Let $t \in R_i$ and A_t and B_t the fibers of $\mathscr{A}_{R_i}$ and $\mathscr{B}_{R_i}$ respectively such that $A_t \cap B_t = t$. Since B_t meets βR_i it must do so in $p_i p_{i+1}$ or $s_i r_i$. Hence by Theorem 3.4 it must meet both. Let $t_1 = B_t \cap p_i p_{i+1}$. Similarly A_t must meet both $p_{i+1} s_i$ and $r_i p_i$ and hence must separate p from q_i in R_i. Thus A_t meets pq_i. Let $t_2 = A_t \cap pq_i$. Define π_j on R_i by $\pi_j(t) = t_j$, $j = 1, 2$ and h_i on R_i by $h_i(t) = (f_i(\pi_1(t)), g_i(\pi_2(t)))$. Then h_i is clearly one-to-one. The continuity of h_i follows from that of π_1 and π_2 which follows from the *USC* of $\mathscr{B}_{R_i}$ and $\mathscr{A}_{R_i}$. Thus since R_i is compact h_i is a homeomorphism.

Let $R = \bigcup_{i \in \mathbf{Z}_n} R_i$. It is easily seen that for $i \neq n - 1$ h_i agrees with h_{i+1} on pp_{i+1}. If n is even this is also true for $i = n - 1$ and gives a single valued map

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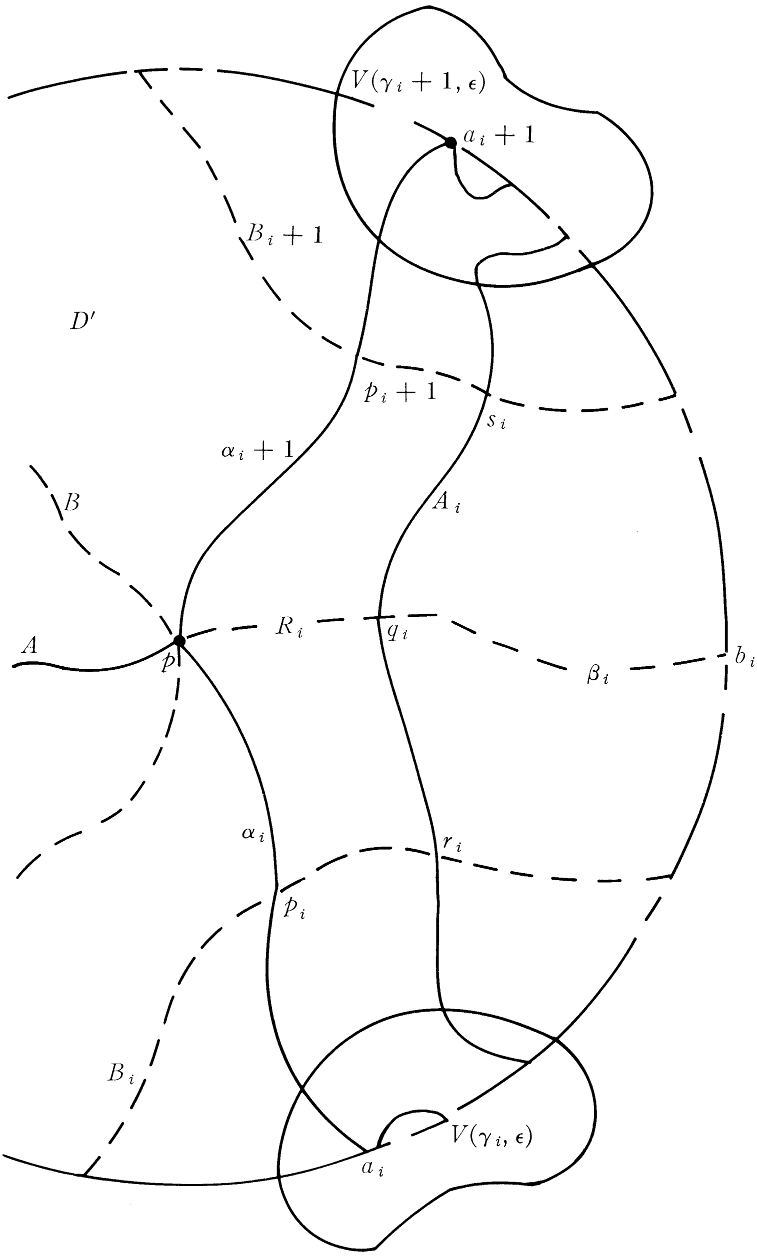

$V(\gamma_i + 1, \epsilon)$
$a_i + 1$
$B_i + 1$
D'
$p_i + 1$
s_i
$\alpha_i + 1$
B
A_i
R_i
q_i
A
p
β_i
b_i
α_i
r_i
p_i
B_i
$V(\gamma_i, \epsilon)$
a_i

Figure 5

$h' : R \to [-1, 1] \times [-1, 1] \subset \mathbf{R}^2$ of index $n/2$ which is clearly topologically equivalent to $z^{n/2}$. Thus restricting to $N = (h')^{-1} (N(0, 1))$ there is a homeomorphism h of N to the unit disk E in the z-plane about 0 such that $h'(t) = (h(t))^{n/2}$ on N which satisfies the requirements of the theorem. If n is odd $h_{n-1} = -h_0$ on pp_0. Thus a double valued maps $h' : R \to [-1, 1] \times [-1, 1]$ can be defined by continuation such that $h' = \pm h_i$ on R_i which is clearly topologically equivalent to the two valued map $z^{n/2}$. As above we get a neighborhood N and homeomorphism h of N to E satisfying the theorem.

4. Conjugate nets on 2-manifolds. In [**13**], I gave a generalized definition of conjugate net as treated by Morse and Jenkins [**6**; **7**; **8**]. Here in view of Theorem 3.1, the definition is extended to include local conditioning by algebraic functions.

Definition. A pair of families $[\mathscr{A}, \mathscr{B}]$ of disjoint locally connected generalized continua on a 2-manifold M forms a *conjugate net* if for each point $p \in M$ there is a neighborhood N about p and a homeomorphism h of N onto the unit disk $N(0, 1)$ in the z-plane such that $h(p) = 0$ and each element of $\mathscr{A}_N$ or $\mathscr{B}_N$ is carried onto a component of a level curve of $\operatorname{Re} z^{n/2}$ or $\operatorname{Im} z^{n/2}$, $n > 1$, respectively. The neighborhood N and homeomorphism h will be termed *canonical*, n the *order of* p, denoted $O(p)$, and $S = \{p \in M | O(p) > 2\}$ the *singular points* of $[\mathscr{A}, \mathscr{B}]$.

THEOREM 4.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two admissible families of generalized continua on a 2-manifold M. If for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$, $A \cap B$ is discrete and $A \cup B$ bounds no relatively compact Jordan domain, then $[\mathscr{A}, \mathscr{B}]$ is a conjugate net.*

Proof. Let $p \in M$ and D be a disk about p. It is easily seen that $\mathscr{A}_D$ and $\mathscr{B}_D$ are admissible families in D. For any $A \in \mathscr{A}_D$ and $B \in \mathscr{B}_D$, the discrete set $A \cap B$, consists of at most one point, else $A \cup B$ would bound a simply connected domain in D [**10**, IV, 20] and hence in M contrary to the hypothesis. Thus $[\mathscr{A}_D, \mathscr{B}_D]$ is a planar net and Theorem 3.1 yields a canonical neighborhood for p. It remains to show the fibers of $\mathscr{A}$ and $\mathscr{B}$ are locally connected.

If, say, A is not locally connected at p, it is clear from the structure of a canonical neighborhood that there is a sequence $\{A_n\}$ of components of $A \cap N$ in $\mathscr{A}_N - \mathscr{A}_p$ limiting in $A_p \in \mathscr{A}_N$. But $B_p \in \mathscr{B}_N$ meets every fiber of $\mathscr{A}_N$ and hence each A_n. Thus if B is the fiber of $\mathscr{B}$ containing B_p, p is a cluster point of $A \cap B$, contrary to the hypothesis.

The converse of Theorem 4.1 is not true. The condition on the unions is global in nature whereas the condition defining a conjugate net is local. For example on the torus represented in the plane by the square $[0, 1] \times [0, 1]$ the lines parallel to the vectors $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$ form two families $\mathscr{A}$ and $\mathscr{B}$ which form a conjugate net, yet $A \cup B$ bounds a relatively compact Jordan domain for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$. In particular, a conjugate net locally structured by z^n need not be generated globally by a light open map. However in the plane, the converse is true.

THEOREM 4.2. *A pair* $[\mathscr{A}, \mathscr{B}]$ *of families of closed generalized continua in the plane form a conjugate net if and only if* $\mathscr{A}$ *and* $\mathscr{B}$ *are admissible and for each* $A \in \mathscr{A}$ *and* $B \in \mathscr{B}$, $A \cap B$ *is discrete and* $A \cup B$ *bounds no relatively compact Jordan domain.*

Proof. It suffices to prove the necessity. Let $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and $p \in A \cap B$. Let N be a canonical neighborhood of p. If p is a cluster point of $A \cap B$, let $\{p_n\}$ be a sequence in $(N \cap A \cap B) - p$ converging to p. Since each element of $\mathscr{A}_N$ meets each element of $\mathscr{B}_N$ in at most one point then there exists infinitely many distinct elements of $\mathscr{A}_N$ or $\mathscr{B}_N$ containing $\{p_n\}$, say of $\mathscr{A}_N$. Let A_n be the element of $\mathscr{A}_N$ containing p_n so that $\{A_n\}$ (components of $A \cap N$) is an infinite subset. But since $p_n \rightarrow p$ it is clear from the structure of $\mathscr{A}_N$ that $\{A_n\}$ converges to a limit continuum in A_p containing p. Hence A is not locally connected at p, contrary to the hypothesis.

Finally, for $A \in \mathscr{A}$ and $B \in \mathscr{B}$ assume $A \cup B$ bounds a relatively compact Jordan domain. By [**15**, VI, 2.51] there is a simple closed curve in $A \cup B$. First it is noted that no fiber of $\mathscr{A}$ or $\mathscr{B}$ contains an end point or an open set by the structure of $\mathscr{A}$ and $\mathscr{B}$ in canonical neighborhoods, and so by Theorem 3.2 neither A nor B contains a simple closed curve. Thus a simple closed curve C can be formed of arcs α and β in A and B respectively joining points $p, q \in A \cap B$, such that $\alpha \cap \beta = \{p, q\}$ (discreteness of $A \cap B$). If D is the bounded domain in $\mathscr{C}(\alpha \cup \beta)$, let $\mathscr{B}' = \{B \in \mathscr{B}_{\mathrm{Cl}D} | B \cap \alpha \neq \emptyset\}$. For each $B \in \mathscr{B}'$, $B \cap \alpha$ consists of more than one point, since if not, B would be a locally connected continuum with at most one end point (necessarily in α), and by Lemma 3.1 would contain a simple closed curve. But this implies that some fiber of $\mathscr{B}$ contains a simple closed curve, contrary to the remark above. Thus, since $B \cap \alpha$ is discrete with more than one point, B contains an arc which spans βD and so separates D [**15**, VI, 3.5]. Consequently, if $p', q' \in B \cap \alpha$, then for any other fiber $B' \in \mathscr{B}'$, either each point of $B' \cap \alpha$ is between p' and q' on α or else none are.

Now $\mathscr{B}'$ is partially ordered as follows: for $B_1, B_2 \in \mathscr{B}'$, define $B_1 \leqq B_2$ if and only if $B_1 = B_2$ or there exists $p_1, q_1 \in B_1 \cap \alpha$ such that each point of $B_2 \cap \alpha$ is between p_1 and q_1 on α. Again, just as in Theorem 3.2, one obtains a contradiction to Zorn's lemma.

COROLLARY. *A pair* $[\mathscr{A}, \mathscr{B}]$ *of families of closed generalized continua on a 2-manifold* M *forms a conjugate net if and only if*

1) $A \cap B$ *is discrete for any* $A \in \mathscr{A}$ *and* $B \in \mathscr{B}$; *and*

2) *for any Jordan domain* $D \subset M$, $A \cup B$ *does not bound a relatively compact Jordan domain for any* $A \in \mathscr{A}_D$ *and* $B \in \mathscr{B}_D$.

Remarks. The necessity in Theorem 4.2 tells us in particular that a conjugate net in the plane is in fact a planar net. If $M = S^2$, then any fiber of a conjugate net on S^2 must be a continuum without an end point and so by Lemma 3.1 contains a simple closed curve contrary to the above corollary. Thus S^2 cannot support a conjugate net, suggesting another view of Liouville's theorem.

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