ON THE COMPLETE CONTINUITY OF DIFFERENTIABLE MAPPINGS

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1. Introduction

Let E_1 and E_2 be real Banach spaces. We denote by E_{12} the Banach space of all bounded linear operators mapping E_1 into E_2 with the usual operator norm. By D_r we denote the open ball of radius r, centred at the origin.

An operator $f : E_1 \to E_2$ is said to be *Fréchet-differentiable at a point* $x \in E_1$ if there exists $l \in E_{12}$ (depending on x) such that for each number $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ so that:

$$f(x+h)-f(x) = lh+w(x, h)$$

where

$$||w(x, h)|| < \varepsilon||h|| \text{ if } ||h|| < \delta(\varepsilon).$$

We denote l by $f'(x)$, the *Fréchet-derivative* of f at x.

In the sequel, an operator f that is Fréchet-differentiable at every point will be referred to be as a Fréchet-differentiable operator. If, for a Fréchet-differentiable operator f, we have for each number $\varepsilon > 0$ a number $\delta(\varepsilon) > 0$ so that:

$$||w(x, h)|| < \varepsilon||h|| \text{ if } ||h|| < \delta(\varepsilon) \text{ and } x, x+h \in D_r,$$

f is said to be *uniformly differentiable* in D_r.

An operator mapping one Banach space into another is said to be *compact* if the image of any bounded set is contained in a compact set. A continuous and compact operator is called a *completely continuous* operator.

It is well-known that ([1], p. 34; [2], p. 51):

[I] If $f : E_1 \to E_2$ is completely continuous and Fréchet-differentiable at x, $f'(x)$ is a completely continuous linear operator.

Therefore, the derivative at every point of a completely continuous Fréchet-differentiable operator is completely continuous. It has been shown in [3] that the converse is not true.

On the other hand, Vainberg ([2], p. 51) has proved:

[II] If

(1) $f'(x)$ is completely continuous for every x,

(2) $f' : E_1 \to E_{12}$ is compact,

then f is completely continuous.

As the converse to [I], this theorem [II] is the best result published so far.

In this paper we shall give a characterization of a uniformly differentiable operator which satisfies (1) and (2) in [II], under the assumption that E_1 has weakly compact unit ball.

2. Results

An operator f mapping one Banach space into another is said to be *strongly continuous* if $x_n \rightharpoonup x_0$ implies that $f(x_n) \to f(x_0)$, where $\rightharpoonup$ and $\to$ denote the *weak* and the *strong* convergence respectively. In general, strong continuity and complete continuity are mutually independent.

However ([2], p. 14):

(3) *If E_1 has weakly compact unit ball, a strongly continuous operator mapping E_1 into another Banach space is completely continuous. The converse is not true.*

Now we can state our main theorem:

THEOREM. *When E_1 has weakly compact unit ball, an operator $f : E_1 \to E_2$, uniformly differentiable in every open ball, is strongly continuous if and only if it satisfies* (1) *and* (2).

3. Proof of the Theorem

LEMMA 1. *If the Fréchet-differentiable operator $f : E_1 \to E_2$ satisfies* (1) *and* (2), *f is strongly continuous.*

PROOF. [II] implies that f is compact. Then, using ([2], p. 47), (2) and the compactness of f imply that f is strongly continuous.

When E_1 has weakly compact unit ball, Lemma 1 is a stronger result than [II], in view of (3).

LEMMA 2. *Let E_1 have weakly compact unit ball. Then, if $f : E_1 \to E_2$ is uniformly differentiable in every open ball and strongly continuous, $f' : E_1 \to E_{12}$ is strongly continuous.*

PROOF. Suppose that f' is not strongly continuous. Then there is a sequence $x_n \rightharpoonup x_0$ and yet $f'(x_n) \nrightarrow f'(x_0)$. This means that there is a subsequence $\{x_{j_n}\}$ and a number $\varepsilon > O$ such that:

$$||f'(x_{j_n}) - f'(x_0)|| > \varepsilon \text{ for all } n.$$

So there is a sequence $\{h_{j_n}\}$ in E_1 such that $||h_{j_n}|| = 1$ and

$$||f'(x_{j_n})h_{j_n} - f'(x_0)h_{j_n}|| > \varepsilon \text{ for all } n.$$

Now, if t is a real number such that $|t| < 1$,

$$\begin{aligned} ||f'(x_{j_n})th_{j_n} - f'(x_0)th_{j_n}|| \leqq\ & ||f'(x_{j_n})th_{j_n} - [f(x_{j_n}+th_{j_n}) - f(x_{j_n})]|| \\ & + ||f'(x_0)th_{j_n} - [f(x_0+th_{j_n}) - f(x_0)]|| \\ & + ||f(x_{j_n}+th_{j_n}) - f(x_{j_n}) - f(x_0+th_{j_n}) + f(x_0)||. \end{aligned} \tag{4}$$

Since $\{x_{j_n}\}$ is weakly convergent to x_0, $||x_{j_n}|| \leqq K$ for all n and $||x_0|| \leqq K$.

Then $||x_{j_n}+th_{j_n}|| \leqq ||x_{j_n}|| + |t| < K+1$ for all m, and, similarly, $||x_0+th_{j_n}|| \leqq ||x_0|| + |t| < K+1$.

Therefore, $x_{j_n}+th_{j_n} \in D_{K+1}$ for all n and $x_0+th_{j_n} \in D_{K+1}$ for all n.

Since f is uniformly differentiable in D_{K+1}, there exists for each number $\varepsilon > O$ a number $\delta(\varepsilon) > O$ such that:

$$||f(x+th) - f(x) - f'(x)th|| < \varepsilon|t|$$

if $|t| < \delta(\varepsilon)$, $||h|| = 1$ and $x, x+h \in D_{K+1}$.

Then, if we fix t at a value t_0 so that $|t_0| < \min\{1, \delta(\varepsilon/3)\}$, the sum of the first two terms on the right hand side of (4) will be less than $2\varepsilon|t_0|/3$.

Now $||h_{j_n}|| = 1$ for all n. Since the unit ball of E_1 is weakly compact, there is a subsequence $h_{K_n} \rightharpoonup h_0$.

Hence, by the strong continuity of f, $f(x_{K_n}) \to f(x_0)$,

$f(x_{K_n}+t_0h_{K_n}) \to f(x_0+t_0h_0)$, $f(x_0+t_0h_{K_n}) \to f(x_0+t_0h_0)$.

So

$$\begin{aligned} ||f(x_{K_n}+t_0h_{K_n}) - f(x_{K_n}) - f(x_0+th_{K_n}) + f(x_0)|| \leqq\ & ||f(x_{K_n}+t_0h_{K_n}) - f(x_0+t_0h_0)|| \\ & + ||f(x_0+t_0h_{K_n}) - f(x_0+t_0h_0)|| \\ & + ||f(x_{K_n}) - f(x_0)|| \\ <\ & \varepsilon|t_0|/3 \text{ if } n \text{ is large enough.} \end{aligned}$$

Hence, returning to (4) and cancelling t_0, we see that

$$||f'(x_{K_n})h_{K_n} - f'(x_0)h_{K_n}|| < \varepsilon \text{ if } n \text{ is large enough.}$$

This contradiction establishes the theorem.

In Lemma 2, the uniform differentiability condition is essential. We

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take each of E_1 and E_2 as the real line and we consider the function f defined by:

$$f(x) = x^2 \sin(1/x) \qquad (x \neq 0);$$
$$f(0) = 0.$$

Then

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \qquad (x \neq 0);$$
$$f'(0) = \lim_{x\to 0} f(x)/x = 0.$$

So f is a Fréchet-differentiable operator and, therefore, strongly continuous since strong and ordinary continuity coincide when E_1 is finite-dimensional. However, f' is not continuous and, therefore, not strongly continuous.

Proof of Theorem. We assume that f is strongly continuous. Then, by Lemma 2, f' is strongly continuous and therefore compact, by (3), so that (2) is satisfied. Again using (3), f is compact. Hence, by [I], $f'(x)$ is completely continuous for every x so that (1) is satisfied.

Now let us assume that f satisfies (1) and (2). Then, by Lemma 1, f is strongly continuous.

In the theorem, the strong continuity of f cannot be replaced by the compactness of f. For if this were so, a compact operator, uniformly differentiable in every ball of a space with weakly compact unit ball, would be strongly continuous. That this is not true is shown by the following example.

We take E_1 as separable Hilbert space and E_2 as the real line R. We consider $f: E_1 \to R$ defined by:

$$f(x) = (x, x)$$

It is easy to show that f is uniformly differentiable in every open ball of E_1 and that f is compact. However, f is not strongly continuous for, if $\{e_n\}$ is an orthonormal basis for E_1, $e_n \rightharpoonup 0$ and yet $f(e_n) \not\to f(0) = 0$ since $f(e_n) = 1$ for all n.

References

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