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Non-rigidity of partially hyperbolic abelian C^1-actions on tori

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Abstract. We prove that every genuinely partially hyperbolic $\mathbb{Z}^r$-action by toral automorphisms can be perturbed in C^1-topology, so that the resulting action is continuously conjugate, but not C^1-conjugate, to the original one.

Key words: Abelian actions, rigidity, Anosoc–Katok
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Contents

1. *Introduction*

1.1. *Statement of results.* In this paper, let $\rho : \mathbb{Z}^r \to \mathrm{GL}_d(\mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^d)$ be a group morphism and denote indifferently by ρ the group action it induces on $\mathbb{T}^d$. Our main result is the following theorem.

THEOREM 1.1. *If an action $\rho : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ by toral automorphisms contains no hyperbolic automorphisms, then for any $\tau > 0$, there exists an action $\alpha : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ by C^1-diffeomorphisms such that:*

(1) $d_{C^1}(\alpha, \rho) < \tau$;
(2) $\alpha^{\mathbf{n}} = \widetilde{H} \circ \rho \circ \widetilde{H}^{-1}$ *for a homeomorphism* $\widetilde{H} : \mathbb{T}^d \to \mathbb{T}^d$ *that is homotopic to* id;
(3) *neither* $\widetilde{H}$ *nor* $\widetilde{H}^{-1}$ *is differentiable.*

Here the C^1-distance d_{C^1} between two actions is defined as $d_{C^1}(\alpha, \rho) = \max_{\mathbf{n}\in\Xi} \|\alpha^{\mathbf{n}} - \rho^{\mathbf{n}}\|_{C^1}$, where $\Xi \in \mathbb{Z}^r$ is the generating set

$$\Xi = \{\pm\mathbf{e}_i \, : \, i = 1, \ldots r\}$$

with $\mathbf{e}_i$ being the ith coordinate vector.

Definition 1.2. [**DK10**, Section 1.3.2] An action $\rho : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ by toral automorphisms is *genuinely partially hyperbolic* if ρ is ergodic with respect to the Haar measure on $\mathbb{T}^d$, but $\rho^{\mathbf{n}}$ is not hyperbolic for any $\mathbf{n}$.

As remarked in [**DK10**], a genuinely partially hyperbolic action contains an element which has no root of unity among its eigenvalues, or equivalently an ergodic toral automorphim.

COROLLARY 1.3. *Suppose* $\rho : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ *is a genuinely partially hyperbolic action by toral automorphisms. Then for any* $\tau > 0$, *there exists an action* $\alpha : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ *by* C^1*-diffeomorphisms such that:*
(1) $d_{C^1}(\alpha, \rho) < \tau$;
(2) α *and* ρ *are not* C^1*-conjugate.*

Corollary 1.3 is deduced from Theorem 1.1 through a standard argument.

Proof. Let α be given by Theorem 1.1 and assume $\widetilde{G} : \mathbb{T}^d \to \mathbb{T}^d$ is a C^1 diffeomorphism such that $\alpha^{\mathbf{n}} \circ \widetilde{G} = \widetilde{G} \circ \rho^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^d$. Then $G := \widetilde{H}^{-1} \circ \widetilde{G}$ is a homeomorphism of $\mathbb{T}^d$ such that

$$\rho^{\mathbf{n}} \circ G = \rho^{\mathbf{n}} \circ \widetilde{H}^{-1} \circ \widetilde{G} = \widetilde{H}^{-1} \circ \alpha^{\mathbf{n}} \circ \widetilde{G} = \widetilde{H}^{-1} \circ \widetilde{G} \circ \rho^{\mathbf{n}} = G \circ \rho^{\mathbf{n}}.$$

Since at least one of the $\rho^{\mathbf{n}}$ is an ergodic toral automorphism, G is affine by [**W70**, Corollary 2]. So $\widetilde{G} = \widetilde{H} \circ G$ cannot be C^1 because $\widetilde{H}$ is not, which contradicts our assumption. □

1.2. *Background.* Faithful linear actions by higher rank abelian groups on tori and nilmanifolds, that is, $\mathbb{Z}^r$-actions generated by automorphisms where $r \geq 2$, have since been long expected to be rigid, in the following sense: under some additional assumptions, a smooth action α in the same homotopy class should be smoothly conjugated to the linear action itself, which we denote by ρ. The issue we address in this paper is whether the conjugacy, denoted by h, should have the same smoothness as α.

One important rigidity phenomenon is the local rigidity of the actions ρ described above, which stands for rigidity under perturbative assumptions. An action ρ is said to be $C^{l,m,n}$-locally rigid if all C^l-actions that are sufficiently close to ρ in C^m topology are C^n-conjugate to ρ. For Cartan actions (that is, faithful linear actions by $\mathbb{Z}^r$ with the largest possible r, modulo restriction to a finite index subgroup) on tori, $C^{\infty,1,\infty}$ local rigidity was proved by Katok and Lewis [**KL91**]. For some more general

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classes of hyperbolic actions, $C^{\infty,1,\infty}$ local rigidity was proved by Katok and Spatzier [**KS94**, **KS97**] and Einsiedler and Fisher [**EF07**]. For global rigidity see [**F69**], [**FKS11**], [**FKS13**] and [**RH07**]. Damjanović and Katok [**DK10**] proved $C^{\infty,r,\infty}$ local rigidity for genuinely partially hyperbolic $\mathbb{Z}^r$-actions by toral automorphisms by the Kolmogorov–Arnold–Moser (KAM) method. For finitely differentiable actions, $C^{l,1,l}$ is not expected to follow from KAM methods because of the loss of regularity when solving a cocycle equation of the form (2.1) below. When $r = 1$, that is, for the dynamics of a single toral automorphism A of $\mathbb{T}^d$ that is partially hyperbolic, such loss of regularity in the cocycle equation was discussed by Veech in [**V86**], where it was shown that, although the cocycle equation $g \circ A - A \circ g = f$ can be solved in C^n if $f \in C^l$ and $n < l - d$, there exists a C^1-function f for which the equation has no C^1-solutions.

Section 3 of this paper will describe similar loss of regularity when solving the cocycle equation for general genuinely partially hyperbolic $\mathbb{Z}^r$-actions by toral automorphisms. In §2, we propose a reversed KAM scheme that allows an accumulation of such losses at certain sequences of periodic points, which leads to the failure of $C^{1,1,1}$-rigidity in Theorem 1.1.

1.3. *Notation.* In the rest of this paper:

- ρ will be fixed;
- all implicit constants in expression of the forms $X \ll Y$ and $X = O(Y)$ will be assumed to be dependent on r, d, ρ, and Ξ, but independent of other variables;
- $e(t)$ will denote the function $e^{2\pi i t}$.

2. *The inductive scheme*

2.1. *Sequence of conjugacies.* We employ a reversed KAM scheme to construct a counterexample. A sequence of conjugacies H_m will be constructed in later sections, where $H_m = \mathrm{id} + h_m$ for a sequence of C^∞ smooth functions $h_m : \mathbb{T}^d \to \mathbb{R}^d$ that are small in C^1 norm. Inductively define

$$\widetilde{H}_m = H_1 \circ \cdots \circ H_m, \tag{2.1}$$

and

$$\alpha_m^{\mathbf{n}} = \widetilde{H}_m \circ \rho^{\mathbf{n}} \circ \widetilde{H}_m^{-1}. \tag{2.2}$$

For $m = 0$, set $\widetilde{H}_0 = \mathrm{id}$ and $\alpha_0 = \rho$.

Notice that as H_m is homotopic to id, all the α_m terms are homotopic to ρ.

Define a twisted coboundary $g_m : \mathbb{Z}^r \times \mathbb{T}^d \to \mathbb{R}^d$ over ρ by

$$g_m^{\mathbf{n}}(x) = h_m \circ \rho^{\mathbf{n}}(x) - \rho^{\mathbf{n}} h_m(x). \tag{2.3}$$

We pose a list of technical conditions on h_m and g_m as follows.

Condition 2.1. The sequence $\{h_m\}_{m=1}^\infty$ will be chosen, together with:

- a positive number $\tau \in (0, 1)$;
- a sequence of positive numbers $\{\theta_m\}_{m=1}^\infty$;
- unit vectors $v, w \in \mathbb{R}^d$, as well as two sequences of non-zero vectors $\{v_m\}_{m=1}^\infty$, $\{v_m^*\}_{m=1}^\infty$ from $\mathbb{R}^d$,

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so that, for all $m \in \mathbb{N}$:

(i) $\sum_{m=1}^{\infty} \theta_m < \tau$;

(ii) $\|h_m\|_{C^1} \ll \tau$ and

$$\Big(\max_{m'=1}^{m-1} \|\widetilde{H}_{m'}^{-1}\|_{C^1}\Big)\Big(\max_{m'=1}^{m-1} \|\widetilde{H}_{m'}\|_{C^1}\Big)\|h_m\|_{C^0} < \theta_m;$$

(iii) $\|\widetilde{H}_{m-1}\|_{C^2}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\|g_m^{\mathbf{n}}\|_{C^1} < \theta_m$;

(iv) $h_m(0) = 0$ and either $(D_0\widetilde{H}_m)v = v + \tau w$ if m is odd or $(D_0\widetilde{H}_m)v = v$ if m is even;

(v) either $w = v$ or $(D_0\widetilde{H}_m)w = w$;

(vi) $h_m(v_{m'}) = h_m(v^*_{m'}) = 0$ for all $1 \le m' \le m-1$, where $v_{m'}$ is identified with its projection in $\mathbb{T}^d$;

(vii) $\|\widetilde{H}_m\|_{C^2}|v_m| < \theta_m$, $\|\widetilde{H}_m\|_{C^1}|v_m/|v_m| - v| < \theta_m$, $\|\widetilde{H}_m\|_{C^2}|v^*_m| < \theta_m$, and $\|\widetilde{H}_m\|_{C^1}$ $|v^*_m/|v^*_m| - (v+\tau w)/|v+\tau w|| < \theta_m$.

Along our proof, it will turn out that v and w may or may not be the same.

2.2. *Sufficient inductive conditions.* We now show the following proposition.

PROPOSITION 2.2. *Given the action ρ, if Condition 2.1 is satisfied and the constant $\tau > 0$ therein is sufficiently small, then:*

(1) *$\{\widetilde{H}_m\}_{m=1}^{\infty}$ converges in C^0 to a homeomorphism $\widetilde{H}$ that is homotopic to* id*;*

(2) *for all $\mathbf{n} \in \Xi$, $\widetilde{H} \circ \rho^{\mathbf{n}} \circ \widetilde{H}^{-1}$ is C^1 differentiable and*

$$\|\widetilde{H} \circ \rho^{\mathbf{n}} \circ \widetilde{H}^{-1} - \rho^{\mathbf{n}}\|_{C^1} \ll \tau;$$

(3) *neither $\widetilde{H}$ nor $\widetilde{H}^{-1}$ is differentiable.*

We first recall a few technical facts regarding C^k norms.

LEMMA 2.3. *For smooth maps $\phi, \psi : \mathbb{T}^d \to \mathbb{T}^d$ and $\Delta : \mathbb{T}^d \to \mathbb{R}^d$:*

(1) $\|\phi \circ \psi\|_{C^2} \ll \|\phi\|_{C^2}(1 + \|\psi\|_{C^0})^2(1 + \|\psi\|_{C^2})$. *If ψ is not homotopically trivial, then $\|\phi \circ \psi\|_{C^1} \le \|\phi\|_{C^1}\|\psi\|_{C^1}$;*

(2) $\|\phi \circ (\psi + \Delta) - \phi \circ \psi\|_{C^1} \ll \|\phi\|_{C^2}(1 + \|\psi\|_{C^1})\|\Delta\|_{C^1}$*;*

(3) *there is $\epsilon = \epsilon(d)$ such that if $\|\phi - \mathrm{id}\|_{C^1} \le \epsilon$, then ϕ is invertible, and $\|\phi^{-1}\|_{C_1} \ll 1 + \|\phi\|_{C_1}$ and $\|\phi^{-1}\|_{C_2} \ll 1 + \|\phi\|_{C_2}$.*

Proof of Lemma 2.3. (1) The C^2 bound is in [**K99**, Proposition A.2.3]. For the C^1 bound, note $\|\phi \circ \psi\|_{C^0} = \|\phi\|_{C^0} \le \|\phi\|_{C^1}\|\psi\|_{C^1}$, where we used $\|\psi\|_{C^1} \le 1$ because ψ is not homotopically trivial. In addition, $\|D(\phi \circ \psi)\|_{C^0} = \|(D\phi \circ \psi)D\psi\|_{C^0} \le \|\phi\|_{C^1}\|\psi\|_{C^1}$.

(2) We have

$$\|\phi \circ (\psi + \Delta) - \phi \circ \psi\|_{C^0} \le \|\phi\|_{C^1}\|\Delta\|_{C^0} \le \|\phi\|_{C^2}(1 + \|\psi\|_{C^1})\|\Delta\|_{C^1}.$$

Moreover,

$$\begin{aligned}&\|D(\phi \circ (\psi + \Delta) - \phi \circ \psi)\|_{C^0}\\ &\quad = \|(D\phi \circ (\psi + \Delta))(D\psi + D\Delta) - (D\phi \circ \psi))D\psi\|_{C^0}\end{aligned}$$

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$$
\begin{aligned}
&= \|(D\phi \circ (\psi + \Delta) - D\phi \circ \psi) D\psi + (D\phi \circ (\psi + \Delta)) D\Delta\|_{C^0} \\
&\leq \|D\phi \circ (\psi + \Delta) - D\phi \circ \psi\|_{C^0} \|D\psi\|_{C^0} + \|D\phi\|_{C^0} \|D\Delta\|_{C^0} \\
&\leq \|D\phi\|_{C^1} \|\Delta\|_{C^0} \|D\psi\|_{C^0} + \|D\phi\|_{C^0} \|D\Delta\|_{C^0} \\
&\leq \|\phi\|_{C^2} \|\psi\|_{C^1} \|\Delta\|_{C^0} + \|\phi\|_{C^1} \|\Delta\|_{C^1} \\
&\leq \|\phi\|_{C^2} (1 + \|\psi\|_{C^1}) \|\Delta\|_{C^1}.
\end{aligned}
$$

(3) Is proven in [**H82**, Lemma 2.3.6]. □

Proof of Proposition 2.2. In the proof below, we will repeatedly use the fact that, because $\widetilde{H}_{m-1}$ is homotopic to id,

$$\|\widetilde{H}_m\|_{C^1} \geq 1, \ \|\widetilde{H}_{m-1}^{-1}\|_{C^1} \geq 1. \tag{2.4}$$

(1) By Lemma 2.3, when τ is sufficiently small depending on the dimension d, $H_m = \mathrm{id} + h_m$ is invertible, and H_m^{-1} is C^1 differentiable and homotopic to id. So every $\widetilde{H}_m$ is invertible in C^1.

By Condition 2.1(ii) and (2.4), for all $x \in \mathbb{T}^d$,

$$
\begin{aligned}
&|\widetilde{H}_m(x) - \widetilde{H}_{m-1}(x)| \\
&\quad = |\widetilde{H}_{m-1}(x + h_m(x)) - \widetilde{H}_{m-1}(x)| \\
&\quad \leq \|\widetilde{H}_{m-1}\|_{C^1} \|h_m\|_{C^0} < \theta_m.
\end{aligned}
$$

It follows that $\{\widetilde{H}_m\}$ is a Cauchy, and hence convergent, sequence in C^0. Its limit, which we denote by $\widetilde{H}$, is a continuous map that is homotopic to id. Note

$$\|\widetilde{H} - \widetilde{H}_m\|_{C^0} \leq \sum_{k=m+1}^{\infty} \|\widetilde{H}_{k-1}\|_{C^1} \|h_k\|_{C^0}. \tag{2.5}$$

However, it is easy to see that $H_m^{-1} = \mathrm{id} + h_m^*$, where $h_m^* = -h_m \circ H_m^{-1}$. In particular, $\|h_m^*\|_{C^0} = \|h_m\|_{C^0}$ and

$$\sum_{m=1}^{\infty} \|h_m^*\|_{C^0} \leq \sum_{m=1}^{\infty} \|\widetilde{H}_{m-1}\|_{C^1} \|h_m\|_{C^0}\| < \sum_{m=1}^{\infty} \theta_m < \tau. \tag{2.6}$$

As $\widetilde{H}_m^{-1} = \widetilde{H}_{m-1}^{-1} + h_m^* \circ \widetilde{H}_{m-1}^{-1}$, it follows that $\{\widetilde{H}_m^{-1}\}$ is a Cauchy sequence in C^0 topology, and thus converges to a continuous map $\widetilde{H}^*$. Additionally, $\widetilde{H}^*$ is homotopic to id. We also have

$$\|\widetilde{H}^* - \widetilde{H}_m^{-1}\|_{C^0} \leq \sum_{k=m+1}^{\infty} \|h_k\|_{C^0}. \tag{2.7}$$

Thus, for all m,

$$
\begin{aligned}
&\|\widetilde{H} \circ \widetilde{H}^* - \mathrm{id}\|_{C^0} \\
&\quad = \|\widetilde{H} \circ \widetilde{H}^* - \widetilde{H}_m \circ \widetilde{H}_m^{-1}\|_{C^0} \\
&\quad \leq \|\widetilde{H} \circ \widetilde{H}^* - \widetilde{H}_m \circ \widetilde{H}^*\|_{C^0} + \|\widetilde{H}_m \circ \widetilde{H}^* - \widetilde{H}_m \circ \widetilde{H}_m^{-1}\|_{C^0} \\
&\quad \leq \|\widetilde{H} - \widetilde{H}_m\|_{C^0} + \|\widetilde{H}_m\|_{C^1} \|\widetilde{H}^* - \widetilde{H}_m^{-1}\|_{C^0}
\end{aligned}
$$

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$$\le \sum_{k=m+1}^{\infty} \|\widetilde{H}_{k-1}\|_{C^1}\|h_k\|_{C^0} + \|\widetilde{H}_m\|_{C^1} \sum_{k=m+1}^{\infty} \|h_k\|_{C^0}$$
$$\le \sum_{k=m+1}^{\infty} \theta_k + \sum_{k=m+1}^{\infty} \theta_k = 2\sum_{k=m+1}^{\infty} \theta_k, \tag{2.8}$$

where we used equation (2.7) and the parts (i), (ii) of Condition 2.1. As $\sum_{m=1}^{\infty}\theta_m < \tau$, it follows that $\|\widetilde{H}\circ\widetilde{H}^* - \mathrm{id}\|_{C^0} = 0$. Therefore, $\widetilde{H}\circ\widetilde{H}^* = \mathrm{id}$.

Similarly, for all m,

$$\begin{aligned}
&\|\widetilde{H}^*\circ\widetilde{H} - \mathrm{id}\|_{C^0}\\
&\quad = \|\widetilde{H}^*\circ\widetilde{H} - \widetilde{H}_m^{-1}\circ\widetilde{H}_m\|_{C^0}\\
&\quad \le \|\widetilde{H}^*\circ\widetilde{H} - \widetilde{H}_m^{-1}\circ\widetilde{H}\|_{C^0} + \|\widetilde{H}_m^{-1}\circ\widetilde{H} - \widetilde{H}_m^{-1}\circ\widetilde{H}_m\|_{C^0}\\
&\quad \le \|\widetilde{H}^* - \widetilde{H}_m^{-1}\|_{C^0} + \|\widetilde{H}_m^{-1}\|_{C^1}\|\widetilde{H} - \widetilde{H}_m\|_{C^0}\\
&\quad \le \sum_{k=m+1}^{\infty}\|h_k\|_{C^0} + \|\widetilde{H}_m^{-1}\|_{C^1}\sum_{k=m+1}^{\infty}\|\widetilde{H}_{k-1}\|_{C^1}\|h_k\|_{C^0}\\
&\quad \le \sum_{k=m+1}^{\infty}\theta_k + \sum_{k=m+1}^{\infty}\theta_k = 2\sum_{k=m+1}^{\infty}\theta_k.
\end{aligned} \tag{2.9}$$

As above, we know $\widetilde{H}^*\circ\widetilde{H} = \mathrm{id}$.

We can now conclude that $\widetilde{H}^* = \widetilde{H}^{-1}$ and $\widetilde{H}$ is a homeomorphism of $\mathbb{T}^d$.

(2) By Lemma 2.3, for $\mathbf{n}\in\Xi$,

$$\begin{aligned}
&\|\alpha_m^{\mathbf{n}} - \alpha_{m-1}^{\mathbf{n}}\|_{C^1}\\
&\quad = \|\widetilde{H}_{m-1}\circ H_m\circ\rho^{\mathbf{n}}\circ\widetilde{H}_m^{-1} - \widetilde{H}_{m-1}\circ\rho^{\mathbf{n}}\circ H_m\circ\widetilde{H}_m^{-1}\|_{C^1}\\
&\quad \le \|\widetilde{H}_{m-1}\circ H_m\circ\rho^{\mathbf{n}} - \widetilde{H}_{m-1}\circ\rho^{\mathbf{n}}\circ H_m\|_{C^1}\|\widetilde{H}_m^{-1}\|_{C^1}\\
&\quad \le \|\widetilde{H}_{m-1}\|_{C^2}(1 + \|H_m\circ\rho^{\mathbf{n}}\|_{C^1})\\
&\qquad \cdot \|H_m\circ\rho^{\mathbf{n}} - \rho^{\mathbf{n}}\circ H_m\|_{C^1}\|H_m^{-1}\|_{C^1}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\\
&\quad \lll \|\widetilde{H}_{m-1}\|_{C^2}(1 + \|H_m\|_{C^1}\|\rho^{\mathbf{n}}\|_{C^1})\\
&\qquad \cdot \|(\rho^{\mathbf{n}} + h_m\circ\rho^{\mathbf{n}}) - (\rho^{\mathbf{n}} + \rho^{\mathbf{n}}h_m)\|_{C^1}\|H_m\|_{C^1}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\\
&\quad \ll \|\widetilde{H}_{m-1}\|_{C^2}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\|g_m\|_{C^1} < \theta_m.
\end{aligned}$$

Because $\sum_{m=1}^{\infty}\theta_m < \tau$, the sequence $\{\alpha_m^{\mathbf{n}}\}$ is Cauchy in C^1 topology. Denote the limit by $\alpha^{\mathbf{n}}$. Since $\rho^{\mathbf{n}} = \alpha_0^{\mathbf{n}}$,

$$\|\alpha^{\mathbf{n}} - \rho^{\mathbf{n}}\|_{C^1} \ll \sum_{m=1}^{\infty}\theta_m < \tau \quad \text{for all } \mathbf{n}\in\Xi. \tag{2.10}$$

Finally, we want to show that $\alpha^{\mathbf{n}} = \widetilde{H}\circ\rho^{\mathbf{n}}\circ\widetilde{H}^{-1}$. For all $m\in\mathbb{N}$ and $\mathbf{n}\in\Xi$,

$$\begin{aligned}
&\|\alpha_m^{\mathbf{n}} - \widetilde{H}\circ\rho^{\mathbf{n}}\circ\widetilde{H}^{-1}\|_{C^0}\\
&\quad \le \|\widetilde{H}_m\circ\rho^{\mathbf{n}}\circ\widetilde{H}_m^{-1} - \widetilde{H}_m\circ\rho^{\mathbf{n}}\circ\widetilde{H}^{-1}\|_{C^0} + \|\widetilde{H}_m\circ\rho^{\mathbf{n}}\circ\widetilde{H}^{-1} - \widetilde{H}\circ\rho^{\mathbf{n}}\circ\widetilde{H}^{-1}\|_{C^0}
\end{aligned}$$

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$$
\begin{aligned}
&\leq \|\widetilde{H}_m \circ \rho^{\mathbf{n}}\|_{C^1} \|\widetilde{H}_m^{-1} - \widetilde{H}^{-1}\|_{C^0} + \|\widetilde{H}_m - \widetilde{H}\|_{C^0} \\
&\ll \|\widetilde{H}_m\|_{C^1} \sum_{k=m+1}^{\infty} \|h_k\|_{C^0} + \sum_{k=m+1}^{\infty} \|\widetilde{H}_{k-1}\|_{C^1} \|h_k\|_{C^0} \\
&\ll \sum_{k=m+1}^{\infty} (\max_{k'=1}^{k-1} \|\widetilde{H}_{k'}\|_{C^1}) \|h_k\|_{C^0} \\
&< \sum_{k=m+1}^{\infty} \theta_k,
\end{aligned} \tag{2.11}
$$

which decays to 0 as $m \to \infty$. Thus, $\widetilde{H} \circ \rho^{\mathbf{n}} \circ \widetilde{H}^{-1}$ is the C^0 limit of $\alpha_m^{\mathbf{n}}$, which coincides with $\alpha^{\mathbf{n}}$.

The extension of the definition $\alpha^{\mathbf{n}} = \widetilde{H} \circ \rho^{\mathbf{n}} \circ \widetilde{H}^{-1}$ to general $\mathbf{n} \in \mathbb{Z}^r$ forms a C^1 action generated by $\{\alpha^{\mathbf{n}} : \mathbf{n} \in \Xi\}$.

(3) Since $H_m(0) = 0 + h_m(0) = 0$,

$$
\widetilde{H}_m(0) = 0 \quad \text{for all } m \text{ and } \widetilde{H}(0) = 0.
$$

In addition, for all positive integers $m' > m \geq 1$, $h_{m'}(v_m) = 0$ and thus $H_{m'}(v_m) = v_m + h_{m'}(v_m) = v_m$. Therefore, for all $k > m \geq 1$,

$$
\begin{aligned}
\widetilde{H}_{m'}(v_m) &= \widetilde{H}_m \circ H_{m+1} \circ \cdots \circ H_{m'-1} \circ H_{m'}(v_m) \\
&= \widetilde{H}_m \circ H_{m+1} \circ \cdots \circ H_{m'-1}(v_m) \\
&= \cdots = \widetilde{H}_m(v_m),
\end{aligned}
$$

and

$$
\widetilde{H}(v_m) = \lim_{m' \to \infty} \widetilde{H}_{m'}(v_m) = \widetilde{H}_m(v_m). \tag{2.12}
$$

Set $y_m = v_m + \sum_{m'=1}^{m} h_{m'} \circ H_{m'+1} \circ \cdots \circ H_m(v_m)$. Then $\widetilde{H}(v_m) = \widetilde{H}_m(v_m)$ is the projection of y_m to $\mathbb{T}^d$, which we indifferently denote by y_m.

We first claim that $\widetilde{H}$ is not differentiable at 0. To show this, it is helpful to study the asymptotic behavior of the sequence of vectors $y_m/|v_m|$.

Remark that since $\sum_{m=1}^{\infty} \theta_m < \tau$, $\theta_m \to 0$. Moreover, as $\widetilde{H}_m$ is homotopic to id, $\|\widetilde{H}_m\|_{C^2} \geq \|\widetilde{H}_m\|_{C^1} \geq 1$. Thus, Condition 2.1(vii) shows $|v_m| \leq \theta_m$ and $|v_m/|v_m| - v| \leq \theta_m$. Thus, $v_m \to 0$ and $v_m/|v_m| \to v$ as $m \to \infty$.

As $\widetilde{H}_m(v_m) = y_m$, by Condition 2.1(vii),

$$
\begin{aligned}
&\frac{y_m}{|v_m|} - (D_0\widetilde{H}_m)v \\
&\quad = \left(\frac{\widetilde{H}_m(v_m)}{|v_m|} - \frac{(D_0\widetilde{H}_m)v_m}{|v_m|}\right) + \left((D_0\widetilde{H}_m)(\frac{v_m}{|v_m|} - v)\right) \\
&\quad = \frac{O(\|\widetilde{H}_m\|_{C^2}|v_m|^2)}{|v_m|} + O\left(\|\widetilde{H}_m\|_{C^1}|\frac{v_m}{|v_m|} - v|\right) \\
&\quad = O(\theta_m).
\end{aligned} \tag{2.13}
$$

This shows, using Condition 2.1(iv),

$$\lim_{l\to\infty}\frac{y_{2l+1}}{|v_{2l+1}|}=\lim_{l\to\infty}(D_0\widetilde{H}_{2l+1})v=v+\tau w,\tag{2.14}$$

and similarly,

$$\lim_{l\to\infty}\frac{y_{2l}}{|v_{2l}|}=\lim_{l\to\infty}(D_0\widetilde{H}_{2l})v=v.\tag{2.15}$$

Non-differentiability of $\widetilde{H}$: Assume for the sake of contradiction that $\widetilde{H}$ is differentiable at 0. Then, as $\widetilde{H}(v_m)=y_m$ as well,

$$\begin{aligned}\frac{y_m}{|v_m|}&-(D_0\widetilde{H})\\&=\bigg(\frac{\widetilde{H}(v_m)}{|v_m|}-\frac{(D_0\widetilde{H})v_m}{|v_m|}\bigg)+\bigg((D_0\widetilde{H})(\frac{v_m}{|v_m|}-v)\bigg)\\&=\frac{o_{\widetilde{H}}(|v_m|)}{|v_m|}+O_{\widetilde{H}}\bigg(\bigg|\frac{v_m}{|v_m|}-v\bigg|\bigg)\to 0\end{aligned}\tag{2.16}$$

as $m\to\infty$. This contradicts equations (2.14) and (2.15) where different subsequences of $y_m/|v_m|$ have different limits. Therefore, $\widetilde{H}$ cannot be differentiable at 0.

Non-differentiability of $\widetilde{H}^{-1}$: By equation (2.14), $\lim_{l\to\infty}(|y_{2l+1}|/|v_{2l+1}|)=|v+\tau w|$. Thus,

$$\lim_{l\to\infty}\frac{y_{2l+1}}{|y_{2l+1}|}=\lim_{l\to\infty}\frac{|v_{2l+1}|}{|y_{2l+1}|}\cdot\lim_{l\to\infty}\frac{y_{2l+1}}{|v_{2l+1}|}=\frac{v+\tau w}{|v+\tau w|}\tag{2.17}$$

and

$$\lim_{l\to\infty}\frac{v_{2l+1}}{|y_{2l+1}|}=\lim_{l\to\infty}\frac{|v_{2l+1}|}{|y_{2l+1}|}\cdot\lim_{l\to\infty}\frac{v_{2l+1}}{|v_{2l+1}|}=\frac{v}{|v+\tau w|}.\tag{2.18}$$

However, using v_m^* instead, we can define $y_m^*=\widetilde{H}(v_m^*)=\widetilde{H}_m(y_m^*)$ as in equation (2.12). Then $|v_m^*|\to 0$ and $|y_m^*|\to 0$ as $m\to\infty$. The same computations in equations (2.13), (2.14), and (2.15) give rise to, in lieu of equation (2.16),

$$\begin{aligned}\lim_{l\to\infty}&\frac{y_{2l}^*}{|v_{2l}^*|}\\&=\lim_{l\to\infty}(D_0\widetilde{H}_{2l})\frac{v+\tau w}{|v+\tau w|}\\&=\lim_{l\to\infty}\frac{(D_0\widetilde{H}_{2l})v+\tau(D_0\widetilde{H}_{2l})w}{|v+\tau w|}.\end{aligned}\tag{2.19}$$

If $w=v$, then

$$\begin{aligned}\lim_{l\to\infty}\frac{y_{2l}^*}{|v_{2l}^*|}&=\frac{(1+\tau)\lim_{l\to\infty}(D_0\widetilde{H}_{2l})v}{|(1+\tau)v|}\\&=\frac{(1+\tau)v}{|(1+\tau)v|}=v.\end{aligned}$$

Therefore, $\lim_{l\to\infty}(y_{2l}^*/|v_{2l}^*|) = 1$ and

$$\begin{cases} \lim_{l\to\infty} \dfrac{y_{2l}^*}{|y_{2l}^*|} = v = \lim_{l\to\infty} \dfrac{y_{2l+1}}{|y_{2l+1}|} \\ \lim_{l\to\infty} \dfrac{v_{2l}^*}{|y_{2l}^*|} = \lim_{l\to\infty} \dfrac{v_{2l}^*}{|v_{2l}^*|} = v \neq \dfrac{v}{1+\tau} = \lim_{l\to\infty} \dfrac{v_{2l+1}}{|y_{2l+1}|}. \end{cases} \tag{2.20}$$

If $w \neq v$, then by equation (2.19) and properties (iv), (v) of Condition 2.1,

$$\lim_{l\to\infty} \frac{y_{2l}^*}{|v_{2l}^*|} = \lim_{l\to\infty} \frac{(v+\tau w)}{|v+\tau w|} = \frac{v+\tau w}{|v+\tau w|},$$

and therefore, $\lim_{l\to\infty}(y_{2l}^*/|v_{2l}^*|) = 1$ and

$$\begin{cases} \lim_{l\to\infty} \dfrac{y_{2l}^*}{|y_{2l}^*|} = \dfrac{v+\tau w}{|v+\tau w|} = \lim_{l\to\infty} \dfrac{y_{2l+1}}{|y_{2l+1}|} \\ \lim_{l\to\infty} \dfrac{v_{2l}^*}{|y_{2l}^*|} = \lim_{l\to\infty} \dfrac{v_{2l}^*}{|v_{2l}^*|} = \dfrac{v+\tau w}{|v+\tau w|} \neq \dfrac{v}{|v+\tau w|} = \lim_{l\to\infty} \dfrac{v_{2l+1}}{|y_{2l+1}|}. \end{cases} \tag{2.21}$$

As $v_{2l}^* = \widetilde{H}^{-1}(y_{2l}^*)$ and $v_{2l+1} = \widetilde{H}^{-1}(y_{2l+1})$, in both the cases of equations (2.20) and (2.21), the same argument as in equation (2.16) shows $\widetilde{H}^{-1}$ is not differentiable at 0 either. □

2.3. *Fulfillment of the inductive conditions.* We will construct the sequence $\{h_m\}_{m=1}^\infty$ based on the following proposition.

PROPOSITION 2.4. *If the linear action $\rho : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ contains no hyperbolic automorphism, then there exist unit vectors $v, w \in \mathbb{R}^d$, such that for all $\delta > 0$ and $Q \in \mathbb{N}$, there exists a C^∞ function $h : \mathbb{T}^d \to \mathbb{R}^d$, such that:*

(1) $h(x) = 0$ *for all* $x \in ((1/Q)\mathbb{Z}^d)/\mathbb{Z}^d \subseteq \mathbb{T}^d$;
(2) $(D_0 h)v = w$; *in addition, either* $v = w$ *or* $(D_0 h)w = 0$;
(3) $\|h\|_{C^0} < \delta$ *and* $\|h\|_{C^1} \ll 1$;
(4) *for all* $\mathbf{n} \in \Xi$, $g^{\mathbf{n}} := \rho^{\mathbf{n}} h - h \circ \rho^{\mathbf{n}}$ *satisfies* $\|g^{\mathbf{n}}\|_{C^1} < \delta$.

The proof of the proposition will be deferred to §3.

PROPOSITION 2.5. *Suppose the linear action $\rho : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ contains no hyperbolic automorphism and v, w are as in Proposition 2.4. Then for all sufficiently small $\tau > 0$ and positive numbers $\{\theta_m\}_{m=1}^\infty$ that satisfy $\sum_{m=1}^\infty \theta_m < \tau$, there exist sequences $\{h_m\}_{m=1}^\infty$, $\{v_m\}_{m=1}^\infty$ and $\{v_m^*\}_{m=1}^\infty$ that satisfy Condition 2.1.*

Proof. Part (i) is already assumed. So we only need to fulfill the remaining assumptions from Condition 2.1.

To inductively construct h_m, assume for all $1 \le m' \le m-1$, there exist a C^∞ function $h_{m'}$, and non-zero vectors $v_{m'}, v_{m'}^* \in \mathbb{Q}^d$ that satisfy, together with v, w, the remaining properties from Condition 2.1. Then the diffeomorphism $\widetilde{H}_{m'}$ is also determined for all $1 \le m' \le m-1$ by equation (2.1). Remark that with the convention $\widetilde{H}_0 = \mathrm{id}$, the

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requirements of $(D_0\widetilde{H}_m)v = v$ and $(D_0\widetilde{H}_m)w = w$ from parts (iv) and (v) of the condition are satisfied at the initial step $m = 0$.

Let

$$\delta_m = \frac{\theta_m}{\max\big(\big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}^{-1}\|_{C^1}\big)\big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}\|_{C^1}\big), \|\widetilde{H}_{m-1}\|_{C^2}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\big)} \tag{2.22}$$

and Q_m be the least common multiple of the denominators of $v_1, \ldots, v_{m-1}, v_1^*, \ldots, v_{m-1}^* \in \mathbb{Q}^d$. We obtain a C^∞ function $\mathring{h}_m$ by applying Proposition 2.4 with parameters δ_m and Q_m, and define

$$h_m = \begin{cases} \tau\mathring{h}_m & \text{if } m \text{ is odd,} \\ \dfrac{-\tau}{1+\tau}\mathring{h}_m & \text{if } v = w \text{ and } m \text{ is even,} \\ -\tau\mathring{h}_m & \text{if } v \neq w \text{ and } m \text{ is even.} \end{cases} \tag{2.23}$$

It in turn determines $H_m = \mathrm{id} + h_m$ and $\widetilde{H}_m = \widetilde{H}_{m-1} \circ H_m$. Remark that $|-\tau/(1+\tau)| < \tau$.

We claim h_m, H_m, and $\widetilde{H}_m$ satisfy the clauses (ii)–(vii) in Condition 2.1:

(ii) $\|h_m\|_{C^1} \le \tau\|\mathring{h}_m\|_{C^1} \ll \tau$ and

$$\begin{aligned} &\Big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}^{-1}\|_{C^1}\Big)\Big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}\|_{C^1}\Big)\|h_m\|_{C^0} \\ &\quad\le \Big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}^{-1}\|_{C^1}\Big)\Big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}\|_{C^1}\Big)\cdot\tau\|\mathring{h}_m\|_{C^1} \\ &\quad< \tau\Big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}^{-1}\|_{C^1}\Big)\Big(\max_{m'=1}^{m-1}\|\widetilde{H}_{m'}\|_{C^1}\Big)\delta_m = \tau\theta_m < \theta_m. \end{aligned}$$

(iii) For all $\mathbf{n} \in \Xi$, with $\mathring{g}_m^{\mathbf{n}} = \mathring{h}_m \circ \rho^{\mathbf{n}} - \rho^{\mathbf{n}}\mathring{h}_m$,

$$\begin{aligned} &\|\widetilde{H}_{m-1}\|_{C^2}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\|g_m^{\mathbf{n}}\|_{C^1} \\ &\quad\le \tau\|\widetilde{H}_{m-1}\|_{C^2}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\|\mathring{g}_m^{\mathbf{n}}\|_{C^1} \\ &\quad\le \tau\|\widetilde{H}_{m-1}\|_{C^2}\|\widetilde{H}_{m-1}^{-1}\|_{C^1}\delta_m < \tau\theta_m < \theta_m. \end{aligned}$$

(iv) Since $0 \in ((1/Q)\mathbb{Z}^d)/\mathbb{Z}^d$, $\mathring{h}_m(0) = 0$ and thus $h_m(0) = 0$. As it was assumed that $h_1(0) = \cdots = h_{m-1}(0) = 0$, we know $H_1(0) = \cdots = H_m(0) = 0$ and $\widetilde{H}_m(0) = \widetilde{H}_{m-1}(0) = 0$. So

$$(D_0\widetilde{H}_m)v = (D_0\widetilde{H}_{m-1})(D_0H_m)v = (D_0\widetilde{H}_{m-1})(v + (D_0h_m)v).$$

If m is odd and $v = w$, then $v + (D_0h_m)v = v + \tau(D_0\mathring{h}_m)v = (1+\tau)v$, and by inductive assumption, $(D_0\widetilde{H}_{m-1})v = v$. So $(D_0\widetilde{H}_m)v = (D_0\widetilde{H}_{m-1})((1+\tau)v) = v + \tau v = v + \tau w$.

If m is even and $v = w$, then $v + (D_0h_m)v = v - \tau/(1+\tau)(D_0\mathring{h}_m)v = v - (\tau/(1+\tau))v = v/(1+\tau)$, and by inductive assumption, $(D_0\widetilde{H}_{m-1})v = v + \tau w = (1+\tau)v$. So $(D_0\widetilde{H}_m)v = (D_0\widetilde{H}_{m-1})(v/(1+\tau)) = v$.

If m is odd and $v \neq w$, then $v + (D_0h_m)v = v + \tau(D_0\mathring{h}_m)v = v + \tau w$, and by inductive assumption, $(D_0\widetilde{H}_{m-1})v = v$, $(D_0\widetilde{H}_{m-1})w = w$. So $(D_0\widetilde{H}_m)v = (D_0\widetilde{H}_{m-1})(v + \tau w) = v + \tau w$.

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If m is even and $v \neq w$, then $v + (D_0 h_m)v = v - \tau(D_0 \mathring{h}_m)v = v - \tau w$, and by inductive assumption, $(D_0\widetilde{H}_{m-1})v = v + \tau w$, $(D_0\widetilde{H}_{m-1})w = w$. So $(D_0\widetilde{H}_m)v = (D_0\widetilde{H}_{m-1})(v - \tau w) = (v + \tau w) - \tau \cdot w = v$.

Therefore, we have proved that property (iv) continues to hold at the mth step in all cases.

(v) Suppose $v \neq w$. Then $(D_0\mathring{h}_m)w = 0$ and, thus, $(D_0 h_m)w = 0$ too. So $(D_0 H_m)w = (\mathrm{id} + (D_0 h_m))w = w$. Since by inductive assumption $(D_0\widetilde{H}_{m-1})w = w$, we still have $(D_0\widetilde{H}_m)w = (D_0\widetilde{H}_{m-1})(D_0 H_m)w = w$.

(vi) By the choice of Q_m, we know $v_{m'}$, $v^*_{m'}$ are in $((1/Q_m)\mathbb{Z})^d$ for all $1 \leq m' \leq m-1$. By Proposition 2.4, $\mathring{h}_m(v_{m'}) = \mathring{h}_m(v^*_{m'}) = 0$. So $h_m(v_{m'}) = h_m(v^*_{m'}) = 0$ as h_m is proportional to $\mathring{h}_m$.

(vii) Now that h_m and $\widetilde{H}_m$ have been constructed, to finish the inductive step, it remains to choose rational vectors v_m, v^*_m that meet the requirement of property (vii), which can obviously be achieved. In fact, it suffices to take any rational vector $u \in \mathbb{Q}^d$ such that $|u - v| < \theta_m/2\|\widetilde{H}_m\|_{C^1}$, and set $v_m = u/L$ for any sufficiently large integer $L > 2\|\widetilde{H}_m\|_{C^1}/\theta_m$. Additionally, v^*_m can be similarly chosen near the direction of $v + \tau w$. □

Proof of Theorem 1.1. Theorem 1.1 immediately follows from Propositions 2.2, 2.4, and 2.5. □

3. *Cocycles with small coboundaries*

In this section, we complete the only still missing component of the argument, namely the proof of Proposition 2.4.

3.1. *The linear algebra of commuting integer matrices.*

The linear algebra of the action ρ is characterized by the following basic fact.

LEMMA 3.1. *Suppose $\rho : \mathbb{Z}^r \to \mathrm{GL}_d(\mathbb{Z})$ is a representation of $\mathbb{Z}^r$ in the group of toral automorphism of $\mathbb{T}^d$. Then for some $J_1, J_2 \geq 0$ and every $1 \leq j \leq J_1 + 2J_2$, there exist:*

- *a number field $\mathbb{F}_j$ embedded in $\mathbb{L}_j$, where $\mathbb{L}_1 = \cdots = \mathbb{L}_{J_1} = \mathbb{R}$ and $\mathbb{L}_{J_1+1} = \cdots = \mathbb{L}_{J_1+2J_2} = \mathbb{C}$;*
- *a positive dimension $d_j \geq 1$;*
- *a group morphism $\zeta_j : \mathbf{n} \to \zeta_j^{\mathbf{n}}$ from $\mathbb{Z}^r$ to the multiplicative group $\mathbb{F}_j^\times$ of $\mathbb{F}_j$;*
- *a group morphism $A_j : \mathbf{n} \to A_j^{\mathbf{n}}$ from $\mathbb{Z}^r$ to the group $\mathrm{N}_{d_j}(\mathbb{F}_j)$ of upper triangular nilpotent matrices in $\mathrm{SL}_{d_j}(\mathbb{F}_j)$;*
- *a linear transform $\mu_j \in \mathrm{Mat}_{d_j \times d}(\mathbb{F}_j)$;*

such that:

(1) *$\{\zeta_j^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^r\} \not\subseteq \mathbb{R}$ generates $\mathbb{F}_j$ as a number field, and spans $\mathbb{L}_j$ over $\mathbb{R}$;*

(2) *$\zeta_1, \ldots, \zeta_{J_1+2J_2}$ are distinct and this list is invariant under the action by the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Actually, for all $1 \leq j \leq J_1 + 2J_2$ and $\sigma \in \mathrm{Gal}(\mathbb{F}_j/\mathbb{Q})$, there exists a unique $1 \leq j' \leq J_1 + 2J_2$ such that $\sigma(\mathbb{F}_j) = \mathbb{F}_{j'}$, $d_j = d_{j'}$, $\sigma(\zeta_j^{\mathbf{n}}) = \zeta_{j'}^{\mathbf{n}}$, $\sigma(A_j^{\mathbf{n}}) = A_{j'}^{\mathbf{n}}$ and $\sigma(\mu_j) = \sigma(\mu_{j'})$;*

(3) *$\overline{\zeta_j^{\mathbf{n}}} = \zeta_{J_2+j}^{\mathbf{n}}$ for all $J_1 \leq j \leq J_1 + J_2$, $\mathbf{n} \in \mathbb{Z}^r$;*

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(4) *with* $\iota_j = \mu_j$ *for* $1 \le j \le J_1$ *and* $\iota_j = 2\,\mathrm{Re}\,\mu_j$ *for* $J_1 + 1 \le j \le J_1 + J_2$*, the linear transform* $\iota = \bigoplus_{j=1}^{J_1+J_2} \iota_j$ *from* $\bigoplus_{j=1}^{J_1+J_2} \mathbb{L}_j^{d_j}$ *to* $\mathbb{R}^d$ *is an* $\mathbb{R}$*-linear isomorphism and satisfies*

$$\iota \circ \bigoplus_{j=1}^{J_1+J_2} \zeta_j^{\mathbf{n}} A_j^{\mathbf{n}} = \rho^{\mathbf{n}} \circ \iota.$$

The lemma should be a standard fact for experts. However, we still include the proof for completeness.

Proof. Thanks to the commutativity of $\mathbb{Z}^r$, it is easy to show (see e.g. the proof of [**RHW14**, Lemma 2.2]) that $\mathbb{C}^d = (\mathbb{R}^d) \otimes_{\mathbb{R}} \mathbb{C}$ splits as a direct sum $\bigoplus_{j=1}^{\widetilde{J}} E_j^{\mathbb{C}}$, where each $E_j^{\mathbb{C}}$ is a maximal common generalized eigenspace of all the $\rho^{\mathbf{n}}$ terms. More precisely, for every j, there exists a group morphism from $\mathbb{Z}^r$: ζ_j to $\mathbb{C}^\times$ such that

$$E_j^{\mathbb{C}} = \bigcap_{\mathbf{n} \in \mathbb{Z}^r} \ker_{\mathbb{C}^d} (\rho^{\mathbf{n}} - \zeta_j^{\mathbf{n}} \mathrm{id})^d = \bigcap_{\mathbf{n} \in \Xi} \ker_{\mathbb{C}^d} (\rho^{\mathbf{n}} - \zeta_j^{\mathbf{n}} \mathrm{id})^d. \tag{3.1}$$

(1) Because $\rho^{\mathbf{n}} \in \mathrm{GL}_d(\mathbb{Z})$, every eigenvalue $\zeta_j^{\mathbf{n}}$ is an algebraic integer. Denote by $\mathbb{F}_j$ the field generated by $\{\zeta_j^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^r\}$, which is a number field as $\mathbb{Z}^r$ is finitely generated. Let $\mathbb{L}_j \in \{\mathbb{R}, \mathbb{C}\}$ be the $\mathbb{R}$-span of $\mathbb{F}_j$.

(2) As the $\rho^{\mathbf{n}}|_{E_j^{\mathbb{C}}}$ terms commute, they can be triangularized simultaneously over $\mathbb{C}$. Actually, equation (3.1) asserts that $E_j^{\mathbb{C}}$ is a linear subspace defined over $\mathbb{F}_j$. Together with the fact that the $\rho^{\mathbf{n}} \in \mathrm{GL}_d(\mathbb{Z})$, this shows that the simultaneous triangularization can be carried out over $\mathbb{F}_j$. In other words, one can find a basis $y_{j1}, \ldots, y_{jd_j} \in E_j^{\mathbb{C}} \cap \mathbb{F}_j^d$ of $E_j^{\mathbb{C}}$, such that the linear isomorphism $\mu_j : \mathbb{C}^{d_j} \to E_j^{\mathbb{C}}$ sending the kth coordinate vector to y_{jk} satisfies

$$\rho^{\mathbf{n}} \circ \mu_j = \mu_j \circ (\zeta_j^{\mathbf{n}} A_j^{\mathbf{n}}). \tag{3.2}$$

Note that μ_j is actually a matrix with coefficients in $\mathbb{F}_j$.

Moreover, we can make the choices above equivariant under Galois conjugacies. Indeed, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the correspondence $\mathbf{n} \to \sigma(\zeta_j^{\mathbf{n}})$ is a group morphism from $\mathbb{Z}^r$ to $\sigma(\mathbb{F}_j)^\times$. By equation (3.1), $\sigma(E_j^{\mathbb{C}} \cap \overline{\mathbb{Q}}^d) = \bigcap_{\mathbf{n} \in \mathbb{Z}^r} \ker_{\overline{\mathbb{Q}}^d} (\rho^{\mathbf{n}} - \sigma(\zeta_j^{\mathbf{n}})\mathrm{id})^d$ is a non-empty $\overline{\mathbb{Q}}$ subspace of dimension $\dim_{\mathbb{C}} E_j^{\mathbb{C}}$ and its $\mathbb{C}$-span is $\bigcap_{\mathbf{n} \in \mathbb{Z}^r} \ker_{\mathbb{C}^d} (\rho^{\mathbf{n}} - \sigma(\zeta_j^{\mathbf{n}})\mathrm{id})^d$, which is $E_{j'}^{\mathbb{C}}$ for some $1 \le j' \le \widetilde{J}$. (Note $j = j'$ if and only if σ fixes every $\zeta_j^{\mathbf{n}}$, or equivalently σ acts trivially on $\mathbb{F}_j$.) In this case, $d_{j'} = d_j$ and $\zeta_{j'}^{\mathbf{n}} = \sigma(\zeta_j^{\mathbf{n}})$. Furthermore, one may choose the basis $y_{j1}, \ldots, y_{jd_j}$ for all the indices j in such a way that, in the situation above, $y_{j'k} = \sigma(y_{jk})$ for $1 \le k \le d_j$, or equivalently $\mu_{j'} = \sigma(\mu_j)$. Then applying σ to equation (3.2) yields

$$\rho^{\mathbf{n}} \circ \mu_{j'} = \mu_{j'} \circ (\zeta_{j'}^{\mathbf{n}} \sigma(A_j^{\mathbf{n}})).$$

Since $\mu_{j'}$ is a linear embedding, this forces $A_{j'}^{\mathbf{n}} = \sigma(A_j^{\mathbf{n}})$.

(3) By choice, $\zeta_1, \ldots, \zeta_{\widetilde{J}}$ are distinct. Additionally, the previous paragraph shows that, by letting $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the complex conjugation, each $\overline{\zeta_j}$ is also in the list.

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Remark that $\zeta_j = \overline{\zeta_j}$ if and only if $\{\zeta_j^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^r\} \subseteq \mathbb{R}$, or equivalently $\mathbb{F}_j = \mathbb{R}$. After rearranging the list, we may assume that there are J_1, J_2 such that $J_1 + 2J_2 = \widetilde{J}$, $\mathbb{F}_j = \mathbb{R}$ assume real values for $j = 1, \ldots J_1$; and that $\mathbb{F}_{J_2+j} = \mathbb{F}_j = \mathbb{C}$ and $\zeta_{J_2+j} = \overline{\zeta_j}$ for $j = J_1 + 1, \ldots, J_1 + J_2$.

(4) As in the statement, set $\iota_j = \mu_j$ for $1 \le j \le J_1$ and $\iota_j = 2\,\mathrm{Re}\,\mu_j$ for $J_1 + 1 \le j \le J_1 + J_2$. To show $\iota \circ \bigoplus_{j=1}^{J_1+J_2} \zeta_j^{\mathbf{n}} A_j^{\mathbf{n}} = \rho^{\mathbf{n}} \circ \iota$, we need for each $1 \le j \le J_2$ that

$$\rho^{\mathbf{n}} \circ \iota_j = \iota_j \circ (\zeta_j^{\mathbf{n}} A_j^{\mathbf{n}}). \tag{3.3}$$

For $1 \le j \le J_1$, this is just equation (3.2). For $J_1 + 1 \le j \le J_1 + J_2$, let $u \in \mathbb{C}^{d_j}$, because $\rho^{\mathbf{n}}$ is a real matrix, for all $\mathbf{n} \in \mathbb{Z}^r$ and $z \in \mathbb{C}^{d_j}$,

$$\begin{aligned}\rho^{\mathbf{n}}(\iota_j(z)) &= \rho^{\mathbf{n}}(2\,\mathrm{Re}\,\mu_j(z)) = 2\,\mathrm{Re}\,\rho^{\mathbf{n}}(\mu_j(z))\\ &= 2\,\mathrm{Re}\,\mu_j(\zeta_j^{\mathbf{n}} A_j^{\mathbf{n}} z) = \iota_j(\zeta_j^{\mathbf{n}} A_j^{\mathbf{n}} z).\end{aligned}$$

So equation (3.3) holds for all $1 \le j \le J_1 + J_2$.

It remains to show that ι is an isomorphism. Recall that $\mathbb{C}^d = \bigoplus_{j=1}^{J_1+2J_2} E_j^{\mathbb{C}}$ is a direct sum. However, the image of $\iota_j = \mu_j$ is contained in $E_j^{\mathbb{C}}$ for $1 \le j \le J_1$; and the image of $\iota_j = 2\,\mathrm{Re}\,\mu_j = \mu_j + \overline{\mu_j} = \mu_j + \mu_{J_2+j}$ is contained in $E_j^{\mathbb{C}} \oplus E_{J_2+j}^{\mathbb{C}}$ for $J_1 + 1 \le j \le J_1 + J_2$. Hence, the images of ι is the direct sum $\bigoplus_{j=1}^{J_1+J_2} \iota_j(\mathbb{L}_j^{d_j})$.

In addition, we claim each ι_j is injective. This is obvious in the case $1 \le j \le J_1$, where $\iota_j = \mu_j$. For $J_1 + 1 \le j \le J_1 + J_2$, if $\iota_j = 2\,\mathrm{Re}\,\mu_j$ is not injective, then $\mu_j(z) = -\overline{\mu_j(z)}$ for some non-zero $z \in \mathbb{C}^{d_j}$. However, $\mu_j(z) \ne 0$, as μ_j is an embedding. This shows $E_j^{\mathbb{C}} \cap E_{J_1+j}^{\mathbb{C}} \ne \{0\}$ as $\mu_j(z) \in E_j^{\mathbb{C}}$ and $\overline{\mu_j(z)} \in E_{J_2+j}^{\mathbb{C}}$, which contradicts the fact that $\bigoplus_{j=1}^{J_1+2J_2} E_j^{\mathbb{C}}$ is a direct sum. Hence, ι_j is injective for all $1 \le j \le J_1 + J_2$.

So we may conclude that $\iota = \bigoplus_{j=1}^{J_1+J_2} \iota_j$ is injective from $\bigoplus_{j=1}^{J_1+J_2} \mathbb{L}_j^{d_j}$ to $\mathbb{R}^d$. As

$$\begin{aligned}\dim_{\mathbb{R}} \bigoplus_{j=1}^{J_1+J_2} \mathbb{L}_j^{d_j} &= \sum_{j=1}^{J_1} d_j + \sum_{j=J_1+1}^{J_1+J_2} 2d_j = \sum_{j=1}^{J_1+2J_2} d_j = \dim_{\mathbb{C}} \bigoplus_{j=1}^{J_1+2J_2} E_j^{\mathbb{C}}\\ &= \dim_{\mathbb{C}} \mathbb{C}^d = d,\end{aligned}$$

ι must be a linear isomorphism. The proof is completed. □

COROLLARY 3.2. *Suppose $1 \le k \le J_1 + J_2$ and P is a $\mathbb{L}_k$-vector subspace defined over $\mathbb{Q}$ of the kth component $\mathbb{L}_k^{d_k}$ in $\bigoplus_{j=1}^{J_1+J_2} \mathbb{L}_j^{d_j}$, then there exists a subspace $P' \subset \mathbb{R}^d$ defined over $\mathbb{Q}$ such that $P = \iota_k^{-1}(P')$.*

Proof. Choose a linear basis $\{p_1, \ldots, p_N\}$ of P from $\mathbb{Q}^{d_k} \subset \mathbb{L}_k^{d_k}$.

There are $j_1, \ldots, j_{M_1} \in \{1, \ldots, J_1\}$ and $j_{M_1+1}, \ldots, j_{M_1+M_2} \in \{J_1 + 1, \ldots, J_1 + J_2\}$ such that, after defining $j_{M_2+m} = J_2 + j_m$ for every $M_1 + 1 \le m \le M_1 + M_2$, $\{\zeta_{j_1}, \ldots, \zeta_{j_{M_1+2M_2}}\}$ form the orbit of ζ_k under the action by the Galois group $\mathrm{Gal}(\mathbb{F}_k/\mathbb{Q})$. For each m, let $\sigma_m \in \mathrm{Gal}(\mathbb{F}_k/\mathbb{Q})$ be the element such that $\sigma_m(\zeta_k) = \zeta_{j_m}$.

Define $(P')^{\mathbb{C}} \subseteq \mathbb{C}^d$ as the $\mathbb{C}$-linear span of

$$\{\mu_{j_m}(p_n) : 1 \le m \le M_1 + 2M_2,\ 1 \le n \le N\}. \tag{3.4}$$

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Because $\mu_{j_m} = \sigma_m(\mu_k)$ and has image in $E^{\mathbb{C}}_{j_m}$, these vectors have algebraic entries and are linearly independent, and this set is invariant by Galois conjugacies from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence, $(P')^{\mathbb{C}}$ is defined over $\mathbb{Q}$ of dimension $(M_1 + 2M_2)N$. The intersection $P' := (P')^{\mathbb{C}} \cap \mathbb{R}^d$ is a real vector space defined over $\mathbb{Q}$ over the same dimension.

For each p_n, $\iota_k(p_n)$ is either $\mu_k(p_n)$ if $1 \le k \le J_1$ or $2\,\mathrm{Re}\,\mu_k(p_n) = \mu_k(p_n) + \mu_{J_2+k}(p_n)$ if $J_1 + 1 \le k \le J_1 + J_2$. In these cases, either k or both k and $J_2 + k$ are among the list $\{j_1, \ldots, j_{M_1+2M_2}\}$. It follows that $\iota_k(p_n) \in (P')^{\mathbb{C}}$ and hence $\iota_k(p_n) \in P'$. We obtain that $P \subseteq \iota_k^{-1}(P')$.

It remains to show that the equality holds. If $1 \le k \le J_1$, then $\mathbb{L}_k = \mathbb{R}$ and $\iota_k(\mathbb{L}_k^{d_j}) = \mu_k(\mathbb{L}_k^{d_j}) \subseteq E_k^{\mathbb{C}}$. So $\iota_k(\iota_k^{-1}(P')) \subseteq P' \cap E_k^{\mathbb{C}}$. As $(P')^{\mathbb{C}} \cap E_k^{\mathbb{C}}$ is the $\mathbb{C}$-span of $\mu_k(p_1), \ldots, \mu_k(p_N)$, all of which are real vectors, $P' \cap E_k^{\mathbb{C}}$ is contained in the $\mathbb{R}$-span of them. Because ι_k is an embedding, $\dim_{\mathbb{R}} \iota_k^{-1}(P') \le N = \dim_{\mathbb{R}} P$. Assume instead $J_1 + 1 \le k \le J_1 + J_2$. Then $\mathbb{L}_k = \mathbb{C}$ and $\iota_k(\mathbb{L}_k^{d_j}) = (\mu_k + \mu_{J_2+k})(\mathbb{L}_k^{d_j}) \subseteq E_k^{\mathbb{C}} \oplus E_{J_2+k}^{\mathbb{C}}$. So $\iota_k(\iota_k^{-1}(P'))$ is contained in $P' \cap (E_k^{\mathbb{C}} \oplus E_{J_2+k}^{\mathbb{C}})$. As $(P')^{\mathbb{C}} \cap (E_k^{\mathbb{C}} \oplus E_{J_2+k}^{\mathbb{C}})$ is the $\mathbb{C}$-span of $\mu_k(p_1), \ldots, \mu_k(p_N), \mu_{J_2+k}(p_1), \ldots, \mu_{J_2+k}(p_N)$ and has complex dimension $2N$. Here, $P' \cap (E_k^{\mathbb{C}} \oplus E_{J_2+k}^{\mathbb{C}}) = \mathbb{R}^d \cap (P')^{\mathbb{C}} \cap (E_k^{\mathbb{C}} \oplus E_{J_2+k}^{\mathbb{C}})$ has real dimension $2N$. Again, since ι_k is injective, $\dim_{\mathbb{R}}(\iota_k^{-1}(P')) \le 2N = 2\dim_{\mathbb{C}} P = \dim_{\mathbb{R}} P$. We conclude that in both cases, $P = \iota_k^{-1}(P')$. $\square$

For $1 \le j \le J$, $1 \le k \le d_j$, write u_{jk} for the kth coordinate vector in $\mathbb{L}_j^{d_j}$, so that all vectors $s \in \bigoplus_{j=1}^{J} \mathbb{L}_j^{d_j}$ have the form

$$s = \bigoplus_{j=1}^{J} \sum_{k=1}^{d_j} \pi_{jk}(s) u_{jk}, \tag{3.5}$$

where π_{jk} is the projection to the u_{jk} coordinate.

Since none of the $\rho^{\mathbf{n}}$ terms is hyperbolic, there must be at least one j_0 such that $|\zeta_{j_0}^{\mathbf{n}}| = 1$ for all $\mathbf{n} \in \mathbb{Z}^r$. This is because otherwise, the linear functionals $\mathbf{n} \to \log|\zeta_{j_0}^{\mathbf{n}}|$ on $\mathbb{Z}^r$ are all non-zero and one can find one $\mathbf{n}_*$ that is not in the kernel of any of such functionals. Then $|\zeta_j^{\mathbf{n}_*}| \ne 1$ for all j. In other words, $\rho^{\mathbf{n}_*}$ has no eigenvalues in the unit circle, so $\rho^{\mathbf{n}_*}$ is a hyperbolic matrix, which contradicts our assumption.

After renormalizing ι if necessary, we may assume

$$|\iota(u_{j_0 d_{j_0}})| = 1.$$

We define vectors $\mathring{v}, \mathring{w} \in \mathbb{L}_{j_0}^{d_{j_0}}$ and $v, w \in \mathbb{R}^d$ by

$$\mathring{v} = u_{j_0 d_{j_0}}, \mathring{w} = \frac{u_{j_0 1}}{|\iota(u_{j_0 1})|}, \quad v = \iota(\mathring{v}), \quad w = \iota(\mathring{w}); \tag{3.6}$$

as well as projections $\pi_{\mathring{v}} : \bigoplus_{j=1}^{J} \mathbb{L}_j^{d_j} \to \mathbb{L}_{j_0}$ and $\psi_v \in (\mathbb{R}^d)^*$ by

$$\pi_{\mathring{v}} = \pi_{j_0 d_{j_0}}, \quad \psi_v = \mathrm{Re}\,\pi_{\mathring{v}} \circ \iota^{-1}. \tag{3.7}$$

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Note that

$$|v| = |w| = 1, \quad \psi_v(v) = 1. \tag{3.8}$$

In the case where $d_{j_0} = 1$, we have $w = v$ and $\psi_v(w) = \psi_v(v) = 1$. However, when $d_{j_0} > 1$, $\mathring{v} \neq \mathring{w}$ and thus $\pi_{\mathring{v}}(\mathring{w}) = 0$, so $\psi_v(w) = 0$. In summary,

$$\psi_v(w) = \mathbf{1}_{v=w}. \tag{3.9}$$

Let $W = \iota_{j_0}(\mathbb{L}_{j_0}\mathring{w})$, which is isomorphic to $\mathbb{L}_{j_0}$ as a real vector space. For all $\mathbf{n} \in \mathbb{Z}^r$ and $w' \in W$, since $w' = \iota(z\mathring{w})$ for some $z \in \mathbb{L}_{j_0}$, and $A_{j_0}^{\mathbf{n}}$ is an upper triangular nilpotent matrix, $A_{j_0}^{\mathbf{n}}\mathring{w} = \mathring{w}$ and thus

$$\rho^{\mathbf{n}} w' = \iota(\zeta_{j_0}^{\mathbf{n}} A_{j_0}^{\mathbf{n}} z\mathring{w}) = \iota(\zeta_{j_0}^{\mathbf{n}} z\mathring{w}) \in W.$$

So W is ρ-invariant and

$$|\rho^{\mathbf{n}} w'| \leq \|\iota\| |\zeta_{j_0}^{\mathbf{n}}| |z\mathring{w}| = \|\iota\| \cdot |z\mathring{w}| \ll |w'| \quad \text{for all } \mathbf{n} \in \mathbb{Z}^r, \text{ for all } w' \in W. \tag{3.10}$$

Furthermore, for $u \in \mathbb{L}_{d_{j_0}}^{j_0}$, $\pi_{\mathring{v}}(\zeta_{j_0}^{\mathbf{n}} A_{j_0}^{\mathbf{n}} u) = \zeta_{j_0}^{\mathbf{n}} \pi_{\mathring{v}}(u)$ and thus

$$\pi_{\mathring{v}}\bigg(\bigg(\bigoplus_{j=1}^{J} \zeta_j^{\mathbf{n}} A_j^{\mathbf{n}}\bigg) u\bigg) = \pi_{\mathring{v}}(\zeta_{j_0}^{\mathbf{n}} A_{j_0}^{\mathbf{n}} \pi_{\mathring{v}}(u)) = \zeta_{j_0}^{\mathbf{n}} \pi_{\mathring{v}}(u)$$

for all $u \in \bigoplus_{j=1}^{J} \mathbb{L}_j^{d_j}$. So

$$\begin{aligned}(\rho^{\mathbf{n}})^{\mathrm{T}} \psi_v &= \operatorname{Re} \pi_{\mathring{v}} \circ \iota^{-1} \circ \rho^{\mathbf{n}} = \operatorname{Re}\bigg(\pi_{\mathring{v}} \circ \bigoplus_{j=1}^{J} \zeta_j^{\mathbf{n}} A_j^{\mathbf{n}} \circ \iota^{-1}\bigg) \\ &= \operatorname{Re}(\zeta_{j_0}^{\mathbf{n}} \pi_{\mathring{v}} \circ \iota^{-1}). \end{aligned} \tag{3.11}$$

In particular, as $|\zeta_{j_0}^{\mathbf{n}}| = 1$, the size of $(\rho^{\mathbf{n}})^{\mathrm{T}} \psi_v \in (\mathbb{R}^d)^*$ is uniformly bounded by

$$|(\rho^{\mathbf{n}})^{\mathrm{T}} \psi_v| \leq \|\pi_{\mathring{v}} \circ \iota^{-1}\|. \tag{3.12}$$

If $d_{j_0} > 1$, by applying Corollary 3.2 to the $\mathbb{L}_{j_0}$-subspace $\bigoplus_{k=2}^{d_{j_0}} \mathbb{L}_{j_0} u_{j_0 k}$ of $\mathbb{L}_{j_0}^{d_{j_0}}$, there is a subspace $W' \subseteq \mathbb{R}^d$ defined over $\mathbb{Q}$ such that $\iota_{j_0}^{-1}(W') = \bigoplus_{k=2}^{d_{j_0}} \mathbb{L}_{d_{j_0}} u_{j_0 k}$. In particular, W' contains $W = \iota_{j_0}(\mathbb{L}_{j_0} u_{j_0 d_{j_0}})$. Set $\Psi = \{\psi \in (\mathbb{R}^d)^* : \psi|_{W'} = 0\}$. Then Ψ is a subspace defined over $\mathbb{Q}$, and

$$\psi|_W = 0 \quad \text{for all } \psi \in \Psi. \tag{3.13}$$

Moreover,

$$\iota^{-1}(W') \subseteq \bigg(\bigoplus_{k=2}^{d_{j_0}} \mathbb{L}_{d_{j_0}} u_{j_0 k}\bigg) \oplus \bigg(\bigoplus_{\substack{1 \leq j \leq J_1 + J_2 \\ j \neq j_0}} \mathbb{L}_j^{d_j}\bigg) = \ker \pi_{\mathring{v}}.$$

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It follows that $\psi_v = \operatorname{Re}\pi_{\mathring{v}} \circ \iota^{-1}$ annihilates W', or equivalently, $\psi_v \in \Psi$. Furthermore, for all $\mathbf{n} \in \mathbb{Z}^r$, we have

$$\rho^{\mathbf{n}} v = \iota(\zeta_{j_0}^{\mathbf{n}} A_{j_0}^{\mathbf{n}} \mathring{v}) = \iota(\zeta_{j_0}^{\mathbf{n}} \mathring{v}) + \iota(\zeta_{j_0}^{\mathbf{n}} (A_{j_0}^{\mathbf{n}} - \mathrm{id})\mathring{v}).$$

Because $A_{j_0}^{\mathbf{n}}$ is an upper triangular nilpotent matrix, $\zeta_{j_0}^{\mathbf{n}}(A_{j_0}^{\mathbf{n}} - \mathrm{id})\mathring{v} \in \bigoplus_{k=2}^{d_{j_0}} \mathbb{L}_{d_{j_0}} u_{j_0 k}$ and $\iota(\zeta_{j_0}^{\mathbf{n}}(A_{j_0}^{\mathbf{n}} - \mathrm{id})\mathring{v}) \in W'$. Thus,

$$\psi(\rho^{\mathbf{n}} v) = \psi(\iota(\zeta_{j_0}^{\mathbf{n}} \mathring{v})) \quad \text{for all } \psi \in \Psi. \tag{3.14}$$

If $d_{j_0} = 1$, take $\Psi = (\mathbb{R}^d)^*$ instead, which is also a rational subspace that contains ψ_v. Additionally, equation (3.14) remains true in this case, because $A_{j_0}^{\mathbf{n}} = \mathrm{id}$. To summarize, we have in any case the following corollary.

COROLLARY 3.3. *There exists a subspace $\Psi \subset (\mathbb{R}^d)^*$ defined over $\mathbb{Q}$ which contains ψ_v and satisfies equation (3.14). In addition, if $d_{j_0} > 1$, then equation (3.13) holds as well.*

It should be remarked that all the constructions above are determined by the actions ρ.

3.2. *The construction of the cocycle.* The construction is inspired by the construction of Veech in [**V86**, Proposition 1.5].

Let $\epsilon > 0$ be a small parameter to be specified later.

We identify $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$ in the standard way so that $(\mathbb{T}^d)^* \subset (\mathbb{R}^d)^*$ is realized as $\mathbb{Z}^d$. Let Ψ be as in Corollary 3.3. Then $\Psi_{\mathbb{Z}} := \Psi \cap \mathbb{Z}^d$ is a lattice in Ψ. There is a constant $R > 0$ such that for every $\psi \in \Psi$, there exists $\eta \in \Psi_{\mathbb{Z}}$ with $|\psi - \eta| < R$. The choice of R depends only on ρ.

Let η_v be the nearest vector to $(Q/\epsilon)\psi_v$ in the lattice $Q\Psi_{\mathbb{Z}}$. Then

$$\left|\eta_v - \frac{Q}{\epsilon}\psi_v\right| \le QR \ll Q. \tag{3.15}$$

Recall $W = \iota_{j_0}(\mathbb{L}_{j_0}\mathring{w})$, which is isomorphic to $\mathbb{L}_{j_0}$ as an $\mathbb{R}$-vector space and contains w. The function $h : \mathbb{T}^d \to \mathbb{R}^d$ will take value in $W \subseteq \mathbb{R}^d$ and have the form

$$h(x) = c \sum_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ |\mathbf{n}| \le N}} (e((\rho^{\mathbf{n}})^{\mathrm{T}} \eta_v \cdot x) - 1)\rho^{-\mathbf{n}} w + (e(\eta_v \cdot x) - 1) w_{\Delta} \tag{3.16}$$

for some $c > 0$, $N \in \mathbb{N}$, and $w_{\Delta} \in W$, all of which will be defined later. Remark that h is C^{∞} as it is a Fourier series supported on finitely many frequencies.

LEMMA 3.4. *If h has the form in equation (3.16), then property (1) in Proposition 2.4 holds.*

Proof. Since $\eta_v \in Q\Psi_{\mathbb{Z}} \subseteq Q\mathbb{Z}^d$ and $\rho^{\mathbf{n}} \in \mathrm{GL}(d, \mathbb{Z})$, $(\rho^{\mathbf{n}})^{\mathrm{T}} \eta_v \in (Q\mathbb{Z})^d$ for all $\mathbf{n}$. Moreover, if $x \in ((1/Q)\mathbb{Z}^d)/\mathbb{Z}^d$, then $e(\eta_v \cdot x) = 1$ and $e((\rho^{\mathbf{n}})^{\mathrm{T}} \eta \cdot x) = 1$ for all $\mathbf{n} \in \mathbb{Z}^r$. Therefore, $h(x) = 0$. This proves part (1). $\square$

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The derivative of equation (3.16) at $x = 0$ is the matrix

$$\begin{aligned} D_0 h &= c \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} ((\rho^{\mathbf{n}})^{\mathrm{T}} \eta_v) \otimes (\rho^{-\mathbf{n}} w) + \eta_v \otimes w_\Delta \\ &= c \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \bigg((\rho^{\mathbf{n}})^{\mathrm{T}} \frac{Q}{\epsilon} \psi_v \bigg) \otimes (\rho^{-\mathbf{n}} w) \\ &\quad + c \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \bigg((\rho^{\mathbf{n}})^{\mathrm{T}} \Big(\eta_v - \frac{Q}{\epsilon} \psi_v\Big) \bigg) \otimes (\rho^{-\mathbf{n}} w) \\ &\quad + \eta_v \otimes w_\Delta. \end{aligned} \tag{3.17}$$

We first study the values of the first two terms in equation (3.17) with v or w as linear input. By definition of v and w,

$$\begin{aligned} &\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} (((\rho^{\mathbf{n}})^{\mathrm{T}} \psi_v) \otimes (\rho^{-\mathbf{n}} w)) v \\ &\quad = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} (\psi_v \cdot (\rho^{\mathbf{n}} v))(\rho^{-\mathbf{n}} w) \\ &\quad = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \operatorname{Re} \pi_{\mathring{v}} \circ \iota^{-1}(\iota(\zeta_{j_0}^{\mathbf{n}} \mathring{v})) \cdot \iota(\zeta_{j_0}^{-\mathbf{n}} \mathring{w}) = \iota\bigg(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \operatorname{Re}(\zeta_{j_0}^{\mathbf{n}}) \zeta_{j_0}^{-\mathbf{n}} \mathring{w} \bigg) \\ &\quad = \frac{1}{2} \iota\bigg(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \zeta_{j_0}^{-\mathbf{n}} \cdot \zeta_{j_0}^{\mathbf{n}} \mathring{w} + \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \zeta_{j_0}^{-\mathbf{n}} \overline{\zeta_{j_0}^{\mathbf{n}}} \mathring{w} \bigg) \\ &\quad = \frac{1}{2} \iota^{-1}\bigg((2N+1)^r \mathring{w} + \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} \zeta_{j_0}^{-\mathbf{n}} \overline{\zeta_{j_0}^{\mathbf{n}}} \mathring{w} \bigg). \end{aligned} \tag{3.18}$$

If $\mathbb{L}_{j_0} = \mathbb{R}$, then $\mathring{w} \in \mathbb{R}^{d_{j_0}}$, $\zeta_{j_0}^{-\mathbf{n}} \overline{\zeta_{j_0}^{\mathbf{n}}} = 1$, and thus

$$\begin{aligned} &\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} (((\rho^{\mathbf{n}})^{\mathrm{T}} \psi_v) \otimes (\rho^{-\mathbf{n}} w)) v \\ &\quad = \frac{1}{2} \iota^{-1}((2N+1)^r \mathring{w} + (2N+1)^r \mathring{w}) = (2N+1)^r w. \end{aligned} \tag{3.19}$$

If $\mathbb{L}_{j_0} = \mathbb{C}$, then by Lemma 3.1(1), there is at least one $i \in \{1, \ldots, r\}$, say $i = 1$ without loss of generality, such that $\zeta_{j_0}^{\mathbf{e}_i} \notin \mathbb{R}$. Then $\overline{\zeta_{j_0}^{\mathbf{e}_1}} / \zeta_{j_0}^{\mathbf{e}_1}$ is in the unit circle but not equal to 1. In this case, $\sum_{n=-N}^{N} (\overline{\zeta_{j_0}^{\mathbf{e}_1}} / \zeta_{j_0}^{\mathbf{e}_1})^n$ is uniformly bounded when N varies. Therefore,

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$$\begin{aligned}\Bigg|\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\zeta_{j_0}^{-\mathbf{n}}\overline{\zeta_{j_0}^{\mathbf{n}}}\Bigg| &= \Bigg|\sum_{n_1,\ldots,n_r\in\{-N,\ldots,N\}}\prod_{i=1}^{r}(\zeta_{j_0}^{\mathbf{e}_i})^{-n_i}(\overline{\zeta_{j_0}^{\mathbf{e}_i}})^{n_i}\Bigg|\\ &= \Bigg|\prod_{i=1}^{r}\sum_{n=-N}^{N}\bigg(\frac{\overline{\zeta_{j_0}^{\mathbf{e}_i}}}{\zeta_{j_0}^{\mathbf{e}_i}}\bigg)^n\Bigg| = \prod_{i=1}^{r}\Bigg|\sum_{n=-N}^{N}\bigg(\frac{\overline{\zeta_{j_0}^{\mathbf{e}_i}}}{\zeta_{j_0}^{\mathbf{e}_i}}\bigg)^n\Bigg|\\ &\le (2N+1)^{r-1}\Bigg|\sum_{n=-N}^{N}\bigg(\frac{\overline{\zeta_{j_0}^{\mathbf{e}_1}}}{\zeta_{j_0}^{\mathbf{e}_1}}\bigg)^n\Bigg| \ll (2N+1)^{r-1}. \end{aligned}\tag{3.20}$$

So

$$\begin{aligned}&\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}(((\rho^{\mathbf{n}})^{\mathrm{T}}\psi_v)\otimes(\rho^{-\mathbf{n}}w))v\\ &\quad=\frac{1}{2}\iota((2N+1)^r\mathring{w}+O((2N+1)^{r-1})\mathring{w})\\ &\quad=\frac{(2N+1)^r}{2}\iota\bigg(\mathring{w}+O\bigg(\frac{1}{N}\bigg)\bigg)=\frac{(2N+1)^r}{2}\bigg(w+O\bigg(\frac{1}{N}\bigg)\bigg).\end{aligned}\tag{3.21}$$

Both equations (3.19) and (3.21) can be expressed as

$$\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}(((\rho^{\mathbf{n}})^{\mathrm{T}}\psi_v)\otimes(\rho^{-\mathbf{n}}w))v=\frac{(2N+1)^r}{\dim_{\mathbb{R}}\mathbb{L}_{j_0}}\bigg(w+O\bigg(\frac{1}{N}\bigg)\bigg).\tag{3.22}$$

We now attend to the second term in equation (3.17).

Since $\eta_v-(Q/\epsilon)\psi_v\in\Psi$, by equations (3.14), (3.15), and the fact that $|\zeta_{j_0}^{\mathbf{n}}|=1$,

$$\bigg|\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)\bigg)v\bigg|=\bigg|\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)(\iota(\zeta_{j_0}^{\mathbf{n}}\mathring{v}))\bigg|\ll\bigg|\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)\bigg|\ll Q.$$

Moreover, $|\rho^{-\mathbf{n}}w|\ll 1$ by equation (3.10). So

$$\begin{aligned}&\Bigg|\bigg(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)\bigg)\otimes(\rho^{-\mathbf{n}}w)\bigg)v\Bigg|\\ &\quad\le\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\bigg|\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)\bigg)v\bigg|\cdot|\rho^{-\mathbf{n}}w|\\ &\quad\ll(2N+1)^rQ.\end{aligned}\tag{3.23}$$

Choose

$$c=\frac{\epsilon\dim_{\mathbb{R}}\mathbb{L}_{j_0}}{(2N+1)^rQ}.\tag{3.24}$$

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Then by equations (3.22) and (3.23),

$$\begin{aligned}&\bigg(c\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\frac{Q}{\epsilon}\psi_v\bigg)\otimes(\rho^{-\mathbf{n}}w)\\&\quad+c\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)\bigg)\otimes(\rho^{-\mathbf{n}}w)\bigg)v\\&\quad=c\frac{Q}{\epsilon}\frac{(2N+1)^r}{\dim_{\mathbb{R}}\mathbb{L}_{j_0}}\bigg(w+O\bigg(\frac{1}{N}\bigg)\bigg)+cO((2N+1)^rQ)\\&\quad=w+O\bigg(\frac{1}{N}+\epsilon\bigg).\end{aligned}\tag{3.25}$$

To make $(D_0h)v=w$, one needs to find the solution $w_\Delta\in W$ to

$$\begin{aligned}\eta_v(v)w_\Delta&=(\eta_v\otimes w_\Delta)v\\&=-\bigg(\bigg(c\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\frac{Q}{\epsilon}\psi_v\bigg)\otimes(\rho^{-\mathbf{n}}w)\\&\qquad+c\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}\bigg((\rho^{\mathbf{n}})^{\mathrm{T}}\bigg(\eta_v-\frac{Q}{\epsilon}\psi_v\bigg)\bigg)\otimes(\rho^{-\mathbf{n}}w)\bigg)v-w\bigg),\end{aligned}\tag{3.26}$$

which by equation (3.25) is

$$w_\Delta=-\frac{1}{\eta_v(v)}O\bigg(\frac{1}{N}+\epsilon\bigg).$$

Since $\psi_v(v)=1$, by equation (3.15), $\eta_v(v)=Q/\epsilon+O(Q)=(Q/\epsilon)(1+O(\epsilon))$ and thus, we have

$$w_\Delta=\frac{1}{(Q/\epsilon)(1+O(\epsilon))}O\bigg(\frac{1}{N}+\epsilon\bigg)=O\bigg(\frac{\epsilon}{Q}\bigg(\frac{1}{N}+\epsilon\bigg)\bigg)\tag{3.27}$$

as long as $\epsilon\ll 1$. Note that w_Δ is automatically in W because equation (3.25) and $w\in W$.

Moreover, if $w\neq v$, or in other words $d_{j_0}=1$, then by Corollary 3.3 and the fact that $\eta_v\in\Psi$, $\eta_v|_W=0$. As $\rho^{\mathbf{n}}w\in W$ for all $\mathbf{n}$, in this case,

$$(D_0h)w=c\sum_{\substack{\mathbf{n}\in\mathbb{Z}^r\\|\mathbf{n}|\le N}}((\rho^{\mathbf{n}})^{\mathrm{T}}\eta_v)w)\cdot(\rho^{-\mathbf{n}}w)+\eta_v(w)\cdot w_\Delta=0.\tag{3.28}$$

LEMMA 3.5. *Given c and h respectively from equations (3.16) and (3.24), for $N,Q\in\mathbb{N}$ and sufficiently small $\epsilon\ll 1$, there exist $w_\Delta\in W$ of size $O((\epsilon/Q)(1/N+\epsilon))$ such that $(D_0h)v=w$. In addition, $(D_0h)w=0$ if $w\neq v$.*

The first part of part (3) in Property 2.4 is given by the following lemma.

LEMMA 3.6. *Suppose c, w_Δ, and h are chosen as above. Then $\|h\|_{C^0}\ll\epsilon/Q$.*

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Proof. By equations (3.16), (3.24), and Lemma 3.5,

$$\|h\|_{C^0} \ll c \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} |\rho^{-\mathbf{n}} w| + |w_\Delta|$$
$$\ll c(2N+1)^r + \frac{\epsilon}{Q}\bigg(\frac{1}{N}+\epsilon\bigg) \ll \frac{\epsilon}{Q} + \frac{\epsilon}{Q}\bigg(\frac{1}{N}+\epsilon\bigg) \ll \frac{\epsilon}{Q}. \qquad \square$$

To bound the C^1 norms of h and $g^{\mathbf{n}}$, write

$$\|\rho\| = \max_{\mathbf{n}\in\Xi} \|\rho^{\mathbf{n}}\| \ge 1$$

for the matrix norm of the linear action ρ, so that

$$\|\rho^{\mathbf{n}}\| \le \|\rho\|^{|\mathbf{n}|} \quad \text{for all } \mathbf{n} \in \mathbb{Z}^r. \tag{3.29}$$

For $\mathbf{n} \in \mathbb{Z}^r$, we deduce from equations (3.12) and (3.15) that

$$\begin{aligned} |(\rho^{\mathbf{n}})^{\mathrm{T}}\eta_v| &\le \bigg|(\rho^{\mathbf{n}})^{\mathrm{T}}\frac{Q}{\epsilon}\psi_v\bigg| + \bigg|(\rho^{\mathbf{n}})^{\mathrm{T}}\bigg(\eta_v - \frac{Q}{\epsilon}\psi_v\bigg)\bigg| \\ &\le \frac{Q}{\epsilon}|(\rho^{\mathbf{n}})^{\mathrm{T}}\psi_v| + \|\rho\|^{|\mathbf{n}|}\bigg|\eta_v - \frac{Q}{\epsilon}\psi_v\bigg| \\ &\ll \frac{Q}{\epsilon}(1+\|\rho\|^{|\mathbf{n}|}\epsilon). \end{aligned} \tag{3.30}$$

By the construction in equation (3.16) of h, Lemma 3.6, as well as the bounds in equations (3.10), (3.12), (3.27), and (3.30),

$$\begin{aligned} \|h\|_{C^1} &\ll \|h\|_{C^0} + c \sum_{\substack{\mathbf{n}\in\mathbb{Z}^r \\ |\mathbf{n}|\le N}} |(\rho^{\mathbf{n}})^{\mathrm{T}}\eta_v||\rho^{-\mathbf{n}}w| + |\eta_v||w_\Delta| \\ &\ll \frac{\epsilon}{Q} + c(2N+1)^r\frac{Q}{\epsilon}(1+\|\rho\|^N\epsilon) + \frac{Q}{\epsilon}\cdot\bigg(\frac{\epsilon}{Q}\bigg(\frac{1}{N}+\epsilon\bigg)\bigg) \\ &\ll \frac{\epsilon}{Q} + (1+\|\rho\|^N\epsilon) + \bigg(\frac{1}{N}+\epsilon\bigg) \ll 1 + \|\rho\|^N\epsilon. \end{aligned} \tag{3.31}$$

For every $\mathbf{n} \in \Xi$, $g^{\mathbf{n}} = \rho^{\mathbf{n}}h - h\circ\rho^{\mathbf{n}}$ is linearly controlled by h in C^0 norm:

$$\|g^{\mathbf{n}}\|_{C^0} \le |\rho^{\mathbf{n}}|\|h\|_{C^0} + \|h\|_{C^0} \ll \|h\|_{C^0} \ll \frac{\epsilon}{Q}. \tag{3.32}$$

In addition, $g^{\mathbf{n}}$ has the form

$$\begin{aligned} g^{\mathbf{n}} &= \bigg(c \sum_{\substack{\mathbf{a}\in\mathbb{Z}^r \\ |\mathbf{a}|\le N}} (e((\rho^{\mathbf{a}})^{\mathrm{T}}\eta_v\cdot x)-1)\rho^{\mathbf{n}-\mathbf{a}}w + (e(\eta_v\cdot x)-1)\rho^{\mathbf{n}}w_\Delta\bigg) \\ &\quad - \bigg(c \sum_{\substack{\mathbf{a}\in\mathbb{Z}^r \\ |\mathbf{a}|\le N}} (e((\rho^{\mathbf{a}})^{\mathrm{T}}\eta_v\cdot \rho^{\mathbf{n}}x)-1)\rho^{-\mathbf{a}}w + (e(\eta_v\cdot \rho^{\mathbf{n}}x)-1)w_\Delta\bigg) \\ &= \bigg(c \sum_{\substack{\mathbf{a}\in\mathbb{Z}^r \\ |\mathbf{a}+\mathbf{n}|\le N}} (e((\rho^{\mathbf{a}+\mathbf{n}})^{\mathrm{T}}\eta_v\cdot x)-1)\rho^{-\mathbf{a}}w + (e(\eta_v\cdot x)-1)\rho^{\mathbf{n}}w_\Delta\bigg) \end{aligned}$$

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$$
\begin{aligned}
&-\bigg(c\sum_{\substack{\mathbf{a}\in\mathbb{Z}^r\\|\mathbf{a}|\le N}}(e((\rho^{\mathbf{a}+\mathbf{n}})^{\mathrm{T}}\eta_v\cdot x)-1)\rho^{-\mathbf{a}}w+(e((\rho^{\mathbf{n}})^{\mathrm{T}}\eta_v\cdot x)-1)w_\Delta\bigg)\\
&=c\bigg(\sum_{\substack{\mathbf{a}\in\mathbb{Z}^r\\|\mathbf{a}|>N,|\mathbf{a}+\mathbf{n}|\le N}}-\sum_{\substack{\mathbf{a}\in\mathbb{Z}^r\\|\mathbf{a}|\le N,|\mathbf{a}+\mathbf{n}|>N}}\bigg)(e((\rho^{\mathbf{a}+\mathbf{n}})^{\mathrm{T}}\eta_v\cdot x)-1)\rho^{-\mathbf{a}}w\\
&\quad+((e(\eta_v\cdot x)-1)\rho^{\mathbf{n}}w_\Delta-(e((\rho^{\mathbf{n}})^T\eta_v\cdot x)-1)w_\Delta).
\end{aligned}\tag{3.33}
$$

Because $\mathbf{n}\in\Xi$, the summations $\sum_{\substack{\mathbf{a}\in\mathbb{Z}^r\\|\mathbf{a}|>N,|\mathbf{a}+\mathbf{n}|\le N}}$ and $\sum_{\substack{\mathbf{a}\in\mathbb{Z}^r\\|\mathbf{a}|\le N,|\mathbf{a}+\mathbf{n}|>N}}$ each has $O(N^{r-1})$ terms. Since $|\mathbf{n}|=1$ for all $\mathbf{n}\in\Xi$, in all the terms in both summations, $|\mathbf{a}|\le N+1$ and $|\mathbf{a}+\mathbf{n}|\le N+1$. For each of these terms, the derivative is bounded by

$$
\begin{aligned}
&\|D(e((\rho^{\mathbf{a}+\mathbf{n}})^{\mathrm{T}}\eta_v\cdot x)-1)\rho^{-\mathbf{a}}w)\|_{C^1}\\
&\quad\le|(\rho^{\mathbf{a}+\mathbf{n}})^{\mathrm{T}}\eta_v|\cdot|\rho^{-\mathbf{a}}w|\\
&\quad\ll\frac{Q}{\epsilon}(1+\|\rho\|^{|\mathbf{a}+\mathbf{n}|}\epsilon)\ll\frac{Q}{\epsilon}(1+\|\rho\|^{N+1}\epsilon)\ll\frac{Q}{\epsilon}(1+\|\rho\|^{N}\epsilon)
\end{aligned}\tag{3.34}
$$

thanks to equations (3.10), (3.12), and (3.30). As $w_\Delta\in W$, $|\rho^{\mathbf{n}}w_\Delta|\ll|w_\Delta|$ by equation (3.10), and the derivative of $((e(\eta_v\cdot x)-1)\rho^{\mathbf{n}}w_\Delta-(e((\rho^{\mathbf{n}})^T\eta_v\cdot x)-1)w_\Delta)$ is bounded by

$$
\begin{aligned}
&\|D((e(\eta_v\cdot x)-1)\rho^{\mathbf{n}}w_\Delta-(e((\rho^{\mathbf{n}})^T\eta_v\cdot x)-1)w_\Delta)\|_{C^1}\\
&\quad\le|\eta_v|\cdot|\rho^{\mathbf{n}}w_\Delta|+|(\rho^{\mathbf{n}})^{\mathrm{T}}\eta_v|\cdot|w_\Delta|\\
&\quad\ll\frac{Q}{\epsilon}\cdot|w_\Delta|+\frac{Q}{\epsilon}(1+\|\rho\|^{|\mathbf{n}|}\epsilon)\cdot|w_\Delta|\ll\frac{Q}{\epsilon}(1+\|\rho\|^{N}\epsilon)|w_\Delta|\\
&\quad\ll\frac{Q}{\epsilon}(1+\|\rho\|^{N}\epsilon)\cdot\frac{\epsilon}{Q}\bigg(\frac{1}{N}+\epsilon\bigg)=(1+\|\rho\|^{N}\epsilon)\bigg(\frac{1}{N}+\epsilon\bigg)
\end{aligned}\tag{3.35}
$$

thanks to equations (3.12) and (3.10).

Combining the above inequalities yields:

$$
\begin{aligned}
\|g^{\mathbf{n}}\|_{C^1}&\ll\|g^{\mathbf{n}}\|_{C^0}+cN^{r-1}\frac{Q}{\epsilon}(1+\|\rho\|^{N}\epsilon)+(1+\|\rho\|^{N}\epsilon)\bigg(\frac{1}{N}+\epsilon\bigg)\\
&\ll\frac{\epsilon}{Q}+\frac{1}{N}(1+\|\rho\|^{N}\epsilon)+(1+\|\rho\|^{N}\epsilon)\bigg(\frac{1}{N}+\epsilon\bigg)\\
&\ll(1+\|\rho\|^{N}\epsilon)\bigg(\frac{1}{N}+\epsilon\bigg).
\end{aligned}\tag{3.36}
$$

To summarize equations (3.31) and (3.36), we have the following lemma.

LEMMA 3.7. *Suppose c, w_Δ, and h are chosen as above. Then $\|h\|_{C^1}\ll1+\|\rho\|^{N}\epsilon$ and $\|g^{\mathbf{n}}\|_{C^1}\ll(1+\|\rho\|^{N}\epsilon)(1/N+\epsilon)$ for all $\mathbf{n}\in\Xi$.*

Proof of Proposition 2.4. The proposition follows directly from Lemmas 3.4, 3.5, 3.6, and 3.7 after choosing N and ϵ appropriately. Indeed, with $C>1$ denoting the largest among the implicit constants from Lemmas 3.6 and 3.7, choose ϵ sufficiently small such that $N:=\lfloor\log_{\|\rho\|}(1/\epsilon)\rfloor>4C/\delta$ and $C\cdot(\epsilon/Q)<\delta$. Then $1+\|\rho\|^{N}\epsilon<2$ and

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$1/N + \epsilon \le 2/N \le \delta/2C$. So $\|h\|_{C^0} \le C \cdot (\epsilon/Q) < \delta$; $\|h\|_{C^1} \le C(1 + \|\rho\|^N \epsilon) < 2C$; and $\|g\|_{C^1} < C(1 + \|\rho\|^N \epsilon)(1/N + \epsilon) < C \cdot 2 \cdot (\delta/2C) = \delta$. □

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