J. Austral. Math. Soc. (*Series A*) **48** (1990), 1–24

THE ASSOCIATED ORDER OF A PREORDER

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(Received 15 February 1988)

Abstract

Any preorder P on a set X has an associated preorder P', and hence an associate sequence of preorders $P, P', P'', P''', \ldots$. The properties of this sequence are studied. When X is finite the sequence is eventually periodic with period $p = 1$ or $p = 2$. If $p = 1$, the eventual constant preorder is full. For $p = 2$ the possible forms which the eventual alternating order can take are examined: first, the possible combinations of components are enumerated; second, the notion of ramification at a caste is used to show that X may in a heuristic sense be of unbounded complexity. If X is orderdense the periodicity starts at P'.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 06 A 99.
Keywords and phrases: preorder, partial order, associated order, caste, ramification.

1. Introduction

Let $\leqq$ be a preorder on a set X. Then there exists another preorder on X, called the *associated order* of $\leqq$ and denoted by $\leqq'$, defined thus: for $x, y \in X, x \leqq' y$ means

$$(1) \qquad (\forall u \in X)(u < x \Rightarrow u < y) \quad \text{and} \quad (\forall t \in X)(t > y \Rightarrow t > x).$$

For example, let X be $\mathbf{R}^2$ and let $\leqq$ be the strong pointwise order: if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ then $\mathbf{x} < \mathbf{y}$ means $x_1 < y_1$ and $x_2 < y_2$, and $\mathbf{x} \leqq \mathbf{y}$ means $\mathbf{x} < \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$. Then the associated order $\leqq'$ is the weak pointwise order: $\mathbf{x} \leqq' \mathbf{y}$ means $x_1 \leqq y_1$ and $x_2 \leqq y_2$. The situation is the same for $\mathbf{R}^n$ and for partial orderings on spaces of functions. More generally, $\leqq$ could be taken to be a hybrid pointwise order; again $\leqq'$ is the weak pointwise order.

This construction has been used by the author to develop a class of groups G which are partially-ordered groups with respect to one partial order $\leqq$, and lattice-ordered groups with respect to the associated order $\leqq'$, and at the same time topological groups and topological lattices with respect to the open-interval topology of $\leqq$, and where the positive cone of $\leqq'$ is the closure in that topology of the positive cone of $\leqq$. The orders are all partial orders. See [3], [5] and also [2], [4], [6].

In [1], some basic properties of the associated order on an arbitrary set are described, and we return to that generality here: no structure is assumed for X other than its order structure. Since the given order $\leqq$ on X determines $\leqq'$ uniquely, it generates a sequence of higher associated orders, which we call the *associate sequence*,

$$\leqq, \leqq', \leqq'', \ldots, \leqq^{(k)}, \ldots \tag{2}$$

(where $\leqq^{(k)}$ denotes $(\leqq^{(k-1)})'$). This paper deals with the behaviour of the associate sequence. We prove that (2), unless its members are pairwise distinct, is eventually periodic with period 1 or 2. If the period is 1 and X is finite, then the eventual order must be full. If the period is 2, the position is much more complicated. We examine in some detail the forms which the eventual periodic preordered set $(X, \leqq)$ may take, showing by means of the process of ramification at castes that there is no bound to the degree of complexity of $(X, \leqq)$; however, we do not find a simple characterization for this case. When $(X, \leqq)$ is orderdense, at most three of the orders in (2) are distinct.

The author is grateful to Drs Alicia Sterna-Karwat and Andrew Wirth for helpful discussions. An application of the modified associate order defined in Section 6, to decision theory, is presented in A. Wirth [9].

2. Definitions and basic properties

Take any set X with at least two elements. A *preorder* $\leqq$ on X is a reflexive transitive relation on X. As usual we write $x < y$ to mean that $x \leqq y$ and $x \neq y$. Also $x \geqq y$ means $y \leqq x$. Thus it never happens that $x < x$. On the other hand it can happen that $x < y$ and $y < x$; if this is the case or $x = y$ we write $x \sim y$, defining thereby an equivalence relation on X. A *partial order* on X is an antisymmetric preorder on X. The sets of all preorders on X, all partial orders on X respectively, are written $N(X)$, $O(X)$. Note that if $\leqq$ is in $N(X) \backslash O(X)$, then $<$ is not transitive.

If $\leqq_1$ and $\leqq_2$ are two preorders on X we say that $\leqq_1$ *is contained in* $\leqq_2$ if $(\forall x, y \in X)(x \leqq_1 y \Rightarrow x \leqq_2 y)$. This can be shown as $\leqq_1 \subseteq \leqq_2$. Then $N(X)$ is partially ordered by $\subseteq$; in fact it is a complete lattice, containing $O(X)$ as a decreasing subset.

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A partial order $\leqq$ (and the partially ordered set X, with respect to $\leqq$) is called *orderdense* if for all $x, y \in X$, if $x < y$ then there exists $z \in X$ such that $x < z < y$. The partial order is *full* if for every pair $x, y \in X$ exactly one of $x < y$, $x = y$, $y < x$ holds; it is *trivial* if for all $x, y \in X$, $x \leqq y$ implies $x = y$. A preorder $\leqq$ is *improper* if $x \leqq y$ for all $x, y \in X$.

When $\leqq$ is a preorder, an element $x \in X$ is called *minimal* if $a < x$ for no $a \in X$, *pseudominimal* if not minimal but $a < x$ implies $a \sim x$. The terms *maximal* and *pseudomaximal* are defined dually. An element y *covers* x, written $x \dot{<} y$, if $x < y$, $y \nless x$, and

$$x < a < y \text{ implies } x \sim a \text{ or } a \sim y. \tag{3}$$

So when $\leqq$ is partial, $x \dot{<} y$ means $x < y$ and $x < a < y$ for no a. As usual $y \nless x$ means not $y < x$. We sometimes write $x \| y$ to mean $x \nleqq y$ and $y \nleqq x$;

$$x <!y \text{ to mean } x < y \text{ and } y \nless x;$$

and we write

$$(a <) = \{x \in X : a < x\}, \quad (a \geqq) = \{x \in X : a \geqq x\},$$

and so on.

Note: $\dot{<}$ is not transitive, $\dot{\leqq}$ is not a preorder; $<!$ is transitive and $\leqq!$ (meaning $<!$ or $=$) is a partial order. All these notations can be invoked for other preorders as well. If $x < y <!z$ then $x <!z$. If $x <!y < z$ then $x <!z$.

Let $\leqq$ be a preorder on X, and let $\leqq'$ be the relation defined by (1); clearly $\leqq'$ is a preorder on X. Unfortunately, $\leqq'$ need not be a partial order, even if $\leqq$ is a partial order; we shall see finite instances of this presently.

The basic facts about the associated order are given in the following theorem.

THEOREM 1. *Let $\leqq$ be any preorder on X and $x, y \in X$.*

(i) *If $x < y$ then either $x <' y$ or $x \sim y$ but not both.*

(ii) *If $x < y$ and not $y < x$, then $x <' y$ and not $y <' x$. That is, $x <!y$ implies $x <'!y$.*

(iii) *$\leqq$ is trivial if and only if $\leqq'$ is improper.*

(iv) *$\leqq'$ is trivial if $\leqq$ is improper. Conversely, if $\leqq'$ is trivial then any two $\sim$-inequivalent elements are unrelated by $\leqq$.*

(v) *$\leqq$ is contained in $\leqq'$ if and only if $\leqq$ is a partial order.*

(vi) *If $\leqq'$ is contained in $\leqq$ then $\leqq'$ is a partial order.*

(vii) *$\leqq$ and $\leqq'$ coincide if $\leqq$ is full, or if $\leqq$ is orderdense and $\leqq'$ is full.*

(viii) *If $\leqq$ and $\leqq'$ coincide, then they are a partial order.*

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Proof. The theorem follows from the definitions; we give proofs of a few of the items, to indicate the flavour of the arguments.

(i) Let $x < y$. If $u < x$ then $u \leqq x$ and $x \leqq y$ so $u \leqq y$. Suppose $u = y$. Then $y < x$, so $x \sim y$. If also $x <' y$ then $u < x$ gives $u < y$, so $y < y$, a contradiction.

Suppose instead $x < y$ and not $x \sim y$. Then $u < x$ implies $u < y$. Similarly $t > y$ implies $t \geqq x$, that is, $t > x$. Thus $x <' y$.

(ii) If $x < y$ and not $y < x$, then $x <' y$ by (i). Suppose also $y <' x$: then $x < y$ gives $x < x$, contradiction.

(vii) The first statement follows from (i). Suppose instead that $\leqq$ is orderdense and $\leqq'$ is full, and for some pair a, b we have neither $a < b$ nor $b < a$. Let $u < a$. There exists v with $u < v < a$. Now either $v <' b$ or $v \geqq' b$, and the latter implies $b < a$, so in fact we must have $v <' b$, whence $u < b$. Thus $u < a$ implies $u < b$. Similarly $t > b$ implies $t > a$. Therefor $a \leqq' b$. In the same way, we prove $b \leqq' a$. Since $\leqq'$ is a partial order, $a = b$. This shows that $\leqq$ is full, and hence it coincides with $\leqq'$.

3. The associate sequence

The first result is

Theorem 2. *Let $\leqq$ be any preorder on X.*

(i) *For all $x, y \in X, x < y$ implies $x <'' y$.*

(ii) *Suppose $\leqq$ is an orderdense partial order. Then also $x <'' y$ implies $x <' y$; moreover the orders $\leqq^{(2n)}, n = 1, 2, \ldots$, all coincide with $\leqq''$, and the orders $\leqq^{(2n+1)}, n = 1, 2, \ldots$, all coincide with $\leqq'$, so there are at most three distinct orders in the sequence* (2).

Proof. (i) Suppose $x < y$. To prove $x \leqq'' y$, we have to prove

$$(\forall u)(u <' x \Rightarrow u <' y) \tag{4}$$

and

$$(\forall t)(t >' y \Rightarrow t >' x). \tag{5}$$

To prove (4), we assume $u <' x$ and prove

$$(\forall a)(a < u \Rightarrow a < y) \tag{6}$$

and

$$(\forall b)(b > y \Rightarrow b > u). \tag{7}$$

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Proof of (6). Let $a < u$. Since $u <' x$, we have $a < x$. But $x < y$, so $a \leqq y$. If $a < y$, we are through. Suppose instead $a = y$. Then $y < u$; therefore $x \leqq u$. If $x < u$ then $x < x$ since $u <' x$; if $x = u$ then $x <' x$; in either case there is a contradiction. Thus $a \neq y$, and we have proved (6). The proof of (7) is analogous. Thus (4) holds. In the same manner we prove (5).

(ii) Suppose $x <'' y$. Let $u < x$; there exists v such that $u < v < x$, and by Theorem 1(i) we have $v <' x$ since $\leqq$ is partial, so $v <' y$; therefore $u < y$. Thus $u < x$ implies $u < y$; and similarly $t > y$ implies $t > x$. Therefore $x <' y$.

We have now shown that $x < y \Rightarrow x <'' y$ and that $x <'' y \Rightarrow x <' y$. By (i), $x <' y \Rightarrow x <''' y$. We prove conversely that $x <''' y \Rightarrow x <' y$, as follows. Suppose $x <''' y$. Let $u < x$; there exist v, w such that $u < v < w < x$. Then $w <'' x$ by (i), so $w <'' y$. Since $v < w$ and $\leqq$ is partial, $v <' w$; therefore $v <' y$. Since $u < v$, we have $u < y$. This proves half of (1), and the other half is proved similarly; so $x <' y$.

Thus $\leqq'$ and $\leqq'''$ coincide. Therefore $\leqq^{(n)}$ and $\leqq^{(n+2)}$ coincide for all $n = 1, 2, \ldots$. This completes the proof.

A sequence $\alpha_1, \alpha_2, \ldots$ of relations on X is called *eventually periodic* if there exist integers $n \geqq 0$ and $p \geqq 1$ such that $\alpha_k = \alpha_{k+p}$ for all $k \geqq n$; then the *eventual period* is the least p for which there exists an n. If $p = 1$ the sequence is called *eventually stationary*; if $p = 2$ it is called *eventually alternating*.

In the associate sequence (2), either all orders are distinct or else the sequence is eventually periodic.

THEOREM 3. *For any preorder $\leqq$ on any set X, the associate sequence* (2), *if it is eventually periodic, is eventually stationary or eventually alternating.*

PROOF. For $n = 1, 2, \ldots$ write

$$x <^{(n)}!y \text{ to mean } x <^{(n)} y \text{ and not } y <^{(n)} x,$$
$$x \sim^{(n)} y \text{ to mean } x <^{(n)} y \text{ and } y <^{(n)} x, \text{ or } x = y,$$

and

$$F_n = \{(x, y) \in X \times X : x <^{(n)}!y\},$$
$$T_n = \{(x, y) \in X \times X : x \sim^{(n)} y\}.$$

Now Theorem 1(ii) applied to $\leqq^{(n)}$ says that

(8) $$F_n \subseteq F_{n+1} \quad \text{for all } n = 0, 1, 2, \ldots$$

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and Theorem 2(i) shows that $x \sim^{(n)} y$ implies $x \sim^{(n+2)} y$, so

(9) $$T_n \subseteq T_{n+2} \quad \text{for all } n = 0, 1, 2, \ldots.$$

Suppose that sequence (2) is eventually periodic, with period $p \geqq 1$. Then for all $n \geqq k$, say,

(10) $$F_n = F_{n+p}, \quad T_n = T_{n+p}.$$

Suppose p is odd. Since

(11) $$T_k \subseteq T_{k+2} \subseteq \cdots \subseteq T_{k+p+1} = T_{k+1} \subseteq T_{k+3} \subseteq \cdots \subseteq T_{k+p} = T_k,$$

T_n is constant for $n \geqq k$. So is F_n; therefore $\leqq^{(n)}$ is unchanging for $n = k, k+1, \ldots$, and hence $p = 1$.

Suppose instead that p is even. We find that $T_n = T_{n+2}$ and $F_n = F_{n+1}$ for all $n \geqq k$. Therefore $p = 2$.

A finite set X can carry only a finite number of preorders, so the sequence (2) must be eventually periodic; thus

COROLLARY 4. *For any preorder $\leqq$ on a finite set X, the sequence* (2) *is eventually stationary or eventually alternating.*

A necessary condition for (2) to be eventually stationary is given in [8].

4. Examples of the sequence (2)

In the following examples we use Hasse diagrams to illustrate the sequences $\leqq, \leqq', \leqq'', \ldots$. If $\leqq$ is a partial order on X, then each large dot in the diagram for $(X, \leqq)$ represents an element of X, and the dot for X is connected to the dot for y by a rising line segment when $x < y$ and $x < z < y$ for no z in X: that is, a single rising line segment denotes the cover relation, $x \dot{<} y$. Generally, $x < y$ is indicated by the presence of a rising polygonal arc from x to y. When X is infinite, these conventions become strained.

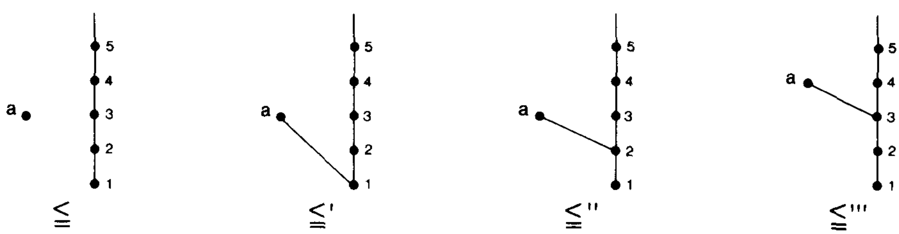
If $\leqq$ is a preorder and not partial, a single rising line segment from x to y again means $x \dot{<} y$, but with the modified interpretation in Section 2. To indicate $x \sim y$ we join the dots for x and y by a horizontal line segment. By a lapse of notation we may omit some 'cover' line segments which are implied by segments not omitted. The Hasse diagram is a (directed) 1-graph with no loops, satisfying certain conditions, and having a required orientation on the page.

Given X and $\leqq$, to determine $\leqq'$ we can check if the two implications in (1) hold, for each pair $(x,y) \in X \times X$ in turn, there being $n(n-1)$ pairs to check if $\operatorname{card}(X) = n$; then $\leqq''$ is found from $\leqq'$ in the same way, and so on. However, Theorems 1 and 2 allow great simplifications. By 1(ii), $x <!y$ implies $x <'!y$. Thus a rising connection from x to y, once established in the Hasse diagram for, say, $\leqq^{(j)}$, remains present in diagrams for all higher associated orders. Again, for distinct elements x and y, $x \sim' y$ is possible only if x and y are unrelated by $\leqq$; and if $x \sim' y$ then x and y are unrelated by $\leqq''$, but by 2(i), $x \sim''' y$. Thus a horizontal connection, once established for say $\leqq^{(j)}$, reappears in every second diagram thereafter, the two elements being unrelated in the intervening diagrams. Also, if a is minimal and b is maximal for $\leqq$, then $a \leqq' b$, by vacuous fulfilment of (1). Thus if $(X, \leqq)$ is a finite partially ordered set and its Hasse diagram is not connected (see Section 7), then each component has a maximal element and a minimal element, and consequently $(X, \leqq')$ is connected.

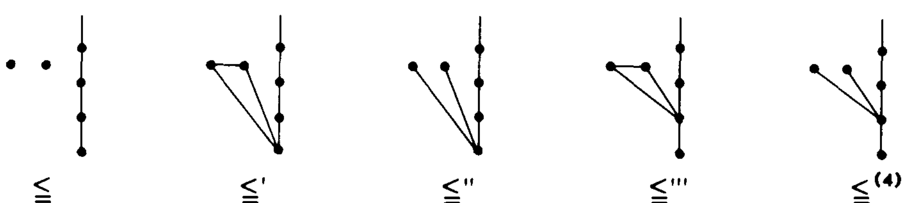
In the following examples the diagrams describe successively $\leqq, \leqq', \leqq'', \ldots$.

Ex. 1°. X is the union of two copies $\mathbf{Z}_1$ and $\mathbf{Z}_2$ of $\mathbf{Z}$; $\leqq$ relativized to $\mathbf{Z}_1$ is the usual ordering on the integers, likewise for $\mathbf{Z}_2$; and a, b are unrelated if $a \in \mathbf{Z}_1$, $b \in \mathbf{Z}_2$. This is an example where $\leqq$ equals $\leqq'$, so (2) is immediately stationary, but $\leqq$ is not full.

Ex. 2°. X is the set $\mathbf{N}$ of natural numbers together with an extra element a unrelated to any element in $\mathbf{N}$. In this example all $\leqq^{(j)}$ are distinct, and all are partial orders. (After M. Bertschi, private communication.)



Ex. 3°.



Here $\leqq^{(j)}$ are all distinct, and $\leqq^{(j)}$ is partial only if j is even.

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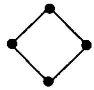
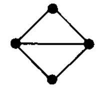
Ex. 4°.



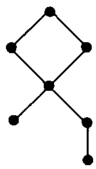
$\leqq$



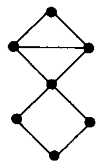
$\leqq'$

Here $\leqq''$ equals $\leqq$; the associate sequence alternates immediately.

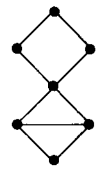
Ex. 5°.



$\leqq$



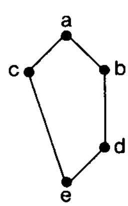
$\leqq'$



$\leqq''$

Here $\leqq'''$ equals $\leqq'$; and $\leqq^{(j)}$ is partial only if $j = 0$.

Ex. 6°.

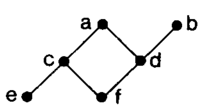


$\leqq$

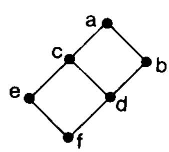
$\leqq'$

Here $\leqq'$ is full, and hence $\leqq''$ equals $\leqq'$.

Ex. 7°.



$\leqq$



$\leqq'$

$\leqq''$

All $\leqq^{(j)}$ are full, for $j \geqq 2$.

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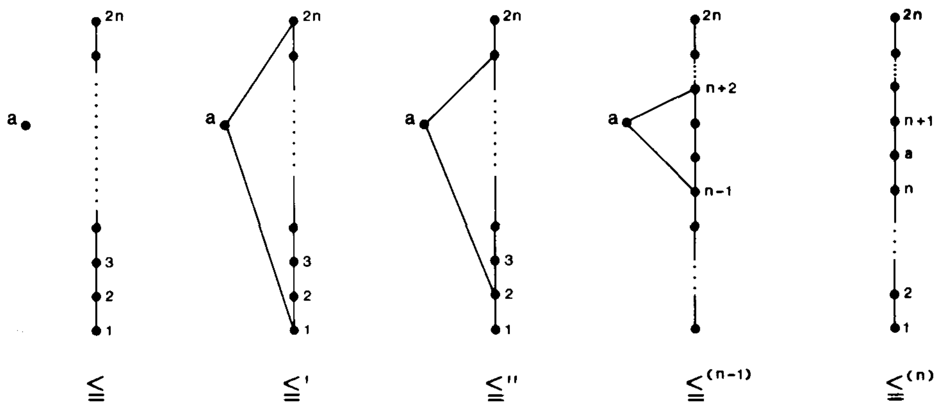
Ex. 8°.



Here $\leqq^{(j)}$ are distinct for $j = 0, 1, \ldots, n$; $\leqq^{(n)}$ is full, and the associate sequence is stationary thereafter.

Instead, take X as above, but with $2n-1$ elements in the chain instead of $2n$. Then the $\leqq^{(j)}$ are distinct for $j = 0, 1, \ldots, n$; $\leqq^{(n+1)}$ equals $\leqq^{(n-1)}$, and the associate sequence alternates thereafter.

5. Stationary associate sequences

If the associate sequence $\leqq, \leqq', \leqq'', \ldots$ is eventually stationary, what forms can the eventual preorder take? The associate sequence is eventually stationary when $\leqq^{(k)}$ equals $\leqq^{(k)\prime}$, for some k, so the question is equivalent to asking for all preorders $\leqq$ for which $\leqq$ equals $\leqq'$. Let us call such a preorder *stationary. A stationary preorder is necessarily a partial order*; for if there exists an equivalent pair $x \sim y$, $x \neq y$, then by Theorem 1(i) we have not $x \sim' y$, so $\leqq$ is not stationary. More generally, we find that *every preorder in an eventually stationary associate sequence is a partial order.*

Lemma 5. *If X is finite and $\leqq$ is a stationary partial order, then X has a least element.*

Proof. Being finite, X has minimal elements. Suppose p, q are two distinct minimal elements. If $(p<) = (q<)$ then $t > p \Leftrightarrow t > q$, and $u < p \Leftrightarrow u < q$ (vacuously), so $p \sim q$, contradiction. Suppose without loss of generality that there exists t such that $p < t$ and $q \nless t$. Then the set

$$S = \{t \in X : p < t \text{ and } q \nless t\} \tag{12}$$

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is nonempty and has a maximal element, t_0 say. We have

$$q <' t_0. \tag{13}$$

For $u < q \Rightarrow u < t_0$ vacuously; if $v > t_0$ then $v > p$, so if $v \not> q$ then $v \in S$, contradicting the maximality of t_0; thus $v > t_0 \Rightarrow v > q$. This proves (13). Since $\leqq$ equals $\leqq'$, we have $q < t_0$, contradicting $t_0 \in S$. Thus X has a unique minimal element, which accordingly is the least element.

LEMMA 6. *If X is finite and $\leqq$ is stationary, then the least element p has a unique cover.*

PROOF. The element p has at least one cover (we assume $\mathrm{Card}(X) \geqq 2$). If a, b are distinct covers of p we argue in much the same way as in the proof of Lemma 5, to obtain a contradiction.

THEOREM 7. *The only stationary partial orders on a finite set are the full orders.*

PROOF. The proof is by induction, starting from the positions described in Lemmas 5 and 6. Suppose we know that

$$X = \{p_0, p_1, \dots, p_n\} \cup Y$$

where

$$p_0 < p_1 < p_2 < \cdots < p_n < y \quad \text{for all } y \in Y.$$

Then the same argument shows that (unless $\mathrm{card}(Y) = 1$) Y contains a unique cover of p_n, call it p_{n+1}, such that

$$p_0 < p_1 < p_2 < \cdots < p_n < p_{n+1} < y \quad \text{for all } y \in Y \backslash \{p_{n+1}\}.$$

After finitely many steps we arrive at a full ordering of X by $\leqq$.

If X is infinite, a stationary partial order need not be full: Section 4, 1° is an example. It is even not the case that each component (see Section 7) need be full; an example is obtained from Section 4, 1°, modified by adding a single least element.

6. The modified associated order $\leqq^*$

In its original use with partially ordered groups $(G, \leqq)$, the associated order was itself partial, and $\leqq'$ and $\leqq''$ were equal in all cases of interest. It could be argued, therefore, that the definition (1) is inappropriate because of the special and awkward role played by nonsingleton equivalence classes in

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forming the associate sequence; and that a more natural definition is

(14) $(\forall u \in X)(u <!x \Rightarrow u < y)$ and $(\forall t \in X)(t >!y \Rightarrow t > x)$.

Let property (14) be denoted by $x \leqq^* y$, and let $x <^* y$ mean $x \leqq^* y$ and $x \neq y$; the relation $\leqq^*$ is called the *modified associated order* of $\leqq$.

We first remark that (14) is equivalent to

(15) $(\forall u \in X)(u <!x \Rightarrow u <!y)$ and $(\forall t \in X)(t >!y \Rightarrow t >!x)$;

indeed, it is easily verified that the first implication of (14) implies the first of (15), and likewise for the second implications. Furthermore, the relation $\leqq^*$ is a preorder; and for all $x, y \in X$,

(16) $x < y$ implies $x <^* y$,

(17) $x <!y$ implies $x <^*!y$,

(18) $x <' y$ implies $x <^* y$;

and if $\leqq$ is partial then $\leqq'$ and $\leqq^*$ coincide.

By (16), the *modified associate sequence*

(19) $\leqq, \leqq^*, \leqq^{**}, \ldots$

is increasing, so either its members are all distinct or it is eventually stationary. If (2) is stationary, then necessarily (19) is stationary.

Suppose that $\leqq$ is stationary for the sequence (19), that is, that $\leqq$ equals $\leqq^*$. First, since $x <' y \Rightarrow x <^* y \Leftrightarrow x < y$, Theorem 1(vi) shows that $\leqq'$ is partial. If $\leqq$ is also partial then by Theorem 1(v), $x < y \Rightarrow x <' y$, so $\leqq, \leqq', \leqq^*$ all coincide and sequence (2) is stationary. Assume that $\leqq$ is not partial. Then

(20) $x <' y \Rightarrow x <^* y \Leftrightarrow x < y \Rightarrow x <'' y \Leftrightarrow x <'^* y$,

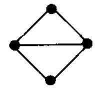
and it can also be verified that

(21) $x <'^* y \Rightarrow x <^* y$;

from (20) and (21) we conclude that $\leqq$ equals $\leqq''$.

Thus the cases where $\leqq$ equals $\leqq^*$ can be found among the cases where $\leqq$ is stationary or alternating for (2), discussed previously.

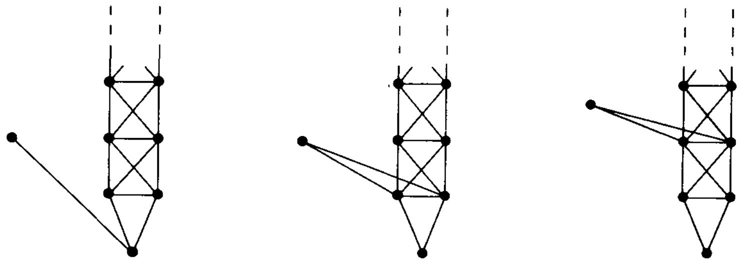
Ex. 9°.



$\leqq$

Here sequence (2) is alternating, but sequence (19) is stationary, $\leqq^*$ is not full.

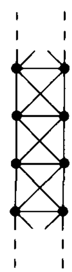
Ex. 10°.



$\leqq$ $\leqq^*$ $\leqq^{**}$

Here $\leqq^{(2n)}$ equals $\leqq^{*(n)}$ for all n; neither (2) nor (19) is eventually periodic.

Ex. 11°.



$\leqq$

Here sequence (2) is alternating, (19) is stationary.

We use the preorder $\leqq^*$ in Section 8, to elucidate the nature of alternating associate sequences (2). Of particular interest are the $\sim^*$ equivalence classes, where $x \sim^* y$ means either $x <^* y$ and $y <^* x$, or $x = y$, that is,

(22) $$(\forall u \in X)(u <!x \Leftrightarrow u <!y) \quad \text{and} \quad (\forall t \in X)(t >!y \Leftrightarrow t >!x).$$

Such an equivalence class will be called a *caste* of $(X, \leqq)$.

7. Alternating associate sequences

What eventual forms can an eventually alternating associate sequence have?

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This is equivalent to asking for a description of all preorders $\leqq$ such that

$$\leqq \text{ equals } \leqq'', \qquad \leqq \text{ differs from } \leqq'. \tag{23}$$

Let us call such a preorder *alternating*, like the associate sequence it starts. We find some partial answers to this question. The set X is not assumed finite.

Some further definitions are needed. Two distinct elements x, y of X are said to be *connected* with respect to $\leqq$, denoted $x\gamma y$, if there exists a sequence $a_0, a_1, \ldots, a_m$ in X such that $a_0 = x$, $a_m = y$ and

$$a_0 \rho a_1 \rho a_2 \rho \cdots \rho a_m, \tag{24}$$

where $c\rho d$ means either $c < d$ or $c > d$ (or both); and (24) means $a_0 \rho a_1$ and $a_1 \rho a_2$ and .... We also write $x\gamma x$ for all x; then γ is an equivalence relation, and its equivalence classes are called the *components* of $(X, \leqq)$. A subset Y of X is called *connected* if $y_1 \gamma y_2$ for all $y_1, y_2 \in Y$; the components are the maximal connected subsets, and correspond to the connected components of the Hasse diagram of $(X, \leqq)$ regarded as a directed graph. An element $x \in X$ is called *isolated* if x is both maximal and minimal, that is, if the singleton $\{x\}$ is a component. A *horizontal component* H is a component of more than one element such that $x \sim y$ for all $x, y \in H$; that is, H is also a $\sim$-equivalence class. A non-singleton, non-horizontal component will be called a *main component.*

LEMMA 8. *Let C be a main component of $(X, \leqq)$; then for all $x, y \in C$ there exists a sequence $b_0, b_1, \ldots, b_n$ in X such that $b_0 = x$, $b_n = y$, and*

$$b_0 \rho! b_1 \rho! b_2 \rho! \cdots \rho! b_n, \tag{25}$$

where $c\rho! d$ means (by abuse of notation) either $c <! d$ or $c >! d$.

PROOF. Let $x, y \in C$, and $x \neq y$. There exists a sequence $x = a_0, a_1, \ldots, a_m = y$ with property (24). Suppose $a_0 \sim a_1 \sim \cdots \sim a_s <! a_{s+1}$. Then we have that $a_0 <! a_{s+1}$ and the sequence can be contracted by omitting $a_1, a_2, \ldots, a_s$, and defining $b_0 = a_0, b_1 = a_{s+1}$. This process leads to a sequence $b_0, \ldots, b_n$ with property (25), unless we have started from $a_0 \sim a_1 \sim \cdots \sim a_m$.

Consider this case. If there exists no sequence $x = b_0, \ldots, b_n = y$ with property (25), then $a_0 <! u$ for no $u \in X$, for if $a_0 <! u$ then we could write

$$a_0 <! u >! a_0 \sim a_1 \sim \cdots \sim a_m$$

and contract this to $a_0 <! u >! a_m$. Similarly $a_0 >! u$ for no $u \in X$. Thus for all $u \in X$, either $a_0 \sim u$ or $a_0 \| u$. Then we find that C consists of the $\sim$-equivalence class of a_0, and is horizontal, a contradiction.

In the following Lemmas 9 to 14, it is part of the premise that $\leqq$ is alternating, that is, that (23) holds. Note from Theorem 1 that when $\leqq$ is alternating then for all $x, y \in X$, $x <!y$ if and only if $x <'! y$; thus the Hasse diagrams for $\leqq$ and $\leqq'$ differ only in the occurrence of horizontal line segments. Moreover, $x \sim y$ implies $x \|' y$. Of course, if $\leqq$ is alternating then so is $\leqq'$.

Lemmas 9 to 14 deal mainly with minimal and maximal elements, and the numbers of components of the three types which may occur.

Lemma 9. *Assume $\leqq$ is alternating. If p is minimal and q is maximal in $(X, \leqq)$ then either $p \leqq q$, or p and q are both isolated.*

Proof. We have $p \leqq' q$; therefore either $p <'' q$ or $p \sim' q$. If $p <'' q$ then $p < q$ by (23). If $p \sim' q$ and $p \neq q$ then $q <' p$; then if p is not isolated there exists $a \in X$, $p < a$, and so $q < a$, contradiction; therefore p is isolated, and similarly q is isolated.

Thus if there exist in $(X, \leqq)$ a minimal element p and a maximal element q, not both isolated, then they belong to the same component. Put another way, if $(X, \leqq)$ has an isolated element, then all non-singleton components have neither maximal nor minimal elements. A non-singleton component in finite X having an isolated element must therefore have pseudomaximals and pseudominimals.

The next three lemmas examine the changes in connectivity in passage from $(X, \leqq)$ to $(X, \leqq')$.

Lemma 10. *Assume $\leqq$ is alternating. If a subset C is a main component of $(X, \leqq)$, then C is a main component of $(X, \leqq')$. Moreover, the restriction $\leqq' | C$ is the associated order of $\leqq | C$; thus $\leqq'$ on C can be calculated without making reference to $X \backslash C$.*

Proof. Let $x, y \in C$. By Lemma 8, there exists in X a sequence $x = b_0, b_1, \ldots, b_n = y$ for which (25) holds. Then from Theorem 1(ii) it follows immediately that

$$b_0 \sigma! b_1 \sigma! b_2 \sigma! \cdots \sigma! b_n.$$

(Here $c \sigma d$ means $c <' d$ or $c >' d$ or both, and $c\sigma!d$ means $c <'!d$ or $c >'!d$.) Therefore x, y are connected in $(X, \leqq')$, and lie in the same $\leqq'$-component, call it D; so $C \subseteq D$, and the same argument shows that D is contained in some $\leqq''$-component, which is however C. Thus $C = D$, a $\leqq'$-component. The statement about $\leqq' | C$ follows easily from definition (1) and the fact that C is a component.

LEMMA 11. *Assume* $\leqq$ *is alternating. If* H *is a horizontal component of* $(X, \leqq)$ *then each* x *in* H *is isolated in* $(X, \leqq')$. *Thus again,* $\leqq' |H$ *is the associated order of* $\leqq |H$.

PROOF. If $x, a \in H$ and $x \neq a$, then $x \sim a$ shows that $x \|' a$. Suppose instead $x \sigma a$ for some $a \in X \backslash H$, say $x <' a$. There exists $y \in H$, $y \neq x$; now $y \| a$ since H is a component. But $y \sim x$, so $y < x, y < a$, contradiction. Thus for every $a \in X \backslash H$, x and a are unrelated by $\leqq'$. Thus $x \|' z$ for every $z \in X \backslash \{x\}$, showing that x is isolated in $(X, \leqq')$.

LEMMA 12. *Assume* $\leqq$ *is alternating. The set* K *of all the isolated elements of* X, *if nonempty, constitutes a horizontal component of* $(X, \leqq')$, *or a singleton component if* K *is a singleton.*

PROOF. Let $x, y \in K, x \neq y$. By vacuous fulfilment of (1), $x \sim' y$. Thus K is connected in $(X, \leqq')$.

By Lemma 9 there are no maximal elements or minimal elements outside of K. Let $x \in K$, $a \in X \backslash K$. There must exist elements $b, c \in X$ with $b < a < c$. We have $a \not<' x$, since otherwise $b < x$, a contradiction; likewise $x \not<' a$. Thus x, a are unrelated by $\leqq'$. If x, a are in the same $\leqq'$-component E of X, there exists a sequence $e_0, e_1, \ldots, e_n$ with

$$x = e_0 \sigma ! e_1 \sigma ! e_2 \sigma ! \cdots \sigma ! e_n = a$$

where σ is as defined in the proof of Lemma 10. Here $x <' ! e_1$ or $x >' ! e_1$; say the former. But $e_1 \in X \backslash K$, for if e_1 were isolated then $x \sim' e_1$, contradiction; therefore x and e_1 are unrelated by $\leqq'$, a contradiction. Thus a cannot be in the same $\leqq'$-component as x. This shows that K is a maximal connected subset of $(X, \leqq')$.

LEMMA 13. *Assume* $\leqq$ *is alternating. The preordered set* $(X, \leqq)$ *has at most one horizontal component. If it has one horizontal component, then it has either no isolated elements, or at least two isolated elements.*

PROOF. Let A, B be two distinct horizontal components. By Lemma 11, their elements are all isolated in $(X, \leqq')$; by Lemma 12, $A \cup B$ is contained in a single horizontal component of $(X, \leqq'')$, which is $(X, \leqq)$, a contradiction. A similar argument proves the impossibility of X having a single isolated point together with a horizontal component.

LEMMA 14. *Assume* $\leqq$ *is alternating. If* p *is minimal in* $(X, \leqq)$ *then it is minimal or pseudominimal in* $(X, \leqq')$. *If* q *is pseudominimal in* $(X, \leqq)$ *then it is minimal in* $(X, \leqq')$. *The dual statements hold for maximals.*

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Proof. Let p be minimal in $(X, \leqq')$. If p is not minimal in $(X, \leqq')$ there exists $a <' p$. Since $a <'!p$ would imply $a <''!p$ and hence $a < p$, we must have $a \sim' p$. Thus p is pseudominimal. The statement about q is proved similarly.

Let Min, Max, PMin, PMax denote the sets of minimal, maximal, pseudominimal, pseudomaximal elements of $(X, \leqq)$ respectively, and Min$'$, ... the corresponding sets for $(X, \leqq')$. Since $\leqq'$ is also alternating, Lemma 14 gives

$$\text{PMin} \subseteq \text{Min}', \quad \text{Min} \subseteq \text{Min}' \cup \text{PMin}', \tag{26}$$

$$\text{PMin}' \subseteq \text{Min}, \quad \text{Min}' \subseteq \text{Min} \cup \text{PMin}, \tag{27}$$

and hence *when* $\leqq$ *is alternating we have*

$$\text{Min} \cup \text{PMin} = \text{Min}' \cup \text{PMin}'; \tag{28}$$

and dually for Max, PMax.

From these various lemmas we can put together an identikit picture of possible forms which $(X, \leqq)$ may take, having regard to the combinations of components. We confine attention to finite sets X, since here one knows that any element dominates either a minimal or a pseudominimal element, and dually.

Theorem 15. *Let X be finite and let $\leqq$ be alternating. Then one of the following situations occurs.*

(1) $(X, \leqq)$ *has main components only, say n of them, and either* (i) $n = 1$; *or* (ii) Min = PMin$'$ $\neq \varnothing$, PMax = Max$'$ $\neq \varnothing$, Max $= \varnothing$, PMin $= \varnothing$, Min$'$ $= \varnothing$, PMax$'$ $= \varnothing$, $n > 1$; *or* (iii) *the dual of* (ii); *or* (iv) Min = PMin$'$, Max = PMax$'$, Min$'$ = PMin, Max$'$ = PMax, *all these sets are nonempty*, $n = 2$, Min $\cup$ Max *is contained in one component, and* PMin $\cup$ PMax *in the other.*

(2) $(X, \leqq)$ *has one main component C, one horizontal component H, and no singleton components,* Min $\cup$ Max $\subseteq C$, PMin = PMax $= H$.

(3) $(X, \leqq)$ *has one main component C, no horizontal component, and two or more singleton components.* Min = Max $= K$, *the set of singleton components*; PMin $\cup$ PMax $\subseteq C$.

(4) $(X, \leqq)$ *has no main components, one horizontal component, and m singleton components with $m = 0$ or $m \geq 2$.*

(5) $(X, \leqq)$ *is a set of singleton components.*

Outline of the proof. Case (1), X has main components only. If Min $= \varnothing =$ Max, then we must have $n = 1$. If Min $\neq \varnothing$ and Max $= \varnothing$, then we must have (1)(ii) or $n = 1$. If Min $= \varnothing$ and Max $\neq \varnothing$, then (1)(iii) or $n = 1$. If Min $\neq \varnothing$ and Max $\neq \varnothing$, then (1)(iv) or $n = 1$.

If X has horizontal components, then it has precisely one, by Lemma 13; moreover, it can then have no more than one main component. If X has one horizontal component and one main component, then it cannot have any singleton components. This gives Case (2).

If X has no horizontal component but has n main components and m singleton components, then we must have $n = 1, m \geq 2$. This gives Case (3).

Cases (4) and (5) are the remaining possibilities.

The details of the above argument use the preceding lemmas and equations (26), (27).

It remains to describe the main components. We cannot give a single characterization of the possible forms taken by main components for alternating $\leqq$, even for finite X. On the other hand, we shall exhibit in the next section a means of building up progressively more elaborate alternating orders.

8. Alternating associate sequences; ramification

To examine the structure of a main component, for alternating $\leqq$, we make use of the notion of caste. The term was defined at (22) in Section 6.

By Lemma 10 it suffices, if C is a main component of $(X, \leqq)$, to treat $(C, \leqq | C)$ as the preordered space, ignoring the rest of X. Equivalently, we could confine attention to connected nonhorizontal sets X. However, we do not need to make that assumption explicitly here.

In what follows, for any nonempty subset A of X the preorder $\leqq$ relativized to A is written $\leqq | A$, or simply $\leqq$ if no ambiguity is likely.

If A is a caste then the components of $(A, \leqq | A)$ are either singletons or horizontal, for we cannot have $a_1 <! a_2$ when $a_1, a_2 \in A$.

Let $\mathscr{C}(X, \leqq)$ (briefly, $\mathscr{C}$) be the set of castes of $(X, \leqq)$. For $A, B \in \mathscr{C}$ write $A \prec B$ to mean there exist $a \in A$, $b \in B$ with $a <! b$. It is easily seen that $A \prec B$ if and only if $a <! b$ for all $a \in A$, $b \in B$. The notation $A \preceq B$ means $A \prec B$ or $A = B$.

LEMMA 16. (i) *$(\mathscr{C}, \preceq)$ is a partially ordered set; the associated order $\preceq'$ of $\preceq$ is also a partial order.*

(ii) *If $\leqq$ is alternating then $(X, \leqq)$ and $(X, \leqq')$ have the same castes, and the partial order $\preceq$ is the same for the two preordered sets. Thus $(\mathscr{C}, \preceq)$ is the caste poset for both $(X, \leqq)$ and $(X, \leqq')$.*

The proof follows from the definitions and earlier results. The defining property of $\preceq'$ is of course that $A \preceq' B$ means

(29) $\quad (\forall U \in \mathscr{C})(U \prec A \Rightarrow U \prec B) \quad \text{and} \quad (\forall V \in \mathscr{C})(V \succ B \Rightarrow V \succ A).$

Result 16(i) shows that not every poset can be the poset of castes of a preordered set.

For $x \in X$ let $\lambda(x)$ denote the caste containing x. Then $\lambda : X \to \mathscr{C}$ is an order-preserving map from $(X, \leqq)$ to $(\mathscr{C}, \preceq) : a < b$ implies $\lambda(a) \preceq \lambda(b)$. In general, λ does not preserve order from $(X, \leqq')$ or from $(X, \leqq^*)$. But from (18), $a \sim' b$ implies $\lambda(a) = \lambda(b)$.

LEMMA 17. *Let $(X, \leqq)$ be any preordered set, and let T be a caste of X. Then the preorders $\leqq' |T$ and $(\leqq |T)'$ coincide. Thus $\leqq' |T$ can be calculated without reference to $X \backslash T$.*

PROOF. For $x, y \in T$ the statement $x \ (\leqq |T)' y$ means

$$(30) \qquad (\forall u \in T)(u < x \Rightarrow u < y) \quad \text{and} \quad (\forall v \in T)(v > y \Rightarrow v > x)$$

whereas $x \leqq' |T y$ means (1). Clearly (1) implies (30). Suppose $x, y \in T$ and (30) holds. Let $u \in X, u < x$. If $u \sim x$ then $u \sim^* x$ by (16), so $u \in T, u < y$. If $u \notin T$ then $u <!x$, so $u <!y$ since $x, y \in T$ and T is a caste. Thus, either way, we deduce $u < y$, proving the first part of (1). Similarly we can prove the second part, and (1) holds.

The next result shows that the behaviour of singleton and horizontal components of castes is like that found for X in Section 7.

LEMMA 18. *Let $\leqq$ be alternating, and let T be a caste of $(X, \leqq)$.*

(i) *If F is a horizontal component of $(T, \leqq)$ then each element of F is isolated in $(T, \leqq')$. (Cf. Lemma* 11.)

(ii) *The set J of all isolated elements of $(T, \leqq)$, if nonempty, constitutes a horizontal component of $(T, \leqq')$, or a singleton component. (Cf. Lemma* 12.)

(iii) *The preordered set $(T, \leqq)$ has at most one horizontal component. If it has one, then it has either no isolated elements, or at least two isolated elements. (Cf. Lemma* 13.)

PROOF. (i) is proved by the same argument as Lemma 12.

(ii) Let $x, y \in J$ and $x \neq y$. Let $u < x$. If $u \sim x$ then $u \in T$, so x is not isolated in T, a contradiction. Therefore $u <!x$, and hence $u <!y$ since $x \sim^* y$. This and a similar argument proves that $x <' y$, and similarly $y <' x$. Thus J is connected in $(T, \leqq')$, and horizontal. Since T is a union of singleton components and horizontal components, (i) implies that J is in fact maximal connected in $(T, \leqq')$, and is therefore a horizontal component.

If J is a singleton, either $T = J = \{x\}$ in which case the assertion follows, or T has a horizontal component. In the latter case, suppose $a <' x$ and $a \in T$. If $a <'!x$ then $x <'!x$ since $a \sim^* x$, a contradiction. Therefore

$a \sim' x$. Now a belongs to some horizontal component G, say, and there exists $b \in G\backslash\{a\}$. Then $b \sim a$ so $b \sim x$, a contradiction. So x is isolated in $(T, \leqq')$.

(iii) Let F, G be distinct horizontal components of $(T, \leqq)$. All elements of $F \cup G$ are isolated in $(T, \leqq')$. Let $x \in F, y \in G$.

Suppose $u \in X, u <' x$. If $u <'!x$ then $u <!x$ since $\leqq$ is alternating, hence $u <!y$ since $x, y \in T$ and T is a caste, and hence $u <'!y$. Assume instead that $u \sim' x$. Then $u \in T$ since x is isolated in $(T, \leqq')$. But for all $a, b \in X$,

$$a <!u \Rightarrow a <!u <' x \Rightarrow a <!x$$

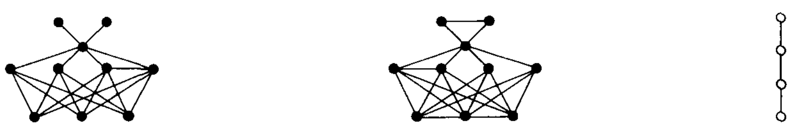
and

$$b >!x \Rightarrow b >!x >' u \Rightarrow b >!u,$$

so $u <^* x$; similarly $x <^* u$. Therefore $u \sim^* x, u \in T$, a contradiction. Therefore $u <' x \Rightarrow u <' y$. Similarly $t >' y \Rightarrow t >' x$, whence $x \leqq'' y$, that is, $x < y$. But this implies that x, y belong to the same component of $(T, \leqq)$, a contradiction.

If $(T, \leqq)$ has exactly one isolated component $F = \{x\}$ and horizontal component G, a similar argument leads again to a contradiction, since $F \cup G$ is again a set of isolated elements in $(X, \leqq')$.

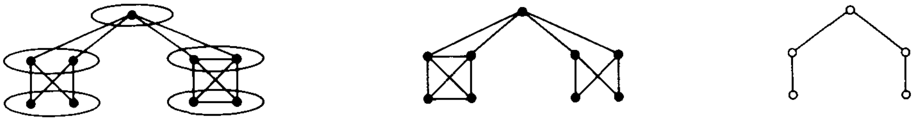
Ex. 12°.



$\leqq$ $\qquad$ $\leqq'$ $\qquad$ $(\mathscr{C}, \precsim)$

Here $\leqq$ is alternating. Each of the four horizontal levels in the first diagram is a caste, and likewise for the second diagram.

Ex. 13°.



$\leqq$ $\qquad$ $\leqq'$ $\qquad$ $(\mathscr{C}, \precsim)$

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Here $\leqq$ is alternating. The five castes of $(X, \leqq)$ are shown circled: $(\mathscr{C}, \preceq)$ is partially ordered and not full. Note that $(X, \leqq)$ and $(X, \leqq')$, although distinct, are order isomorphic.

We shall now describe a method of building preordered sets of greater and greater complexity from initially given preordered sets, which preserves the property of being alternating. (We do not give a formal definition of complexity.)

DEFINITION 19. Let $(X, \leqq_X)$ and $(Y, \leqq_Y)$ be preordered sets. (It is assumed that X and Y are disjoint; the symbols $\leqq_X$ and $\leqq_Y$ may be abbreviated as $\leqq$.) Let G be a caste of X. Write $Z = (X \backslash G) \cup Y$ and for $a, b \in Z$ write $a <_Z b$ when either

(i) $a, b \in X \backslash G$ and $a <_X b$, or
(ii) $a, b \in Y$ and $a <_Y b$, or
(iii) $a \in X \backslash G$, $b \in Y$ and $a <_X !g$ for all $g \in G$, that is, $\lambda(a) \prec G$, or
(iv) $a \in Y$, $b \in X \backslash G$ and $g <_X !b$ for all $g \in G$, that is, $G \prec \lambda(b)$.

Write $a \leqq_Z b$ to mean $a <_Z b$ or $a = b$. Then $(Z, \leqq_Z)$ is the *ramification of* $(X, \leqq_X)$ *at* G *by* $(Y, \leqq_Y)$.

It can be verified that $\leqq_Z$ is a preorder in Z. Further, by enumeration of cases one can prove:

LEMMA 20. *If* $(Z, \leqq)$ *is the ramification of* $(X, \leqq)$ *at caste* G *by* $(Y, \leqq)$, *then the castes of* $(Z, \leqq)$ *consist of the castes of* X *other than* G, *and the castes of* Y. *For* $A \in \mathscr{C}(Y, \leqq)$ *and* $B \in \mathscr{C}(X, \leqq)$, $B \neq G$, *we have* $A \prec B$ *if and only if* $G \prec B$, *and dually* $A \succ B$ *if and only if* $G \succ B$. *In other cases* $A \prec B$ *has the meaning inherited from* $\mathscr{C}(X, \leqq)$ *or* $\mathscr{C}(Y, \leqq)$.

Let $\leqq'_Z$ denote the associated order $(\leqq_Z)'$ of $\leqq_Z$ on Z. We have the following description of $<'_Z$.

LEMMA 21. *Let* X, Y, G, Z *be as in Definition* 19. *Suppose that* $(G, \leqq | G)$ *has no horizontal component, that is, the components of* G *are all isolated points of* G. *Then for* $a, b \in Z$, $a <'_Z b$ *if and only if one of the following conditions holds*:

(i) $a, b \in X \backslash G$ *and* $a <'_X b$;
(ii) $a, b \in Y$ *and* $a <'_Y b$;
(iii) $a \in X \setminus G$, $b \in Y$ *and* $\lambda(a) \prec G$ (*and then* $a <'_X g$ *for all* $g \in G$);
(iv) $a \in X \backslash G$, $b \in \operatorname{Max}(Y)$, $\lambda(a) \| G$ *and* $a <'_X g$ *for all* $g \in G$;
(v) $a \in Y$, $b \in X \backslash G$ *and* $G \prec \lambda(b)$ (*and then* $g <'_X b$ *for all* $g \in G$);
(vi) $a \in \operatorname{Min}(Y)$, $b \in X \backslash G$, $G \| \lambda(b)$ *and* $g <'_X b$ *for all* $g \in G$.

Remarks. In (iii) and (v) the statements in brackets are consequences of the preceding conditions. In (iv) and (vi), $A \| B$ means $A \not\preceq B$ and $B \not\preceq A$. Note that since the order conditions in (iii) do not mention b, we deduce the statement

(30) If $a \in X \backslash G$, then $\lambda(a) \prec G$ implies $a <'_Z b$ for all $b \in Y$.

Similarly, (iv) implies

(31) If $a \in X \backslash G$, then $\lambda(a) \| G$ and $a <'_X g$ for all $g \in G$ together imply $a <'_Z b$ for all $b \in \operatorname{Max}(Y)$.

Dual comments apply to (v), (vi).

The proof of Lemma 21 is rather long, since it involves many cases. We will give the outline, with details for some parts. *Assume $a <'_Z b$*, that is,

$$(32) \qquad (\forall u \in Z)(u <_Z a \Rightarrow u <_Z b)$$

and

$$(33) \qquad (\forall v \in Z)(v >_Z b \Rightarrow v >_Z a).$$

Recall that $x <_X! y$ if and only if $s <_X! t$ for all $s \in \lambda(x), t \in \lambda(y)$. We use repeatedly the cases in Definition 19.

Cases (i) and (ii) are straightforward. *Suppose $a \in X \backslash G$, $b \in Y$.* If $\lambda(a) \prec G$ then we are in Case (iii). We cannot have $G \prec \lambda(a)$ (for otherwise $b <_Z a <'_Z b$, so $b <_Z b$, a contradiction). This leaves the case $\lambda(a) \| G$. Since we cannot have $g <_X a$ $(g \in G)$, (32) reduces to

$$(32)_1 \qquad (\forall g \in G)(\forall u \in X)(u <_X a \Rightarrow u <_X! g).$$

Consider (33). This implies that for $y \in Y$, $y >_Y b \Rightarrow G \succ \lambda(a)$. Thus we have a contradiction unless $b \in \operatorname{Max}(Y)$. Assume $b \in \operatorname{Max}(Y)$; then (33) reduces to

$$(33)_1 \qquad (\forall g \in G)(\forall v \in X \backslash G)(v >_X! g \Rightarrow v >_X a).$$

The premise can be weakened to $v >_X g$. Moreover, although $G \| \lambda(a)$,

$$(33)_2 \qquad (\forall g \in G)(\forall v \in G)(v >_X g \Rightarrow v >_X a)$$

holds, since $v >_X g$ is false because G has no horizontal component. Together, $(32)_1$, $(33)_1$ and $(33)_2$ give $a <'_X g$ for all $g \in G$, and we are in Case (iv).

Cases (v), (vi) are the duals of (iii), (iv) respectively. Thus $a <'_Z b$ implies that one of (i)–(vi) holds.

Converse. If (i), (ii) or (iii) holds, then $a <'_Z b$. Suppose instead (iv). Suppose $u \in Z$, $u <_Z a$. We cannot have $u \in Y$, so $u \in X \backslash G$, $u <_X a <'_X g$, $u <_X! g$ for all $g \in G$, and hence $u <_Z b$. This proves (32). Suppose $v \in Z$,

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$v >_Z b$. We cannot have $v \in Y$, so $v >_X !g >'_X a$ for all $g \in G$, $v >_X a$. This proves (33). Therefore $a <'_Z b$. Cases (v), (vi) are similar. This concludes the proof.

This brings us to the main result concerning ramifications.

THEOREM 22. *Let X, Y, G, Z be as in Definition 19, so that Z is the ramification of X at G by Y. If $\leqq_X$ and $\leqq_Y$ are alternating, if G has no horizontal component, and if* $\mathrm{Min}'(Y) = \mathrm{Max}'(Y) = \varnothing$, *then $\leqq_Z$ is alternating.*

PROOF. Noting Theorem 2(i), we have to show, for arbitrary $a, b \in Z$, that

$$(34) \qquad a \leqq''_Z b \text{ implies } a \leqq_Z b.$$

Assume $a \leqq''_Z b$, that is,

$$(35) \qquad (\forall u \in Z)(u <'_Z a \Rightarrow u <'_Z b),$$

$$(36) \qquad (\forall v \in Z)(v >'_Z b \Rightarrow v >'_Z a).$$

CASE (i), $a, b \in X \backslash G$. Suppose $u \in X \backslash G$ and $u <'_X a$. Then $u <'_Z a$, so $u <'_Z b$ by (35), that is $u <'_X b$. Suppose instead $u \in G$ and $u <'_X a$. We cannot have $u \sim' a$, since this implies $u \sim^* a$ by (18), $a \in G$. Thus $u <'!a$, so $u <!a$ since $\leqq_X$ is alternating. Therefore $G \prec \lambda(a)$, so by Lemma 21(v), $y <'_Z a$ for all $y \in Y$. Since $a \leqq''_Z b, y <'_Z b$. By 21(v), (vi), $g <'_X b$ for all $g \in G$, and hence $u <'_X b$. We have proved

$$(37) \qquad (\forall u \in X)(u <'_X a \Rightarrow u <'_X b).$$

The dual implication is proved similarly, from (36), showing that $a \leqq''_X b$. Therefore $a \leqq_X b$. This says $a \leqq_Z b$. So (34) is proved in this case.

CASE (ii), $a, b \in Y$. The proof of (34) is straightforward.

CASE (iii), $a \in X \backslash G, b \in Y$. Since we assume that $\mathrm{Max}'(Y) = \varnothing$, b cannot be maximal with respect to $\leqq'_Y$, so (36) implies that $v >'_Z a$ for some $v \in Y$. By 21(iii), (iv), $a <'_X g$ for all $g \in G$. Now if $a \sim'_X g$ then $a \sim^* g, \lambda(a) = G$, a contradiction. So $a <'_X !g$, whence $a <_X !g$ for all $g \in G$. By 19(iii), this says $a <_Z b$.

CASE (iv), $a \in Y$, $b \in X \backslash G$. The argument is the dual of that in (iii).

This concludes the verification of (34). Thus $\leqq_Z$ is alternating or stationary. It must be the former, since it relativizes to $\leqq_Y$ on Y, and $\leqq_Y$ is alternating. This concludes the proof.

The conditions in Theorem 22 on G and Y are evidence of the need to have no non-singleton $\sim$-equivalence classes either in G or in the boundary of the preordered set Y replacing G. Here by 'boundary' we mean the part of Y

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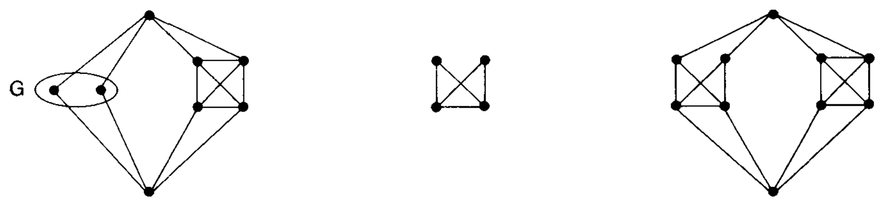
most immediate to $X\backslash G$, that is, $\mathrm{Min}(Y) \cup \mathrm{PMin}(Y) \cup \mathrm{Max}(Y) \cup \mathrm{PMax}(Y)$. In fact, by (26), (27) and their duals the condition on the alternating preordered set $(Y, \leqq)$ implies that

$$\begin{array}{ll} \mathrm{Min}(Y) = \mathrm{PMin}'(Y), & \mathrm{Max}(Y) = \mathrm{PMax}'(Y), \\ \mathrm{PMin}(Y) = \varnothing, & \mathrm{PMax}(Y) = \varnothing. \end{array} \tag{37}$$

The following examples show that the theorem fails if the conditions on G or Y are omitted.

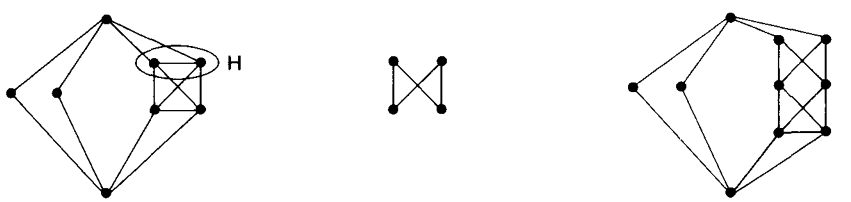
Ex. 14°.



$(X, \leqq)$ $(Y, \leqq)$ $(Z, \leqq)$

Here $\leqq_X, \leqq_Y$ are alternating, G is the circled caste of X, Z is the ramification of X at G by Y; G has no horizontal component, but $\mathrm{Min}'(Y) \neq \varnothing$, and $\leqq_Z$ is not alternating.

Ex. 15°.



$(X, \leqq)$ $(V, \leqq)$ $(U, \leqq)$

Here X is as in 14°, but we ramify at the caste H, which has (is) a horizontal component, by V, for which $\mathrm{Min}'(V) = \mathrm{Max}'(V) = \varnothing$, to get U; $\leqq_U$ is not alternating.

Starting from simple examples and using Theorem 22, we can construct successively more and more complicated alternating preordered sets. This is the sense in which alternating preordered sets were asserted to have unbounded complexity.

Of course, it is also possible to reduce the cardinality of X by ramification, and to simplify X to the extent of replacing a component G by a set Y consisting of isolated elements, with $\mathrm{card}(Y) < \mathrm{card}(G)$. But a proper examination of the process inverse to ramification involves identifying the subsets of an alternating preordered set $(Z, \leqq)$ which can be said to have been created by ramification. These ideas are pursued in [7].

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