# GENERATION OF THE LOWER CENTRAL SERIES

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**0. Introduction.** Let $G$ be a group. The $r$th term $L_rG$ of the lower central series of $G$ is the subgroup generated by the $r$-fold commutators

$$\Gamma_rG=\{[x_0,\ldots,x_r]\mid x_i\in G\},$$

where $[x_0]=x_0$, $[x_0,x_1]=x_0^{-1}x_1^{-1}x_0x_1$, and for $r>1$,

$$[x_0,\ldots,x_r]=[[x_0,\ldots,x_{r-1}],x_r].$$

Dark and Newell [**1**, Theorem 1] proved that if $G$ is nilpotent and $L_rG$ is cyclic, then $L_rG=\Gamma_rG$. In this paper, we generalize this and obtain:

THEOREM A. *Suppose $r\geq 2$. There exists a group $G$ with $L_rG$ cyclic of order $n$ and $L_rG\neq(\Gamma_rG)^k$ if and only if $n=p_1^{\alpha_1}\ldots p_m^{\alpha_m}$, where the $p_i$ are distinct primes and $m\geq 2^{k+1}-1$.*

The case $r=1$ has been studied in great detail (see [**3**], [**7**] and [**8**]). For $r=1$, the condition $m\geq 2^{k+1}-1$ must be replaced by a more complicated set of conditions (see [**4**, Theorem 5]). In section 2, we show that if $L_rG=\langle a\rangle$ with $a\in(\Gamma_rG)^k$, then $L_rG=(\Gamma_rG)^{k+1}$. Some results for $L_rG$ a rank 2 abelian $p$-group are given in section 3. In particular, an example is constructed to show that for $r\geq 2$, this does not imply $L_rG=\Gamma_rG$.

**1. Proof of Theorem A.** We first consider an example. This is a generalization of one of MacDonald [**7**].

EXAMPLE 1.1. Denote the nonempty subsets of $\{1,\ldots,k\}$ by $\alpha_1,\alpha_2,\ldots,\alpha_s$, where $s=2^k-1$. Let $A_1,\ldots,A_s$ be nontrivial abelian groups. Then we can choose abelian groups $B_1,\ldots,B_s$ so that

$$A_i=\{b^{2^r}\mid b\in B_i\}\qquad(i=1,\ldots,s).$$

Let $E=\langle x_1,\ldots,x_k\rangle$ be an elementary abelian group of order $2^k$. Consider $G=(B_1\times\ldots\times B_s)E$ (semidirect), where if $b\in B_j$, then

$$x_ibx_i=\begin{cases}b & \text{if } i\in\alpha_j,\\ b^{-1} & \text{if } i\notin\alpha_j.\end{cases}$$

It is easily seen that $L_rG=A_1\times\ldots\times A_s$. We claim that if $r\geq 2$ and $1\neq a_i\in A_i$ for each $i$, then

$$(a_1,\ldots,a_s)\in L_rG-(\Gamma_rG)^{k-1}.$$

To see this suppose $c\in(\Gamma_2G)^{k-1}\supseteq(\Gamma_rG)^{k-1}$. It is straightforward to verify that this implies

$$c=\prod_{i=1}^{k-1}[(b_{i1},\ldots,b_{is}),y_i],$$

where $b_{ij} \in B_j$ and $y_i \in E$ for $i = 1, \ldots, k-1$. Let $H$ be a hyperplane of $E$ containing $\langle y_1, \ldots, y_{k-1}\rangle$. Now each $C_E(B_j)$ is a hyperplane and $C_E(B_j) \neq C_E(B_k)$ unless $j = k$. Thus since there are $s$ hyperplanes, $H = C_E(B_j)$ for some $j$. Thus the $j$th component of $c$ is 1, proving the claim.

This proves the sufficiency of Theorem $A$ by taking $A_i$ to be cyclic of order $p_i^{\alpha_i}$. For necessity, we need some preliminary results.

LEMMA 1.2. *If $A$ is an abelian normal subgroup of $G$, then $[A, x_1, \ldots, x_r] \subseteq \Gamma_r G$.*

*Proof.* The map sending $a \to [a, x_1, \ldots, x_r]$ is an endomorphism of $A$. Hence its image is $[A, x_1, \ldots, x_r]$.

LEMMA 1.3. *Suppose $G$ is a finite group and $x \in (\Gamma_r G)^k$. If $x$ has order $m$ and $(e, m) = 1$, then $x^e \in (\Gamma_r G)^k$.*

*Proof.* Set

$$\phi(g) = \left|\left\{(g_{ij}) \mid g = \prod_{i=1}^{k} [g_{io}, \ldots g_{ir}]\right\}\right|.$$

Clearly $g \in (\Gamma_r G)^k$ if and only if $\phi(g) \neq 0$. Also $\phi$ is a class function. Hence

$$\phi = \sum a_\chi \chi,$$

where the sum runs over the irreducible complex characters of $G$. Gallagher [**2**, Equation 4] has shown that the $a_\chi$ are rational. Let $\theta$ be a primitive $m$th root of 1 and $\sigma$ an automorphism with $\sigma(\theta) = \theta^e$. Thus

$$\begin{aligned}\phi(x) &= \sigma(\phi(x)) \\ &= \sum a_\chi(\sigma\chi(x)) \\ &= \sum a_\chi \chi(x^e) \\ &= \phi(x^e).\end{aligned}$$

In particular, $\phi(x) = 0$ if and only if $\phi(x^e) = 0$.

LEMMA 1.4. *If $L_r G$ is finite, then there exist a finite group $H$ and an isomorphism $\varphi$ of $L_r G$ and $L_r H$ such that $\varphi(\Gamma_s G) = \Gamma_s H$ for all $s \geq r$.*

*Proof.* By passing to a subgroup, we can assume $G$ is finitely generated. Now $G$ is nilpotent-by-finite. Hence $G$ has a torsion-free characteristic subgroup $T$ of finite index (cf. [**9**, p. 153]). Let $H = G/T$. If $s \geq r$, then

$$L_s H = TL_s G/T \cong L_s G/(T \cap L_s G) \cong L_s G.$$

Clearly $\Gamma_s G$ and $\Gamma_s H$ correspond under this isomorphism.

We need one more lemma to obtain the result of Dark and Newell for $L_r G$ finite cyclic.

LEMMA 1.5. *Let $P$ be a $p$-group with $L_rP$ cyclic. Suppose $P=\langle I\rangle$. If $x\in L_rP$, there exist $t\in G$ and $t_1,\ldots,t_r\in I$ such that $\langle x\rangle=\langle[t,t_1,\ldots,t_r]\rangle$.*

*Proof.* Since $P=\langle I\rangle$, there exist $t_0,\ldots,t_r\in I$ such that $y=[t_0,\ldots,t_r]$ is a generator of $L_rP$. If $x\in L_{r+1}P$, the result follows by induction since $|L_{r+1}P|<|L_rP|$. Otherwise

$$x=y^e\equiv[t_0^e,t_1,\ldots,t_r](\mathrm{mod}\ L_{r+1}P),$$

and so $\langle x\rangle=\langle[t_0^e,t_1,\ldots,t_r]\rangle$.

THEOREM 1.6 (Dark–Newell [**1**]). *If $G$ is nilpotent and $L_rG$ is a finite cyclic group, then $L_rG=\Gamma_rG$.*

*Proof.* By Lemma 1.4, we can assume $G$ is finite, and so we can take $G$ a $p$-group. The result now follows from Lemmas 1.3 and 1.5.

Dark and Newell [**1**] also proved the result for $L_rG$ infinite cyclic without assuming $G$ nilpotent. Rodney [**8**] proved the above results for $r=1$. The assumption that $G$ is nilpotent can be weakened. Set

$$L_\infty G=\bigcap_{i=1}^{\infty}L_iG.$$

THEOREM 1.7. *If $L_rG$ is finite cyclic and $L_\infty G$ has order $p^e$, then $L_rG=\Gamma_rG$.*

*Proof.* By Lemma 1.4 and Theorem 1.6, we can assume $G$ is finite and $e\geq 1$. Let $P\in\mathrm{Syl}_p(G)$. Since $H=L_\infty G\subset P$, $P$ is normal in $G$ and thus has a complement $T$. Note that $K=L_rG\cap P\supset H\neq 1$. As $K$ is cyclic, $L_rP\subset K=[T,K]=H$. So if $x\in L_rG$, then $x=hu$ with $h\in H$ and $u\in L_rT$. Since $T$ is nilpotent, by Lemma 1.5, there exist $t\in T$ and $t_1,\ldots,t_r\in T-C_T(H)$ so that $\langle u\rangle=\langle[t,t_1,\ldots,t_r]\rangle$. Now since $t_i\in T-C_T(H)$ and $(|T|,p)=1$, $[H,t_i]=H$, and so $[H,t_1,\ldots,t_r]=H$. Thus there exists $y\in H$ so that $[y,t_1,\ldots,t_r]=h$. Hence $\langle hu\rangle=\langle[ty,t_1,\ldots,t_r]\rangle$, and so $x=hu\in\Gamma_rG$ by Lemma 1.3.

One more result is needed for Theorem A. Set

$$[H,x;1]=[H,x]\quad\text{and}\quad[H,x;n]=[[H,x;n-1],x].$$

If $|H|=p_1^{\alpha_1}\ldots p_m^{\alpha_m}$, set $\ell(H)=m$.

LEMMA 1.8. *Let $A$ be a group acting on a nontrivial cyclic group $H$ such that $[H,A]=H$. Let $r$ be a positive integer.*

(a) *There exists $x\in A$ such that $\ell(H/[H,x;r])<\ell(H)/2$.*
(b) *If $\ell(H)\leq 2^k-3$, there exists $x\in A$ such that $\ell(H/[H,x;r])\leq 2^{k-1}-3$.*

*Proof.* Without loss of generality, we can assume $|H|$ is squarefree, and so $[H,x;r]=[H,x]$ and $H=[H,x]\oplus C_H(x)$. Choose $x\in A$ with $\ell([H,x])$ maximal. By induction, there exists $y\in A$ with $\ell([C,y])>\ell(C)/2$, where $C=C_H(x)$. Hence,

$$\begin{aligned}\ell([H,x])&\geq\ell([H,xy])\\&\geq\ell([H,x])-\ell([H,x,y])+\ell([C,y])\\&=\ell([H,x])-\ell([H,y])+2\ell([C,y]).\end{aligned}$$

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This yields

$$\ell(C) < 2\ell([C, y]) \leq \ell([H, y]) \leq \ell([H, x]) \tag{$*$}$$

and proves (a).

To prove (b), it suffices to assume that $\ell(H) = 2^k - 3$ and $k \geq 2$. Choose $x$ and $y$ as above. By (a), $\ell([H, x]) \geq 2^{k-1} - 1$. If $\ell([H, x]) = 2^{k-1} - 1$, then $\ell(C) = 2^{k-1} - 2$, and so $\ell([C, y]) \geq 2^{k-2}$. Then $(*)$ implies that

$$\ell([H, x]) \geq 2\ell([C, y]) \geq 2^{k-1},$$

and the result follows.

*Proof of Theorem A.* Suppose $L_rG$ is cyclic of order $n = p_1^{\alpha_1} \ldots p_m^{\alpha_m} (m < 2^{k+1} - 1)$. Set $H = L_\infty G$. First consider the case $H = L_rG$. By Lemma 1.8(a), there exists $x \in G$ with

$$\ell(H/[H, x; r]) < m/2 \leq 2^k - 1.$$

By induction and Lemma 1.1,

$$L_rG = (\Gamma_rG)^{k-1}[H, x; r] \subseteq (\Gamma_rG)^k$$

as desired.

Now assume $H \neq L_rG$. Since $H$ is a summand of $L_rG$, this implies $\ell(H) \leq 2^{k+1} - 3$. We now show that if $\ell(H) \leq 2^{k+1} - 3$, then $(L_rG) = (\Gamma_rG)^k$. This follows from Theorem 1.7 for $k = 1$. If $k \geq 2$, by Lemma 1.8(b), there exists $x \in G$ with $\ell(H/[H, x; r]) \leq 2^k - 3$. As above, we obtain $L_rG = (\Gamma_rG)^k$.

**2. Generators of $L_rG$.** If $L_rG = \langle a \rangle$ and $a \in \Gamma_rG$, then all generators of $L_rG$ are in $\Gamma_rG$ by Lemma 1.3. However, this does not imply $L_rG = \Gamma_rG$. (See [**3**] for examples with $r = 1$.) Similar examples can be constructed for $r > 1$. However, we do obtain:

THEOREM 2.1. *If $L_rG = \langle a \rangle$ and $a \in (\Gamma_rG)^e$, then $L_rG = (\Gamma_rG)^{e+1}$.*

*Proof.* If $L_rG$ is infinite, then $L_rG = \Gamma_rG$ by [**1**, Theorem 4]. If $r = 1$, the result follows by [**4**, Theorem 1]. Thus we can assume $G$ is finite and $r \geq 2$. Let $H = L_\infty G$. Thus

$$a = \prod_{i=1}^{e} [t_{0i}, \ldots, t_{ri}].$$

It follows easily since $r \geq 2$ that

$$H = \prod_{i=1}^{e} [H, t_{ri}] = \prod_{i=1}^{e} [H, t_{ri}; r] \subseteq (\Gamma_rG)^e.$$

Hence by Theorem 1.6, we have $L_rG = (\Gamma_rG)H \subseteq (\Gamma_rG)^{e+1}$.

If $r = 1$ and $e \geq 2$, then in fact $G' = (\Gamma_1G)^e$ (*see* [**4**, Theorem 1]). We do not know if this is true for $r > 1$.

**3. Rank 2 Subgroups.** By Example 1.1, if $A$ is a finitely generated abelian group with $A = A_1 \times A_2 \times A_3$, there exists $G$ with $L_rG = A$ and $L_rG \neq \Gamma_rG$. This leaves open the case where $L_rG$ is a rank 2 $p$-group. Dark and Newell [**1**, Theorem 2] proved in this case that if also $L_{r+1}G=\{1\}$, then $L_rG=\Gamma_rG$. (See [**6**] for the case $r=1$.) The author [**5**, Theorem A] has shown that if $P\in \mathrm{Syl}_p(G')$ is a rank 2 abelian $p$-group, then $P\subseteq \Gamma_1G$. We give an example with $p=2$ and $r=2$ with $L_rG\neq\Gamma_rG$.

EXAMPLE 3.1. Let $G=\langle x, y, a, b\rangle$ with relations

$$[x,y]=b,\qquad b^x=b^{-1},\qquad b^y=ba,$$
$$b^8=a^2=[x,a]=[y,a]=[b,a]=1.$$

Now

$$\Gamma_2G\subset[G',x]\cup[G',y]\cup[G',xy]=\langle b^2\rangle\cup\langle a\rangle\cup\langle ab^2\rangle.$$

Hence $ab^4\in L_2G$, but $ab^4\notin\Gamma_2G$. Similar examples can be constructed for any $r\geq 2$.

Certain assumptions do guarantee that $L_rG=\Gamma_rG$.

THEOREM 3.2. *Suppose $L_rG$ is a rank 2 abelian $p$-group. If any of the following hold, then $L_rG=\Gamma_rG$.*

(i) $r=1$.
(ii) $L_{r+1}G=\{1\}$.
(iii) *$G$ is not nilpotent.*
(iv) *$L_rG$ has exponent $p$.*

*Proof.* As we remarked above, (i) and (ii) are known. We sketch the proof of (iii). So assume $H=L_\infty G\neq\{1\}$. As usual, we take $G$ finite. Let $T$ be a complement of $P\in\mathrm{Syl}_p(G)$. We can choose $t\in T$ with $[H,t;r]=[H,t]=H$ (see [**5**, Lemma 2.6]). Then

$$L_rG=\{[ts_0,\dots,ts_r]\mid s_i\in P\}\subseteq\Gamma_rG.$$

This follows by Theorem 1.6 and induction on $G/[ZP,t]$. Further, (iv) follows since if (ii) and (iii) do not hold, then $L_{r+1}G$ is cyclic and central. Hence $L_{r+1}G\subseteq\Gamma_rG$ and if $x,y\in L_rG-L_{r+1}G$, then $\langle x\rangle$ and $\langle y\rangle$ are conjugate. Hence $L_rG\subseteq\Gamma_rG$ by Lemma 1.3.

We close with a conjecture.

CONJECTURE 3.3. There exists a finite set of primes $\Omega$ (perhaps depending on $r$) such that if $p\notin\Omega$ and $L_rG$ is a rank 2 abelian $p$-group, then $L_rG\doteq\Gamma_rG$.

For $r=1$ and $L_1G=G'$ a rank 3 abelian $p$-group, we can take $\Omega=\{2,3\}$ (see [**5**, Theorem B]).

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