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COEFFICIENT MULTIPLIERS OF BERGMAN SPACES A^p, II

Dedicated to my teachers

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ABSTRACT. We show that the multiplier space $(A^1, X) = \{g : M_\infty(r, g'') = O(1-r)^{-1}\}$, where X is BMOA, VMOA, B, B_0 or disk algebra A. We give the multipliers from A^1 to $A^q(H^q)(1 \le q \le \infty)$, we also give the multipliers from $l^p(1 \le p \le 2)$, C_0, BMOA, and $H^p(2 \le p < \infty)$ into $A^q(1 \le q \le 2)$.

1. **Introduction.** For $0 < p \le \infty$, by H^p we denote the Hardy space (see [5]) of analytic functions $f(z)$ in the unit disk D, for which

$$\|f\|_{H^p} = \lim_{r \to 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\Big)^{1/p},$$

or

$$M_\infty(r, f) = \max_{0 \le \theta \le 2\pi} |f(re^{i\theta})|.$$

The Bergman space [1] $A^p(0 < p \le \infty)$ consists of all analytic functions f in D for which

$$\|f\|_{A^p} = \Big(\int_0^1 M_p(r, f)^p r \, dr\Big)^{1/p} < \infty.$$

and $A^\infty = H^\infty$. Thus A^p and H^p are Banach spaces if $p \ge 1$, and Fréchet spaces if $0 < p < 1$.

Let X and Y be two vector spaces of sequences. A sequence $\lambda = \{\lambda_n\}$ is said to be a *multiplier* from X to Y, if $\{\lambda_n x_n\} \in Y$ whenever $\{x_n\} \in X$. The set of all multipliers from X to Y will be denoted by (X, Y). We regard spaces of analytic functions in the disc as sequence space by identifying a function with its sequence of Taylor coefficients.

It is an important question in function theory to describe the coefficient multipliers between various spaces of analytic functions. This previous way of obtaining information on the Taylor coefficients of functions in certain spaces makes it possible to examine whether a given function is in a particular space. For example: the coefficient multipliers

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between the Hardy spaces H^p and H^q have been studied extensively for a wide range of indices p, q ([5]). However, this does not seem to be the case for Bergman spaces A^P. Multipliers for A^p spaces are studied by several authors, such as Vukotic [18], Wojtaszczyk [19], MacGregor-Zhu [13] and Ahern [1].

In Section 3, we first give an interesting result (Theorem 3.1), which shows that

$$\begin{aligned}(A^1, A) = (A^1, B) = (A^1, B_0) = (A^1, \mathrm{BMOA}) = (A^1, \mathrm{VMOA})\\ = \{g : M_\infty(r, g'') = O(1-r)^{-1}\},\end{aligned}$$

where A denotes the space of all functions which are analytic in D and continuous on $\bar{D}$, B, The Bloch space ([7], [20]), defined by $f \in B$ if and only if f is analytic in D and

$$\|f\|_B = |f(0)| + \sup_{z\in D}(1-|z|^2)|f'(z)| < \infty,$$

B_0, the little Bloch space, the set of analytic functions f in D, for which

$$(1-|z|^2)|f'(z)| \longrightarrow 0, \quad |z| \longrightarrow 1,$$

and BMOA ([7], [20]), the space of analytic functions of Bounded Mean Oscillation. By VMOA we denote the space of analytic functions of Vanishing Mean Oscillation.

As a main theorem, Vukotić has proved the following

THEOREM A ([18, THEOREM 10]).

$$(A^1, A^2) = \Big\{\{\lambda_n\} : \sum_{n=1}^{N} n^2|\lambda_n|^2 = O(N)\Big\}.$$

We extend this result and show that

$$(A^1, A^q) = \{g : M_q(r, g'') = O(1-r)^{-1-\frac{1}{q}}\},$$

for $1 \le q \le \infty$ (Theorem 3.2). The analogue of this result for H^p space is a open problem [4].

In [13], MacGregor and Zhu have given a sufficient condition for multipliers from A^p into $H^p (1 \le p \le 2)$ and from $H^p (2 \le p < \infty)$ into $A^q (1 \le q \le 2)$.

THEOREM B. *(i) For* $1 \le p \le 2$ *we have* $\{n^{-\frac{1}{p}}\}_{n=1}^\infty \in (A^p, H^p)$.
(ii) For $2 \le q < \infty$ *we have* $\{n^{\frac{1}{q}}\}_{n=1}^\infty \in (H^q, A^q)$.

We extend the result and give a necessary and sufficient condition for multipliers from A^1 into H^q for $1 \le q \le \infty$ (Theorem 3.3), we also give a necessary and a sufficient condition for multipliers from A^p into H^q for $1 \le p, q \le 2$.

In Section 4, we describe the multipliers from some spaces into A^q, and give some necessary and sufficient conditions for multipliers from $l^p (1 \le p \le 2)$, C_0 (Theorem 4.2), BMOA (Theorem 4.5), $H^p (2 \le p < \infty)$ (Corollary 4.6) into $A^q (1 \le q \le 2)$.

In this paper, the letter C will denote the constant depending only on the indexes $p, q, \ldots, C$ may differ at different occurrences.

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2. **Preliminaries.** The following lemmas will be used in proving the theorems.

LEMMA 2.1 ([16, THEOREM 5], [5, THEOREM 5.6]). *Let* $1 \le s \le \infty$, $-1 \le b < \infty$ *and* $1 \le a < \infty$. *Then for all* $\beta > 0$

$$\int_0^1 (1-r)^b M_s^a(r,f)\,dr \le C\int_0^1 (1-r)^{a+b} M_s^a(r,f')\,dr + |f(0)|^a.$$

LEMMA 2.2 ([16, LEMMA 5]). *Let* $0 < s \le \infty$, $0 < p < q < \infty$, *and* $\beta > 0$. *Then*

$$\Big(\int_0^1 (1-\rho)^{q\beta-1} M_s^q(r\rho,f)\,d\rho\Big)^{\frac{1}{q}} \le C\Big(\int_0^1 (1-\rho)^{p\beta-1} M_s^p(r\rho,f)\,d\rho\Big)^{\frac{1}{p}}$$

for every $r \in (0,1]$.

LEMMA 2.3 ([12, PROOF OF THEOREM 2.7]). *For* $1 \le p \le 2$,

$$s(A^p) = \big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l(2,p)\big\},$$

where $s(X)$ *is the largest solid subspace of* X *(see [3] for the details).*

LEMMA 2.4. *For* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

(i) $(A^p)^a = \{g : g' \in A^q\} \triangleq G^q$.

(ii) $(A^p)^{aa} \subset A^p$,

where X^a *is the Abel dual of* X *(see [3]).*

(iii) For $1 \le p \le \infty$, *let* $f_z(w) = f(zw)$, $w \in D$, $z \in \bar{D}$, *then*

$$\|f_{z_1} - f_{z_2}\|_{A^p} \longrightarrow 0, \quad |z_1 - z_2| \longrightarrow 0.$$

PROOF. (i) can be found in Taibleson [17]. See also Shapiro [15].

(ii) From (i) $(A^p)^{aa} = \{f : f'' \in A^p\}$. If $\int_0^1 M_p(r,f'')^p\,dr < \infty$, from Lemma 2.1 $\int_0^1 M_p(r,f)^p\,dr < \infty$. That is $f \in A^p$.

(iii) $0 < s < 1$,

$$\begin{aligned}\|f_{z_1} - f_{z_2}\|_{A^p} &= \int_0^1 M_p(r, f_{z_1} - f_{z_2})^p\,dr\\ &= \int_0^s M_p(r, f_{z_1} - f_{z_2})^p\,dr + \int_s^1 M_p(r, f_{z_1} - f_{z_2})^p\,dr\\ &\le \int_0^s M_p(r, f_{z_1} - f_{z_2})^p\,dr + 2^p\int_s^1 M_p(r,f)^p\,dr.\end{aligned}$$

First we choose s so that the second term is as small as we please, independent of z_1, z_2, then the first term goes to zero as $|z_1 - z_2| \longrightarrow 0$. ∎

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LEMMA 2.5. *Let r, s, u and v be real numbers in* $(0,\infty]$, *and define p and q by*
$\frac{1}{p}=\frac{1}{u}+\frac{1}{r}$ *if* $r>u$, $p=\infty$ *if* $r\le u$.
$\frac{1}{q}=\frac{1}{v}+\frac{1}{s}$ *if* $s>v$, $q=\infty$ *if* $s\le v$.

Then $\big(l(r,s),l(u,v)\big)=l(p,q)$, where $l(p,q)$ denotes the set of those sequences $\{a_k\}$ $(k\ge 1)$ for which

$$\Big\{\Big(\sum_{I_n}|a_k|^p\Big)^{\frac{1}{p}}\Big\}_{n=0}^{\infty}\in l^q \quad p<\infty,$$
$$\{\sup_{k\in I_n}|a_k|\}_{n=0}^{\infty}\in l^q \quad p=\infty,$$

and $I_n=\{k:2^n\le k<2^{n+1}\}$.

The lemma was proved in [8, Theorem 1] in the case $1\le r,s,u,v\le\infty$. The proof shows that it holds for all $0<r,s,u,v\le\infty$.

We remark that $l(p,p)=l^p$.

LEMMA 2.6 ([3, LEMMA 1, LEMMA 3]). *If A, B, D are any sequence spaces, then*
(i) $A\subset B\Rightarrow(B,D)\subset(A,D)$.
(ii) $(A,B)\subset(B^a,A^a)$.
(iii) X be a solid space, then $(X,A)=(X,s(A))$.

3. **Multipliers from A^1 to some spaces.** We begin with the following interesting result,

THEOREM 3.1. $(A^1,A)=(A^1,B)=(A^1,B_0)=(A^1,\mathrm{BMOA})=(A^1,\mathrm{VMOA})=\{g:M_\infty(r,g'')=O(1-r)^{-1}\}$.

PROOF. It is well known that ([7], [20]) VMOA $\subset$ BMOA $\subset B$, $A\subset B$, $B_0\subset B$, and VMOA $\subset B_0$. So by Lemma 2.6 it is easy to show the following inclusions

$$(A^1,\mathrm{VMOA})\subset(A^1,\mathrm{BMOA})\subset(A^1,B),$$
$$(A^1,A)\subset(A^1,B),\quad (A^1,B_0)\subset(A^1,B),$$
$$(A^1,\mathrm{VMOA})\subset(A^1,B_0).$$

So it is enough to prove
(a) $\{g:M_\infty(r,g'')=O(1-r)^{-1}\}\subset(A^1,A)$.
(b) $(A^1,B)\subset\{g:M_\infty(r,g'')=O(1-r)^{-1}\}$.
(c) $\{g:M_\infty(r,g'')=O(1-r)^{-1}\}\subset(A^1,\mathrm{VMOA})$.

We first prove (a).

Suppose $f(z)=\sum a_nz^n\in A^1, g(z)=\sum x_nz^n\in\{g:M_\infty(r,g'')=O(1-r)^{-1}\}$ (note that $\|g'\|_B<\infty$) and $h=f*g$, where $(f*g)(z)=\sum a_nx_nz^n$ is the Hadamard product of $f(z)=\sum a_nz^n$ and $g(z)=\sum x_nz^n$. Then

$$h(\rho z)=\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{it})g(ze^{-it})\,dt,\quad 0<r<1. \tag{1}$$

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Differentiation with respect to z in (1) gives

$$\rho^2 h''(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it}) g''(z e^{-it}) e^{-2it}\, dt. \tag{2}$$

Setting $|z| = r = \rho^2$ in (2), we have

$$\begin{aligned}\rho^2 |h''(\rho^3 e^{i\theta})| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})|\, dt M_\infty(\rho^2, g'') \\ &\le C(1-\rho)^{-1} M_1(\rho, f)\|g'\|_B.\end{aligned}$$

Since

$$\int_0^1 (1-r) M_\infty(r, h'')\, dr = 3\int_0^1 (1-r) r^2 M_\infty(r^3, h'')\, dr$$

then

$$\int_0^1 (1-r) M_\infty(r, h'')\, dr \le C\|g'\|_B \|f\|_{A^1}. \tag{3}$$

By Lemma 2.1

$$\int_0^1 M_\infty(r, h')\, dr \le C\int_0^1 (1-r) M_\infty(r, h'')\, dr. \tag{4}$$

From [6, Theorem 5]

$$M_\infty(r, h) \le C\int_0^1 M_\infty(r, h')\, dr. \tag{5}$$

Combine (5) with (4) and (3) we get

$$|(f * g)(z)| \le C\|f\|_{A^1}\|g'\|_B. \tag{6}$$

For $f \in A^1$, $z \in \bar{D}$, set $f_z(w) = f(zw)$. Since the correspondence $z\ (\in \bar{D}) \longrightarrow f_z \in A^1$ is continuous, from (6) and Lemma 2.4(iii) we have, as $|z_1 - z_2| \longrightarrow 0$

$$\begin{aligned}|f * g(z_1) - f * g(z_2)| &= |(f_{z_1} - f_{z_2}) * g(1)| \\ &\le C\|f_{z_1} - f_{z_2}\|_{A^1}\|g'\|_B \longrightarrow 0.\end{aligned}$$

So $f * g \in A$, for all $f \in A^1$. This proves (a).

Now we prove (b).

For $g(z) = \sum x_n z^n \in (A^1, B)$, we define a linear operator $T_g\colon A^1 \longrightarrow B$ by $T_g(f) = f * g$. Then by the closed graph theorem. T_g is a bounded operator from A^1 to B. Let

$$f_r(z) = \frac{2(1-r)(rz)^2}{(1-rz)^3} = (1-r)\sum_{n=2}^{\infty} n(n-1)(rz)^n,$$

by simple computation, $\|f_r\|_{A^1} \le C$, where the constant C is independent of r. Since $f \in A^1$ and T_g is bounded, then

$$\|g * f_r\|_B \le C\|f_r\|_{A^1}. \tag{7}$$

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Let $|z| = r$,

$$g * f_r(z) = (1-r)\sum_{n=2}^{\infty} n(n-1)(1-r)x_n(rz)^n$$
$$= (1-r)(rz)^2 g''(rz).$$

Combine this with (7) we have

$$M_\infty(r, g''') = O(1-r)^{-2}. \tag{8}$$

By [5, Theorem 5.5] (8) is equivalent to

$$M_\infty(r, g'') = O(1-r)^{-1}.$$

That is the proof of (b).

Finally we prove (c).

From $(A^1, A) \subset (A^1, \mathrm{BMOA}) \subset (A^1, B)$, (a) and (b) we find that

$$(A^1, \mathrm{BMOA}) = \{g : M_\infty(r, g'') = O(1-r)^{-1}\}, \tag{9}$$

this will be used in the following proof.

By [7, p. 238 Lemma 3.2], $f \in \mathrm{BMOA}$ if and only if

$$\|f\|_{\mathrm{BMOA}} = \sup_{w\in D}\int_D |f'(z)|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-z\bar{w}|^2}\,dx\,dy < \infty, \quad z = x + yi.$$

Let $f \in A^1$ and $g \in \{g : M_\infty(r, g'') = O(1-r)^{-1}\}$. By (9)

$$\|F * g\|_{\mathrm{BMOA}} \le C\|F\|_{A^1}, \quad F \in A^1. \tag{10}$$

Substitute $F = f_r - f$ in (10), we get

$$\|f * g_r - f * g\|_{\mathrm{BMOA}} \le C\|f_r - f\|_{A^1}.$$

From Lemma 2.4(iii)

$$\|f_r - f\|_{A^1} \longrightarrow 0, \quad r \longrightarrow 1,$$

it follows from [7, p. 250 Theorem 5.1], that $f * g \in \mathrm{VMOA}$ for all $f \in A^1$. This proves (c). Which completes the proof of Theorem 3.1. ∎

Now we give a necessary and sufficient condition for multiplier from A^1 into A^q.

THEOREM 3.2. *$(A^1, A^q) = \{g : M_q(r, g'') = O(1-r)^{-1-\frac{1}{q}}\}$, where $1 \le q \le \infty$.*

PROOF. Suppose $g(z) = \sum x_n z^n \in (A^1, A^q)$, and $f(z) = \sum a_n z^n \in A^1$, We define a linear operator $T_g : A^1 \longrightarrow A^q$ by $T_g(f) = f * g$. Then by the closed graph theorem. T_g is a bounded operator from A^1 to A^q. Let

$$f_r(z) = \frac{2(1-r)(rz)^2}{(1-rz)^3}.$$

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By computation, we get $\|f_r\|_{A^1} \le C$, where the constant C is independent of r, and

$$g * f_r(z) = (rz)^2 g''(rz)(1-r). \tag{11}$$

Since T_g is bounded, we have

$$\|g * f_r\|_{A^q} \le C\|f_r\|_{A^1}. \tag{12}$$

Let $|z| = \rho$, from (11) and (12) we get

$$\int_0^1 (1-r)^q M_q(r\rho, g'')^q \, d\rho \le C.$$

Hence

$$M_q(r\rho, g'') \le C(1-r)^{-1}(1-\rho)^{-\frac{1}{q}}.$$

Taking $\rho = r$, we obtain

$$M_q(r^2, g'') \le C(1-r)^{-1-\frac{1}{q}}.$$

So

$$(A^1, A^q) \subset \{g : M_q(r, g'') = O(1-r)^{-1-\frac{1}{q}}\}.$$

To prove the converse, for $1 \le q < \infty$, let $h = f * g$, where g satisfies the condition

$$M_q(r, g'') = O(1-r)^{-1-\frac{1}{q}},$$

$f \in A^1$, then

$$h(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}) g(z e^{-it}) \, dt. \tag{13}$$

Differentiation with respect to z in (13), we get

$$\rho^2 h''(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}) g''(z e^{-it}) e^{-2it} \, dt.$$

Setting $|z| = r = \rho$ this gives

$$\begin{aligned} r^2 M_q(r^2, h'') &\le C M_1(r, f) M_q(r, g'') \\ &\le C M_1(r, f)(1-r)^{-1-\frac{1}{q}}. \end{aligned}$$

So

$$\int_0^1 (1-r)^{2q} M_q(r, h'')^q \, dr \le C \int_0^1 (1-r)^{q-1} M_1(r, f)^q \, dr. \tag{14}$$

Setting $s = 1$, $\beta = 1$, $p = 1$ in Lemma 2.2, we have

$$\left(\int_0^1 (1-r)^{q-1} M_1(r, f)^q \, dr \right)^{1/q} \le C \int_0^1 M_1(r, f) \, dr. \tag{15}$$

Hence from (14), (15) and $\int_0^1 M_1(r, f) \, dr < \infty$, we get

$$\int_0^1 (1-r)^{2q} M_q(r, h'')^q \, dr < \infty.$$

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We use Lemma 2.1 to obtain $\int_0^1 M_q(r,h)^q\,dr < \infty$, That is $h \in A^q$.

For $q = \infty$, since $A \subset H^\infty \subset B$, by Theorem 3.1, we have

$$(A^1, A^\infty) = (A^1, H^\infty) = \{g : M_\infty(r, g'') = O(1-r)^{-1}\}.$$

Hence

$$\{g : M_\infty(r, g'') = O(1-r)^{-1-\frac{1}{q}}\} \subset (A^1, A^q),$$

for $1 \le q \le \infty$. This proves Theorem 3.2. ∎

By [5, Theorem 5.5],

$$M_2(r, g'') = O(1-r)^{-1-\frac{1}{2}}$$

is equivalent to

$$M_2(r, g') = O(1-r)^{-\frac{1}{2}}. \tag{16}$$

With the similar discussion to that of [3, p. 261], (16) is equivalent to

$$\sum_{n=1}^{N} n^2 |x_n|^2 = O(N) \quad \Big(g(z) = \sum x_n z^n\Big).$$

So Theorem 3.2 extends Vukotić's result Theorem A, it also extends Lemma 9 of [18].

The following theorem extends Theorem B partly; its proof is similar to that of Theorem 3.2.

THEOREM 3.3. *$(A^1, H^q) = \{g : M_q(r, g'') = O(1-r)^{-1}\}$, where $1 \le q \le \infty$.*

PROOF. Let $f \in A^1$, $M_q(r, g'') = O(1-r)^{-1}$ and $h = f * g$. Then

$$\rho^2 h''(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}) g''(z e^{-it}) e^{-2it}\,dt.$$

So

$$\begin{aligned} r^2 M_q(r^2, h'') &\le C M_1(r, f) M_q(r, g'') \\ &\le C M_1(r, f)(1-r)^{-1}. \end{aligned}$$

Hence

$$\int_0^1 (1-r) M_q(r^2, h'')\,dr \le C \int_0^1 M_1(r, f)\,dr. \tag{17}$$

Using Lemma 2.1, from (17) and $\int_0^1 M_1(r,f)\,dr < \infty$ we have $\int_0^1 M_q(r,h')\,dr < \infty$. It follows from [14, p. 74 (2.4)] that $h \in H^q$.

To prove the converse, for $1 \le q < \infty$, suppose that $g = \sum x_n z^n \in (A^1, A^q)$, whenever $f \in A^1$. Applying the closed graph theorem in the standard way, we conclude

$$\|f * g\|_{H^q} \le C \|f\|_{A^1}, \quad f \in A^1. \tag{18}$$

We substitute

$$f_r(z) = \frac{2(1-r)(rz)^2}{(1-rz)^3}$$

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for f in the inequality (18), we have

$$\|f_r * g\|_{H^q} \le C\|f_r\|_{A^1} = O(1). \tag{19}$$

Since

$$(f_r * g)(z) = (1-r)(rz)^2 g''(rz)$$

then from (19) we have

$$\|f_r * g\|_{H^q} = (1-r)r^2 \lim_{\rho\to 1} M_q(r\rho, g'') = O(1).$$

Hence

$$M_q(r, g'') = O(1-r)^{-1}.$$

For $q = \infty$, it follows from Theorem 3.1. This completes the proof. ∎

THEOREM 3.4. *For* $1 \le p, q \le 2$

$$\Big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in l\Big(\frac{2p}{2-p}, \infty\Big)\Big\} \subset (A^p, H^q) \subset \Big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in l^\infty\Big\}.$$

PROOF. By Lemma 2.3 and [9, Lemma 4.5]

$$s(A^p) = \big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l(2,p)\big\}, \quad s(H^p) = H^2.$$

From Lemma 2.6, we have

$$\begin{aligned}
(A^p, H^q) \subset \big(s(A^p), H^q\big) &= \Big(\big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l(2,p)\big\}, H^q\Big)\\
&= \Big(\big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l(2,p)\big\}, s(H^q)\Big)\\
&= \Big(\big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l(2,p)\big\}, l(2,2)\Big)\\
&= \Big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in \big(l(2,p), l(2,2)\big)\Big\}\\
&= \big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in l^\infty\big\}.
\end{aligned}$$

Here we have used the result: $\big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in l(2,p)\big\}$ is solid space [3].

From [10, Theorem 1]

$$A^p \subset \Big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l\Big(\frac{p}{p-1}, p\Big)\Big\},$$

we have

$$\begin{aligned}
(A^p, H^q) &\supset \Big(\Big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l\Big(\frac{p}{p-1}, p\Big)\Big\}, H^q\Big)\\
&= \Big(\Big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l\Big(\frac{p}{p-1}, p\Big)\Big\}, s(H^q)\Big)\\
&= \Big(\Big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in l\Big(\frac{p}{p-1}, p\Big)\Big\}, l(2,2)\Big)\\
&= \Big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in \Big(l\Big(\frac{p}{p-1}, p\Big), l(2,2)\Big)\Big\}\\
&= \Big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in l\Big(\frac{2p}{2-p}, \infty\Big)\Big\}.
\end{aligned}$$

This proves the theorem. ∎

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COROLLARY 3.5. *For* $1 \le q \le 2$

$$(A^2, H^q) = \big\{\{a_k\} : \{k^{\frac{1}{2}} a_k\} \in l^\infty\big\}.$$

4. **Multipliers into A^q.**

THEOREM 4.1. *For* $0 < p \le \infty, 1 \le q \le 2$,

$$(l^p, A^q) = \big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in \big(l(p,p), l(2,q)\big)\big\}. \tag{20}$$

PROOF. By Lemma 2.3 and 2.6, we have

$$\begin{aligned}(l^p, A^q) &= \big(l(p,p), s(A^q)\big)\\ &= \big(l(p,p), \big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l(2,q)\big\}\big)\\ &= \big\{\{a_k\} : \{k^{-\frac{1}{p}} a_k\} \in \big(l(p,p), l(2,q)\big)\big\}.\end{aligned}$$

Using Lemma 2.5 to calculate $\big(l(p,p), l(2,q)\big)$, we can get (20). ∎

The following theorem gives a necessary and sufficient condition for multipliers from C_0 into A^q:

THEOREM 4.2. *For* $1 < q \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}(C_0, A^q) &= \Big\{\{a_k\} : \{k^{\frac{1}{p}-1} a_k\} \in l\Big(2, \frac{p}{p-1}\Big)\Big\},\\ (C_0, A^1) &= \big\{\{a_k\} : \{k^{-1} a_k\} \in l(2,1)\big\},\end{aligned}$$

where $C_0 = \big\{\{a_k\} : \{a_k\} \in l^\infty, a_k \longrightarrow 0, k \longrightarrow \infty\big\}$.

PROOF. For $1 < q \le 2$, by [3, p. 257], $(C_0)^a = l^1$, $(l^1)^a = l^\infty$, $C_0 \subset l^\infty$. From Lemma 2.4 and 2.6, we have

$$\begin{aligned}(C_0, A^q) &\subset \big((A^q)^a, l^1\big) = (G^p, l^1)\\ &\subset \big(l^\infty, (A^q)^{aa}\big) \subset (l^\infty, A^q)\\ &\subset (C_0, A^q).\end{aligned}$$

So

$$(C_0, A^q) = (G^p, l^1), \quad 2 \le p < \infty. \tag{21}$$

From [1, Theorem 4], for $2 \le p \le \infty$

$$(A^p, l^1) = \Big\{\{a_k\} : \{k^{\frac{1}{p}} a_k\} \in l\Big(2, \frac{p}{p-1}\Big)\Big\}. \tag{22}$$

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Therefore from (21) and (22) we get

$$(C_0, A^q) = (G^p, l^1) = \Big\{\{a_k\} : \{k^{\frac{1}{p}-1}a_k\} \in l\Big(2, \frac{p}{p-1}\Big)\Big\}.$$

For $q = 1$, by [2] $B^a = G^1$, $(G^1)^a = B$, it follows from Lemma 2.6 that

$$(C_0, G^1) \subset (B, l^1) \subset (l^\infty, G^1) \subset (C_0, G^1),$$

by [3, Corollary 1] $(B, l^1) = l(2, 1)$, so

$$(C_0, G^1) = (B, l^1) = l(2, 1).$$

Hence

$$(C_0, A^1) = \big\{\{a_k\} : \{k^{-1}a_k\} \in l(2, 1)\big\}.$$

We conclude the proof. ∎

THEOREM 4.3. *For* $1 \le p, q \le 2$

$$\Big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in l\Big(\frac{2p}{2-p}, \frac{2q}{2-q}\Big)\Big\} \subset (H^p, A^q) \subset \Big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in l\Big(\infty, \frac{2q}{2-q}\Big)\Big\}.$$

PROOF. By Lemma 2.3 and 2.6, we have

$$\begin{aligned}(H^p, A^q) \subset \big(s(H^p), A^q\big) &= \big(l(2,2), A^q\big) = \big(l(2,2), s(A^q)\big)\\ &= \Big(l(2,2), \big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in l(2,q)\big\}\Big)\\ &= \Big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in \big(l(2,2), l(2,q)\big)\Big\}\\ &= \Big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in l(\infty, \frac{2q}{2-q})\Big\}.\end{aligned}$$

By [8, Theorem 2] $H^p \subset l(\frac{p}{p-1}, 2)$, from Lemma 2.6, we have

$$\begin{aligned}(H^p, A^q) &\supset \Big(l\Big(\frac{p}{p-1}, 2\Big), A^q\Big) = \Big(l\Big(\frac{p}{p-1}, 2\Big), s(A^q)\Big)\\ &= \Big(l\Big(\frac{p}{p-1}, 2\Big), \big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in l(2,q)\big\}\Big)\\ &= \Big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in \Big(l\Big(\frac{p}{p-1}, 2\Big), l(2,q)\Big)\Big\}\\ &= \Big\{\{a_k\} : \{k^{-\frac{1}{q}}a_k\} \in l\Big(\frac{2p}{2-p}, \frac{2q}{2-q}\Big)\Big\}.\end{aligned}$$

the theorem follows. ∎

COROLLARY 4.4. *For* $1 \le q \le 2$,

$$(H^2, A^q) = l\Big(\infty, \frac{2q}{2-q}\Big).$$

We finish the paper with the following theorem, which describes the multipliers from BMOA into $A^q (1 \le q \le 2)$:

THEOREM 4.5. *For* $1 \le q \le 2$

$$(\text{BMOA}, A^q) = \Big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l\Big(\infty, \frac{2q}{2-q}\Big)\Big\}.$$

PROOF. In general $H^1 \subset l(\infty, 2)$ ([21, Chapter XII Theorem 7.8]. This implies that $l(1,2) \subset$ BMOA, so by Lemma 2.6, we have

$$\begin{aligned}
(\text{BMOA}, A^q) &\subset \big(l(1,2), A^q\big) = \big(l(1,2), s(A^q)\big) \\
&= \Big(l(1,2), \big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l(2,q)\big\}\Big) \\
&= \Big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in \big(l(1,2), l(2,q)\big)\Big\} \\
&= \Big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l\Big(\infty, \frac{2q}{2-q}\Big)\Big\}, \\
(\text{BMOA}, A^q) &\supset \big(l(2,2), A^q\big) = \big(l(2,2), s(A^q)\big) \\
&= \Big(l(2,2), \big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l(2,q)\big\}\Big) \\
&= \Big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in \big(l(2,2), l(2,q)\big)\Big\} \\
&= \Big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l\Big(\infty, \frac{2q}{2-q}\Big)\Big\}.
\end{aligned}$$

This proves the theorem. ∎

COROLLARY 4.6. *For* $2 \le p < \infty, 1 \le q \le 2$,

$$(H^p, A^q) = \Big\{\{a_k\} : \{k^{-\frac{1}{q}} a_k\} \in l\Big(\infty, \frac{2q}{2-q}\Big)\Big\}.$$

PROOF. For $2 \le p < \infty$, since BMOA $\subset H^p \subset H^2$, it follows from Corollary 4.4 and Theorem 4.5. ∎

Corollary 4.6 is a supplement of Theorem B.

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