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ON DISCRETE STOCHASTIC PROCESSES WITH DISJUNCTIVE OUTCOMES

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Abstract

We introduce a class of discrete-time stochastic processes, called disjunctive processes, which are important for reliable simulations in random iteration algorithms. Their definition requires that all possible patterns of states appear with probability 1. Sufficient conditions for nonhomogeneous chains to be disjunctive are provided. Suitable examples show that strongly mixing Markov chains and pairwise independent sequences, often employed in applications, may not be disjunctive. As a particular step towards a general theory we shall examine the problem arising when disjunctiveness is inherited under passing to a subsequence. An application to the verification problem for switched control systems is also included.

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1. Introduction

Markov chains, ergodic stochastic processes and pairwise independent sequences of random variables play an important role in practical applications of probability to engineering, biology, geography, and so on; see [9, 12, 16, 17, 19–21]. Nevertheless, the properties enjoyed by these processes may not be sufficient for some simulations [10]. One interesting instance of this phenomenon is the random iteration algorithm known as the 'chaos game' [1] used in image processing and DNA classification. It has been observed in [2, 8] that a successful execution of the algorithm requires that the underlying stochastic process generates every finite sequence of states. Sequences exhausting all patterns are called 'disjunctive' [4]. The purpose of this note is to start a discussion of more detailed studies of processes with disjunctive outcomes. We believe that these kinds of processes may be responsible for the reliability of many probabilistic simulations (as in [24]).

In Section 2 we define the notion of a discrete-time disjunctive stochastic process. Such a process is nonhomogeneous and nonstationary in general, a case little studied so far (see [3, 9, 21, 22]).

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In Section 3 we state sufficient conditions for time-nonhomogeneous chains to be disjunctive. Some examples are also provided. It might appear surprising that strongly mixing Markov chains need not be disjunctive, since one usually imagines an attractor of the iterated function system to be the support of an invariant measure of the ergodic Markov–Feller operator [1, 22, 23].

Section 4 is devoted to an answer to the following question: under which conditions is the property of disjunctiveness preserved when passing from the original sequence to its subsequence? Thus, at least some parts of the general theory of disjunctive processes are possible to construct. Finally, Section 5 proposes an application of disjunctiveness to switched control systems.

2. Definitions

Let Σ be a finite alphabet of symbols (discrete space of states). We say that an infinite word $(\sigma_1, \sigma_2, \ldots) \in \Sigma^\infty$ is *disjunctive* [4] if it contains all possible finite words, that is,

$$\forall m \geq 1 \ \forall w \in \Sigma^m \ \exists j \ \forall l = 1, \ldots, m, \quad \sigma_{(j-1)+l} = w_l.$$

Note that any finite word appears arbitrarily far in the disjunctive sequence of symbols, hence infinitely often (because it reappears as a part of longer and longer words).

Example 2.1 (Champernowne word). Let us write down all possible words over the alphabet $\Sigma = \{1, 2, \ldots, N\}$: first one-letter words $1, 2, \ldots, N$, then two-letter words $11, 12, \ldots, NN$, and so on. An infinite word made by concatenating these words successively creates a disjunctive sequence of symbols: $12\ldots N1112\ldots NN\ldots \in \Sigma^\infty$. Note that all normal sequences [13, Ch. 2.9, page 65] are disjunctive, but the sporadic inclusion of a word in a sequence, such that the word appears increasingly rarely, shows that the converse is not true. The shortest finite sequence containing all words up to a given length is called a de Bruijn sequence.

A Σ-valued (discrete time) stochastic process $(Z_n)_{n\geq 1}$ is said to be *disjunctive* provided every finite word appears in the outcome almost surely:

$$\forall m \geq 1 \ \forall \tau \in \Sigma^m, \quad \Pr(Z_{(n-1)+l} = \tau_l, \ l = 1, \ldots, m, \ \text{for some } n) = 1.$$

In general a disjunctive process is not stationary by any means (see [9]).

Proposition 2.2 [2]. *A stochastic process $(Z_n)_{n\geq 1}$ is disjunctive if and only if it generates a disjunctive sequence $(\sigma_n)_{n=1}^\infty \in \Sigma^\infty$ with probability* 1.

For want of any name for the stochastic processes distinguished in [2] and in the present work, we have simply termed them 'disjunctive processes', based on the deterministic prototype [4] which itself is sometimes called *self-reading*, or *rich*, as in, for example, [11]. Other plausible names for disjunctive processes could be *dense* or *thick processes*, as proposed by Ö. Stenflo.

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3. Examples and criteria

The first example of a disjunctive process that comes to mind is the classic Bernoulli scheme (employed in the 'chaos game' in [1]). This observation extends to Markov chains of an arbitrary order.

EXAMPLE 3.1 (Chain with complete connections [2, 3]). Let $(Z_n)_{n\geq 1}$ be a sequence of random variables with conditional marginal distributions

$$\forall n \geq 1\ \forall \sigma_1, \dots, \sigma_n \in \Sigma, \quad \Pr(Z_n = \sigma_n \,|\, Z_{n-1} = \sigma_{n-1}, \dots, Z_1 = \sigma_1) \geq \alpha \tag{3.1}$$

and initial distribution

$$\forall \sigma_1 \in \Sigma, \quad \Pr(Z_1 = \sigma_1) \geq \alpha$$

for some $\alpha > 0$. Then the process $(Z_n)_{n\geq 1}$ is disjunctive.

We shall discuss below the nonhomogeneous Bernoulli scheme, our test field for general discrete time-inhomogeneous stochastic processes.

Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables (with values in Σ) distributed according to

$$\forall n \geq 1\ \forall \sigma \in \Sigma, \quad \Pr(Z_n = \sigma) \geq \alpha_n, \tag{3.2}$$

where $\alpha_n > 0$. We investigate the impact of decay conditions for $\alpha_n \searrow 0$ on achieving the series of successes (encoded as finite words). The tool we repeatedly employ is the classical Borel–Cantelli lemma (see, for example, [13, Theorem 2.2.3]).

EXAMPLE 3.2. Let $\Sigma = \{1, 2\}$, and let $\Pr(Z_n = 1) = \alpha_n := (n+1)^{-c}$ and $\Pr(Z_n = 2) = 1 - \alpha_n$, $n \geq 1$, $c > 0$. We show that such a decrease of chance for success yields a nondisjunctive outcome sequence.

Case $c = 2$. We calculate

$$\Pr(Z_n \neq 1 \text{ for all } n) = \prod_{n=1}^{\infty} (1 - \alpha_n) = \prod_{n=1}^{\infty} \frac{n(n+2)}{(n+1)^2} = \frac{1}{2} > 0.$$

Thus with a positive probability the symbol '1' never occurs.

Case $c = 1$. By the Borel–Cantelli lemma, the symbol '1' almost surely appears infinitely often, because $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} 1/(n+1) = \infty$. However,

$$\Pr((Z_{2n}, Z_{2n+1}) \neq (1,1) \text{ for all } n) = \prod_{n=1}^{\infty} (1 - \alpha_{2n}\alpha_{2n+1}) \geq \frac{1}{2} > 0,$$
$$\Pr((Z_{2n-1}, Z_{2n}) \neq (1,1) \text{ for all } n) > 0.$$

Thus with a positive probability the word '11' occurs neither at an even place nor at an odd place.

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Case $c = 2^{-k}$. Similarly, as above, there is a positive probability that the 2^{k+1}-letter word '1...1' never appears. For instance,

$$\begin{aligned}&\Pr(\text{not}\,(Z_{2^{k+1}n+i} = 1 \text{ for } i = 0, 1, \ldots, 2^{k+1} - 1) \text{ for all } n)\\ &\quad = \prod_{n=1}^{\infty}\Big(1 - \prod_{i=0}^{2^{k+1}-1} \alpha_{2^{k+1}n+i}\Big) \geq \prod_{n=1}^{\infty}(1 - 2^{-2k-2} \cdot n^{-2}) > 0.\end{aligned}$$

The moral is, to ensure success in passing an arbitrary sequence of exams, the chances of succeeding in a single trial should not diminish too quickly.

Theorem 3.3. *Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables with distributions given by (3.2) and minorising sequence $(\alpha_n)_{n=1}^{\infty}$ obeying*

$$\forall m \geq 1, \quad \sum_{n=1}^{\infty} \prod_{l=1}^{m} \alpha_{(n-1)+l} = \infty. \tag{3.3}$$

Then the process is disjunctive.

Proof. Fix the finite word $(\tau_1, \ldots, \tau_m) \in \Sigma^m$, $m \geq 1$. We are interested in the probability of the event $\{E_n$ infinitely often$\}$, where

$$E_n := \{Z_{(n-1)+1} = \tau_1, \ldots, Z_{(n-1)+m} = \tau_m\}.$$

By basic properties of nonnegative series, the condition (3.3) can be transformed into the equivalent

$$\forall m \geq 1\ \exists\, k = 0, 1, \ldots, m-1, \quad \sum_{n=k \pmod m} \prod_{l=1}^{m} \alpha_{(n-1)+l} = \infty. \tag{3.4}$$

Fix an appropriate k for further purposes.

It is enough to consider

$$\Pr(E_n \text{ for infinitely many } n = k \pmod m),$$

which is easier to calculate since $(Z_{n+l-1})_{l=1}^{m}$, $n = k \pmod m$, constitutes a sequence of independent random vectors.

Due to independence and (3.2),

$$\Pr(E_n) = \prod_{l=1}^{m} \Pr(Z_{n+l-1} = \tau_l) \geq \prod_{l=1}^{m} \alpha_{(n-1)+l}.$$

So, by (3.4), $\sum_{n=k \pmod m} \Pr(E_n) = \infty$. Therefore we can apply the Borel–Cantelli lemma to find out that $\Pr(E_n$ infinitely often$) = 1$. □

The above proof follows [13, Example, page 37].

Theorem 3.4. *Let $(Z_n)_{n\geq 1}$ be a sequence of random variables with distributions satisfying the Barnsley–Vince condition (compare with Example 3.1)*

$$\forall n \geq 1\ \forall \sigma_1, \ldots, \sigma_n \in \Sigma, \quad \Pr(Z_n = \sigma_n \mid Z_{n-1} = \sigma_{n-1}, \ldots, Z_1 = \sigma_1) \geq \alpha_n, \tag{3.5}$$

and minorising sequence $(\alpha_n)_{n=1}^{\infty}$ obeying (3.3). Then the process is disjunctive.

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PROOF. Fix an arbitrary disjunctive sequence $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ and an arbitrary positive integer $m_0 \geq 1$. We shall prove that the trajectory of Z_n will include $(\sigma_n)_{n=1}^{m_0}$ almost surely.

Let $(Y_n)_{n\geq 0}$ be the length of the longest subsequence of $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ in the path of $(Z_n)_{n\geq 1}$ most recently visited at 'time' n. More precisely, let $Y_0 := 0$. Let $Y_1 := 1$ if $Z_1 = \sigma_1$, and $Y_1 := 0$ otherwise. If for some m and n, $Z_{n-k} = \sigma_{m-k}$, for all $k = 0, 1, \ldots, m-1$, and m is the largest integer with this property, then we define $Y_n := m$; otherwise let $Y_n := 0$.

We shall show that $(Y_n)_{n\geq 0}$ will reach 'state' m_0 by a coupling argument. See [25] for more on the coupling method.

Let $(I_n)_{n\geq 0}$ be a sequence of independent random variables, uniformly distributed on the unit interval. Define $\tilde{Y}_0 := 0$, and for $n \geq 0$

$$\tilde{Y}_{n+1} := \begin{cases} \tilde{Y}_n + 1 & \text{if } I_n < \alpha_n, \\ 0 & \text{otherwise.} \end{cases}$$

Put $E_n := \{\tilde{Y}_n < \tilde{Y}_{n+1} < \cdots < \tilde{Y}_{n+m_0}\}$, $n \geq 0$. Note that E_n and E_m are independent if $|n-m| \geq m_0$. By assumption, $\sum_{n=0}^{\infty} \Pr(E_{m_0 n+k}) = \infty$, for some $0 \leq k \leq m_0 - 1$, and it therefore follows by the Borel–Cantelli lemma that $\Pr(E_n \text{ infinitely often}) = 1$ and thus $(\tilde{Y}_n)$ reaches m_0 almost surely.

We are now going to prove that we may regard the random sequence $(Y_n)_{n\geq 0}$ as being defined on the same probability space as $(\tilde{Y}_n)_{n\geq 0}$ with $\tilde{Y}_n \leq Y_n$ almost surely, from which one infers that (Y_n) also reaches 'state' m_0 almost surely.

Let $(\tilde{Z}_n)_{n\geq 1}$ be a random sequence defined inductively in the following way. Let $p_1(j) = \Pr(Z_1 = j)$, for any $j \in \Sigma$. Let

$$\tilde{Z}_1 := \begin{cases} \sigma_1 & \text{if } I_1 \leq p_1(\sigma_1), \\ j & \text{if } I_1 \in (p_1(\sigma_1) + \sum_{i\leq j-1, i\neq\sigma_1} p_1(i),\ p_1(\sigma_1) + \sum_{i\leq j, i\neq\sigma_1} p_1(i)],\ j \neq \sigma_1. \end{cases}$$

Then it follows, by construction, that $\tilde{Z}_1$ and Z_1 are identically distributed. Suppose for some $n \geq 1$, $\tilde{Z}_{n-k+1} = \sigma_1, \ldots, \tilde{Z}_n = \sigma_k$, for some $k \geq 1$, and k is the largest integer with this property. If we put

$$p_n(j) := \Pr(Z_n = j \mid Z_{n-1} = \sigma_{k-1}, \ldots, Z_{n-k+1} = \sigma_1, Z_{n-k} = \cdot, \ldots, Z_1 = \cdot) \geq \alpha_n,$$

$$\tilde{Z}_n := \begin{cases} \sigma_k & \text{if } I_n \leq p_n(\sigma_k), \\ j & \text{if } I_n \in (p_n(\sigma_k) + \sum_{i\leq j-1, i\neq\sigma_k} p_n(i),\ p_n(\sigma_k) + \sum_{i\leq j, i\neq\sigma_k} p_n(i)], \end{cases}$$

for any $j \neq \sigma_k$, then the n-dimensional vectors $(Z_1, Z_2, \ldots, Z_n)$ and $(\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_n)$ are identically distributed. Hence, by Kolmogorov's extension theorem, the processes $(\tilde{Z}_n)_{n\geq 1}$ and $(Z_n)_{n\geq 1}$ are identical. We may therefore without loss of generality regard the process $(Y_n)_{n\geq 0}$ as having been generated from the process $(\tilde{Z}_n)_{n\geq 1}$, and thus it follows that $\tilde{Y}_n \leq Y_n$ for any n, by construction since $\tilde{Y}_0 = Y_0 = 0$, and, if $\tilde{Y}_n < \tilde{Y}_{n+1}$, then $Y_n < Y_{n+1}$, for any $n \geq 0$. □

Now we would like to offer an effective criterion for condition (3.3).

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LEMMA 3.5. *If $\alpha_n^{-1} \ll n^c$ for all $c > 0$, that is,*

$$\lim_{n\to\infty} \frac{1}{\alpha_n \cdot n^c} = 0, \tag{3.6}$$

then (3.3) is fulfilled.

PROOF. From (3.6), starting from $n \geq m$,

$$\prod_{l=1}^{m} \alpha_{(n-1)+l}^{-1} \leq \prod_{l=1}^{m} (n-1+l)^c \leq (n+m)^{mc} \leq 2n$$

after putting $c := 1/m$. Thus we arrive at

$$\sum_{n=1}^{\infty} \prod_{l=1}^{m} \alpha_{(n-1)+l} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \infty,$$

and the lemma follows. □

EXAMPLE 3.6 (Logarithmic decay). If $\alpha_n^{-1} \approx (\log n)^b$ for some $b > 0$, that is,

$$\lim_{n\to\infty} \frac{1}{\alpha_n \cdot (\log n)^b} > 0, \tag{3.7}$$

then by L'Hôpital's rule (3.6) holds. Consequently (3.3) is satisfied.

In particular $\alpha_n \equiv \alpha > 0$, as in the Barnsley–Vince condition (3.1), fits logarithmic decay (3.7).

One might hope that in the case of identically distributed random variables, independence could be substituted by pairwise independence or ergodicity, since these properties are robust in applications. However, as the following discussion points out, this would yield combinatorially poor processes.

EXAMPLE 3.7 (Pairwise independence is not enough: [14, Example 2.2]). Let $(Z_{2n-1})_{n\geq 1}$ be a sequence of independent random variables with values in $\Sigma = \{0, 1\}$ distributed according to $\Pr(Z_{2n-1} = \sigma) = 1/2$, $n \geq 2$, $\sigma \in \Sigma$. Put $Z_{2n} := Z_{2n+1} + Z_1 \pmod 2$ for $n \geq 1$. Then it is easy to see that $(Z_n)_{n\geq 1}$ forms a sequence of identically distributed pairwise independent random variables.

By construction

$$Z_{2n+3} - Z_{2n+1} = Z_{2n+2} - Z_{2n} = Z_1 \pmod 2.$$

Hence, for $i = 0, 1$,

$$\Pr(Z_{2n+i} = 1,\ Z_{2n+1+i} = 1,\ Z_{2n+2+i} = 1,\ Z_{2n+3+i} = 0 \text{ for some } n) = 0.$$

Thus the word '1110' is forbidden.

PROPOSITION 3.8. *A homogeneous finite Markov chain is disjunctive if and only if its transition matrix has positive entries (and initial distribution is positive).*

PROOF. Put α in condition (3.1) to be the minimum over positive entries in the transition matrix (and in the initial distribution) to see why positivity suffices for disjunctiveness. Necessity is evident by construction as in [16, Example 1.8.1]. □

We see (by the above and in [21, Proposition I.2.10]) that a Markov chain may be strongly mixing/ergodic yet nondisjunctive (see [2]).

We end this section by writing a simple necessary condition for disjunctiveness of a time-nonhomogeneous Bernoulli scheme.

PROPOSITION 3.9. *Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables with values in Σ. If $(Z_n)_{n\geq 1}$ forms a disjunctive process, then necessarily for any state $\sigma \in \Sigma$*

$$\prod_{n=1}^{\infty}(1-\gamma_n) = 0,$$

where $\gamma_n := \Pr(Z_n = \sigma)$.

PROOF. Suppose $\prod_{n=1}^{\infty}(1-\gamma_n) > 0$ for some $\sigma \in \Sigma$. Then

$$\Pr(Z_1 \neq \sigma, Z_2 \neq \sigma, \ldots) > 0,$$

which ascribes a positive probability for the symbol 'σ' to yield a forbidden word. □

4. Subsequences

We know that a subsequence of independent and identically distributed random variables is again independent and identically distributed. It is no longer true for disjunctive processes. This 'defect' has an analogy in algorithmic information theory (see, for example, [18]). We shall now investigate this problem in more detail.

EXAMPLE 4.1. If $(Z_n)_{n\geq 1}$ is a disjunctive process, then $(Z_{2n})_{n\geq 1}$ is disjunctive too, but $(Z_{n^2})_{n\geq 1}$ not necessarily so.

Write $M = \{m \geq 1 : \forall n,\ m \neq n^2\}$. Take $(Z_m)_{m\in M}$ to be a sequence of independent and identically distributed random variables and Z_{n^2}, $n \geq 1$, to be the trivial random variable assuming a.s. the same constant value. Then, so defined, $(Z_n)_{n\geq 1}$ is disjunctive because, roughly speaking, the gaps between squares become larger and larger, which allows arbitrarily long finite words to appear (as in the case of a Bernoulli scheme).

Now let $(Z_n)_{n\geq 1}$ be a disjunctive Σ-valued process and fix a finite word $(\tau_1, \tau_2, \ldots, \tau_m) \in \Sigma^m$. Since $\tau_1\tau_1\tau_2\tau_2\ldots\tau_m\tau_m$ appears in the outcome of $(Z_n)_{n\geq 1}$, then also $\tau_1\tau_2\ldots\tau_m$ appears a.s. in $(Z_{2n})_{n\geq 1}$.

We say that an injection $\iota : \mathbb{N} \to \mathbb{N}$ *preserves disjunctive sequences* if for every disjunctive sequence $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ over any alphabet Σ the permuted subsequence $(\sigma_{\iota(n)})_{n=1}^{\infty} \in \Sigma^{\infty}$ is still disjunctive.

PROPOSITION 4.2. *Let an injection $\iota : \mathbb{N} \to \mathbb{N}$ preserve disjunctive sequences. Then:*

(a) *the gaps are bounded, that is, $\sup_n |\iota(n+1) - \iota(n)| < \infty$;*
(b) *if $(Z_n)_{n\geq 1}$ is a disjunctive process, then $(Z_{\iota(n)})_{n\geq 1}$ is disjunctive too.*

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As shown in Example 4.1 the presence of arbitrarily large gaps invites nondisjunctive outcomes. Nevertheless, the bounded gap condition (a) is not sufficient for preserving disjunctiveness.

Example 4.3. Let $(\sigma_n)_{n=1}^{\infty}$ be the Champernowne sequence over $\Sigma = \{1, 2\}$ (Example 2.1). Define

$$M := \mathbb{N}\backslash\{n \geq 2 \,:\, \sigma_{n-1} = 1,\ \sigma_n = 2,\ \sigma_{n+1} = 1\}$$

and $\iota : \mathbb{N} \to M \subset \mathbb{N}$ to be an ordered enumeration of M. Then ι has bounded gaps but the subsequence $(\sigma_{\iota(n)})_{n=1}^{\infty}$ lacks the subword '121' by construction.

One concrete criterion for the preservation of disjunctiveness, which extends an insight gained in Example 4.1, reads as follows.

Lemma 4.4. *Let $\lambda : \mathbb{N} \to \mathbb{N}$ be periodic and let $\iota : \mathbb{N} \to \mathbb{N}$ be given by*

$$\forall n \in \mathbb{N}, \quad \iota(n) := \sum_{k \leq n} \lambda(k).$$

Then ι preserves disjunctive sequences.

Proof. Let $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ be a disjunctive sequence. Let t denote the minimal period of λ, that is, the smallest natural number satisfying

$$\forall n \in \mathbb{N}, \quad \lambda(n + t) = \lambda(n).$$

Fix $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \Sigma^m$ to be the word to appear as a subword in $(\sigma_{\iota(n)})_{n=1}^{\infty}$. Choose the smallest $l \in \mathbb{N}$ such that $l \cdot t \geq \iota(m)$. Define $\upsilon \in \Sigma^{mlt}$ by

$$\forall k = 1, \ldots, m\ \forall j = 0, \ldots, m-1, \quad \upsilon\iota(k) + jlt - j := \tau_k.$$

Regardless of how the word υ is 'pierced' when passing to a subsequence via ι it always contains the subword τ. □

Example 4.5. If $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ is disjunctive:

(a) subsequences of the form $(\sigma_{l \cdot n})_{n=1}^{\infty}$, $l \in \mathbb{N}$, are disjunctive. Just put $\lambda(n) := l$ in Lemma 4.4;

(b) the subsequence $(\sigma_2, \sigma_5, \sigma_7, \sigma_{10}, \sigma_{12}, \ldots)$ is disjunctive. Just put $\lambda(n) := 2 + (1 + (-1)^n)/2$ in Lemma 4.4.

5. Application to switched systems

Let (X, d) be a compact metric space. We consider a discrete *control system* [5, 15]

$$x_{n+1} = f_{\alpha(n)}(x_n, u(n)) \tag{5.1}$$

with switching $\alpha : \mathbb{N} \to \{1, \ldots, N\} =: I$ and signals $u : \mathbb{N} \to U$ changing over a finite set $U = \{v_j : j = 1, \ldots, l\}$. The maps $f_i : X \times U \to X$, $i \in I$, are assumed to be Edelstein contractions in the first variable, that is,

$$d(f_i(x, u), f_i(\overline{x}, u)) < d(x, \overline{x})$$

for $x, \overline{x} \in X$.

The crucial observation concerning Edelstein contractions is that, to some extent, they behave like Lipschitz contractions; namely, there exists a comparison function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for $t > 0$, φ is non-decreasing right continuous, and

$$d(f_i(x,u), f_i(\overline{x},u)) \le \varphi(d(x,\overline{x}))$$

for $x, \overline{x} \in X$. (The common φ for the maps f_i is the maximum over their individual comparison functions.) As a consequence,

$$d(f_i^n(x,u), f_i^n(\overline{x},u)) \le \varphi^n(d(x,\overline{x})) \to 0, \tag{5.2}$$

where the superscript n denotes an n-fold composition of $f_i(\cdot,u)$ and φ, respectively. See, for example, [7, (6.10) and (6.15), page 18].

Here we would like to address the *verification problem* as delineated in [6]: Given $X_0 \subset X$ compute all the states visited by trajectories of (5.1) starting from $x_0 \in X_0$ as controls α, u vary in every possible way. We reveal the rough image of the set of attainable states by looking at the omega-limit of a *single* trajectory.

Recall that the *omega-limit* set of $(x_n)_{n=0}^{\infty}$ is

$$\omega((x_n)_{n=0}^{\infty}) := \bigcap_{m=0}^{\infty} \overline{\{x_n : n \ge m\}} = \{y \in X : \exists\, k_n \nearrow \infty\ x_{k_n} \to y\}$$

(see [2]). The ε-neighbourhood of $A \subset X$ is denoted by

$$N_\varepsilon A := \{x \in X : \exists\, a \in A\ d(x,a) < \varepsilon\}.$$

Theorem 5.1. *Let*

$$X_p := \{x_n : n \ge p, x_0 \in X_0, \alpha \in I^\infty, u \in U^\infty\}$$

be the set of states attained by the system (5.1) at the time $p \ge 0$ and later on. Let $C = \omega((x_n)_{n=0}^{\infty})$, where $(x_n)_{n=0}^{\infty}$ is a particular trajectory starting from some arbitrarily chosen $x_0 \in X_0$ and the controls $\alpha \in I^\infty$, $u \in U^\infty$ are simulated according to a $(I \times U)$-valued disjunctive process. Then, almost surely, $\bigcap_{p\ge 0} \overline{X_p} = C$, and $X_p \subset N_\varepsilon C$, $C \subset N_\varepsilon X_p$ for suitably large $p \ge 0$ given $\varepsilon > 0$.

Proof. Define $f_{ij}(x) := f_i(x, v_j)$, $i \in I$, $j \in \{1, \dots, l\} =: J$. Thus we have a finite system of contractions and $x_{n+1} = f_{\alpha(n),\beta(n)}(x_n)$, if we put $\beta(n) := j$ exactly when $u(n) = v_j \in U$. By assumption $(\alpha,\beta) \in (I \times J)^\infty$ is disjunctive with probability 1.

We shall show that an omega-limit set $\omega((y_n)_{n=0}^{\infty})$ of any trajectory $y_{n+1} = f_{\widetilde{\alpha}(n),\widetilde{\beta}(n)}(y_n)$ following some $(\widetilde{\alpha},\widetilde{\beta}) \in (I \times J)^\infty$ is already included in C. (In particular $\omega((y_n)_{n=0}^{\infty}) = C$ for disjunctive $(\widetilde{\alpha},\widetilde{\beta}) \in (I \times J)^\infty$.)

Let $y_\infty \in \omega((y_n)_{n=0}^{\infty})$, that is, $y_{k_n} \to y_\infty$ for a subsequence $k_n \nearrow \infty$. The finite word

$$(\widetilde{\alpha}(1),\widetilde{\beta}(1)) \dots (\widetilde{\alpha}(k_n),\widetilde{\beta}(k_n)) \in (I \times J)^{k_n}$$

appears in (α,β). Moreover, $\omega((x_n)_{n=0}^{\infty})$ does not depend on the choice of x_0 due to (5.2). Hence the limit point y_∞ can be realised as an accumulation point of $(x_n)_{n=0}^{\infty}$. Thanks to standard properties of set-convergences the thesis follows (see [2]). □

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We make a brief comment on the preceding result. The compactness of X bears no problem for a Euclidean space of states, since every closed bounded set is compact in this case. If the controls α and u are simulated from two disjunctive processes, then obviously the joint process generating (α, u) need not be disjunctive. However, highly dependent marginal processes can still yield a disjunctive joint process; this leads to further questions concerning disjunctive processes. Similarly to [2], one can employ a deterministic disjunctive sequence in Theorem 5.1.

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