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RANDOM INTEGRAL REPRESENTATIONS FOR CLASSES OF LIMIT DISTRIBUTIONS SIMILAR TO LÉVY CLASS L_0, II

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§ 0. Introduction

Let $\xi(t)$ and $\eta(t)$ be two stochastic processes such that ξ has stationary independent increments and $\xi(0) = 0$ a.s. Suppose that $\xi(1) \stackrel{d}{=} t\xi(t^\beta) + \eta(t)$ for each $0 < t \leq 1$, with $\xi(t^\beta)$ independent of $\eta(t)$ and a fixed parameter $\beta \in (-2, 0)$. It is shown that $\xi(1)$ satisfies the above equation if and only if $\xi(1)$ is a sum of two independent r.v.'s: strictly stable one with the exponent $-\beta$ and the one given by a random integral $\int_{(0,1)} t dY(t^\beta)$, where Y has stationary independent increments and $\boldsymbol{E}\, [\|Y(1)\|^{-\beta}] < \infty$.

The aim of this paper is to find a random integral representation for some classes of limit distributions. Such representations give a very natural connection between theory of limit distributions and theory of stochastic processes. In some sense this note complements the subject, with a long history, of characterizations of stochastic processes by random integrals; cf. B.L.S. Praksa Rao (1983). On the other hand, this is a continuation of the study begun in Jurek (1988) but basically in a case of a Hilbert space and the identity operator. Recall that an infinitely divisible measure μ belongs to the class $\mathscr{U}_\beta$ if and only if

$$\forall (0<c<1) \exists \mu_c \in ID, \qquad \mu = T_c \mu^{*c^\beta} * \mu_c . \tag{0.1}$$

Here T_c is the linear operator of multiplying by a scalar c. In terms of stochastic processes the equation (0.1) can be rewritten as follows: There exist processes $\xi(t)$ and $\eta(t)$ such that ξ has stationary independent increments ($\mu = \mathscr{L}(\xi(1))$ and $\xi(1) \stackrel{d}{=} c\xi(c^\beta) + \eta(c)$ for $0 < c \leq 1$, with $\eta(c)$

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independent of $\xi(c^\beta)$ and infinitely divisible distribution. Note that $\mathscr{U}_0$ coincides with the Lévy class L_0 of so called *self-decomposable measures*. Class $\mathscr{U}_1$ consists of so called *s-self-decomposable measures*, which are obtained as limit distributions when the summands in partial sums are deformed by some *nonlinear* transformation; cf. Jurek (1985). In any case, the classes $\mathscr{U}_\beta$ defined by (0.1) are classes of limit distributions. Moreover, assuming that μ is non-degenerate measure, we have that $\beta \geq -2$. The main characterization of elements from $\mathscr{U}_\beta$, with $\beta \geq 0$, is the following:

$$\mu \in \mathscr{U}_\beta \quad \text{if and only if } \mu = \mathscr{L}\left(\int_{(0,1)} t dY(\tau_\beta(t))\right), \tag{0.2}$$

with $\tau_\beta(t) := t^\beta$ for $\beta \neq 0$ and $\tau_0(t) := -\ln t$ and a process Y which has stationary independent increments. The random integral (0.2) exists for all Y's in the case of $\beta > 0$. For $\beta = 0$, the existence of the integral in (0.2) is equivalent to $E[\log(1 + \|Y(1)\|)] < \infty$, cf. Jurek (1988) and Jurek-Vervaat (1983) respectively. In the present paper we discuss the case of $-2 \leq \beta < 0$. We show that $\mathscr{U}_{-2}$ consists only of Gaussian measures and each $\mu \in \mathscr{U}_\beta$, for $-1 < \beta < 0$, is a convolution of a strictly stable measure with the exponent $-\beta$ and a distribution of a random integral like this in (0.2). The existence of these integrals is equivalent to the condition $E[\|Y(1)\|^{-\beta}] < \infty$. Similar characterizations hold true for the class $\mathscr{U}_\beta$ with $-2 < \beta \leq -1$ provided the measure μ in (0.1) is symmetric. Expressing the characterization in Theorem 1.2 in terms of the characteristic function, we will get the conjectured formula for the classes $L_\alpha := \mathscr{U}_{1-\alpha}$, with $1 < \alpha < 3$; cf. O'Connor (1979) p. 268 and Jurek (1988), Section 4.

Finally we would like to emphasize that the classes $\mathscr{U}_\beta$ with $-2 < \beta < 0$ are essentially different from those with $\beta > 0$. This has led us to conviction to present these results (for $\mathscr{U}_\beta$, with $\beta < 0$), although they are complete only in the case of a Hilbert space.

The paper is organized as follows: Section 1 contains notations and main results. The existence of the pathwise random integrals (like those in the formula (0.2)) is discussed in Section 2. Section 3 basically gives the proofs, especially the main construction of the process Y needed in the proof of Theorem 1.2. All theorems, lemmas and formulas are numbered separately in each section.

§ 1. Notations and results

Let E and H denote a real separable Banach and Hilbert space respectively. Let ID(E) (ID(H)) denote the closed topological convolution semigroup of all infinitely divisible measures on E (or H). Recall that $\mu \in \mathrm{ID}(E)$ if and only if its characteristic functional $\hat{\mu}$ (Fourier transform) is of the following form

$$\hat{\mu}(y) = \exp\left\{i\langle y, a\rangle - \frac{1}{2}\langle y, Ry\rangle + \int_{E\setminus\{0\}} K_E(y, x)M(dx)\right\}, \qquad y \in E'. \tag{1.1}$$

(E' is the topological dual of E; $\langle\cdot,\cdot\rangle$ is a bilinear form between E' and E; $K_E(y, x) := \exp i\langle y, x\rangle - 1 - i\langle y, x\rangle 1_B(x)$, where $1_B(x)$ is the indicator of the unit ball in E; cf. Araujo-Giné (1980)). Since $\hat{\mu}$ uniquely determines a vector a, a Gaussian covariance operator R and a Lévy spectral measure M in (1.1), in the sequel, we will write $\mu = [a, R, M]$. In the case of a Hilbert space, the kernel $K_H(y, x) := \exp\langle y, x\rangle - 1 - i\langle y, x\rangle \cdot(1 + \|x\|^2)^{-1}$ and the parameters R and M are completely characterized; cf. Parthasarathy (1967), Theorem VI. 4.10. In particular, M is a Lévy spectral measure if and only if $\int_H \min(1, \|x\|^2)M(dx) < \infty$. This characterization is very crucial for the proof of Lemma 2.3.

For a measure ν and a measurable mapping f, we write $f\nu = \nu f^{-1}$ for the measure defined by the means of the formula

$$(f\nu)(F) := \nu(f^{-1}(F)) \qquad \text{for all measurables sets } F. \tag{1.2}$$

In particular, if A is bounded linear operator on E then

$$(A\nu)^\wedge(y) = \hat{\nu}(A^*y) \qquad \text{for } y \in E'.$$

Furthermore, if μ and ν are infinitely divisible on E, A is a bounded linear operator and $t \geq 0$, then

$$(A(\mu * \nu))^{*t} = A((\mu * \nu)^{*t}) = A\mu^{*t} * A\nu^{*t}. \tag{1.3}$$

In fact, we will use (1.3) for the operators $T_c x := cx$ (multiplication by a scalar c), $x \in E$ and $c \in \boldsymbol{R}$.

Let $\beta \in \boldsymbol{R}$ be fixed and define subsets $\mathscr{U}_\beta$ of ID(E) as follows:

$$\mu \in \mathscr{U}_\beta \quad \text{if and only if } \forall(0 < c < 1)\exists \mu_c \in \mathrm{ID}(E) \qquad \mu = T_c\mu^{*c^\beta} * \mu_c. \tag{1.4}$$

The classes $\mathscr{U}_\beta$ coincide with some classes of limit distributions and form closed subsemigroups of ID(E). Furthermore, if $\mu \neq \delta(x_0)$ (i.e. μ is non-

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degenerate) then $\beta \geq -2$; cf. Jurek (1988). The class $\mathscr{U}_{-2}$ is well-known because of the following proposition.

PROPOSITION 1.1. *The class $\mathscr{U}_{-2}$ on a Banach space E coincides with the class of all Gaussian measures.*

Proof. Of course, Gaussian measures $[a, R, 0]$ belong to $\mathscr{U}_{-2}$. Conversely, if $\mu = [a, R, M]$ is in $\mathscr{U}_{-2}$, then $M \geq c^{-2} T_c M$ for all $0 < c < 1$. Hence, for each $\varepsilon > 0$, $y \in E'$ and $B_{\varepsilon,y} := \{x : |\langle y, x\rangle| \leq \varepsilon\}$

$$\infty > \int_{B_{\varepsilon,y}} \langle y, x\rangle^2 M(dx) \geq c^{-2} \int_{B_{\varepsilon,y}} \langle y, x\rangle^2 M(c^{-1}dx)$$
$$= \int_{B_{\varepsilon,y}} \langle y, x\rangle^2 M(dx) + \int_{\{x:\varepsilon < |\langle y,x\rangle| \leq c^{-1}\varepsilon\}} \langle y, x\rangle^2 M(dx) \geq 0 .$$

Thus M vanishes on the sets $B^c_{\varepsilon,y}$. Since $\|x\| = \sup_{\|y_n\| \leq 1} |\langle y_n, x\rangle|$, cf. [1], p. 34, Problem 9, we conclude that $M(\|x\| > \varepsilon) = 0$ for $\varepsilon > 0$. Hence $M \equiv 0$ and therefore μ is a Gaussian measure which completes the proof.

The next theorem gives the conditions for the existence of some random integrals: deterministic integrands and Lévy processes as the integrators. For our purposes we adopt the formal integration by parts formula as the definition of random integrals, cf. Section 2 for details. Also, cf. Prakasa Rao (1983), Section 2.

THEOREM 1.1. *Let Y be a $D_H[0, \infty)$-valued r.v. with stationary independent increments, $Y(0) = 0$ a.s., $\mathscr{L}(Y(1)) = [\tilde{x}, R, M]$ and let $Z^\beta(t) := \int_{[e^{-t},1)} s dY(s^\beta)$ for $t > 0$.*

(a) *For $-1 < \beta < 0$ the following conditions are equivalent:*
   (1) $E[\|Y(1)\|^{-\beta}] < \infty$;
   (2) $\lim_{t\to\infty} \mathscr{L}(Z^\beta(t))$ *exists in weak topology;*
   (3) *there exist $y_t \in H$ such that $(\mathscr{L}(Z^\beta(t) + y_t))_{t\geq 0}$ is conditionally compact in weak topology as $t \to \infty$.*

(b) *For $-2 < \beta \leq -1$ and symmetric $Y(1)$ the following are equivalent:*
   (1) $E[\|Y(1)\|^{-\beta}] < \infty$;
   (2) $\lim_{t\to\infty} \mathscr{L}(Z^\beta(t))$ *exists in weak topology;*
   (3) $(\mathscr{L}(Z^\beta(t)))_{t\geq 0}$ *is conditionally compact in weak topology as $t \to \infty$.*

Remark 1.1. Since the processes Z^β are with independent increments we can add an equivalent condition

(4) $\lim_{t\to\infty} Z^\beta(t)$ exists in probability;

to those in the above Theorem 1.1.

Now we are in a position to give the description of elements from classes $\mathscr{U}_\beta$, on a Hilbert space, in terms of random integrals as it is stated in the title of this note. It might be worthwhile to mention here that $\mathscr{U}_\beta$ form an increasing family of measures i.e., for $-2 \leq \beta_1 \leq 0 \leq \beta_2 \leq 1 \leq \beta_3 < \infty$ we have

$$\mathscr{U}_{-2} \subseteq \mathscr{U}_{\beta_1} \subseteq \mathscr{U}_0 = L_0 \subseteq \mathscr{U}_{\beta_2} \subseteq \mathscr{U}_1 \subseteq \mathscr{U}_{\beta_3} \subseteq \mathrm{ID}(H),$$

where $\mathscr{U}_0 = L_0$ is the Lévy class of self-decomposable measures and $\mathscr{U}_1$ was originally defined as limit distributions for some *nonlinear* deformations of random variables, cf. Jurek (1985).

THEOREM 1.2. (a) *Let* $-1 < \beta < 0$. *Then an infinitely divisible measure* μ *on a Hilbert space* H *belongs to the class* $\mathscr{U}_\beta$ *if and only if there exists a strictly stable measure* γ_β *with the exponent* $-\beta$ *and a* $D_H[0, \infty)$*-valued r.v.* Y *with stationary independent increments such that* $E[\|Y(1)\|^{-\beta}] < \infty$ *and*

$$\mu = \gamma_\beta * \mathscr{L}\left(\int_{(0,1)} t\,dY(t^\beta)\right). \tag{1.5}$$

(b) *Let* $-2 < \beta \leq -1$. *Then a symmetric infinitely divisible measure* μ *on* H *belongs to* $\mathscr{U}_\beta$ *if and only if* μ *is of the form* (1.5), *where* γ_β *is symmetric stable measure with the exponent* $-\beta$ *and* Y *is symmetric* $D_H[0, \infty)$*-valued r.v. with stationary independent increments and* $E[\|Y(1)\|^{-\beta}] < \infty$.

§ 2. A pathwise random integral

Let $D_E[a, b]$ denote the set of all E-valued *cadlag* functions on an interval $[a, b]$, i.e., functions that are right-continuous on $[a, b)$ and have left-hand limits on $(a, b]$. Recall that $D_E[a, b]$ becomes a complete separable metric space under Skorohod metric. Similarly we define $D_E[0, \infty)$; cf. Pollard (1984). Let r be a strictly monotone function from $(a, b]$ into $[0, \infty)$, Y be a $D_E[0, \infty)$-valued random variable and $f \in D_{\mathbf{R}}[a, b]$ has bounded variation. Then we define $\int_{(a,b]} f(t)dY(r(t))$ by formal integration by parts:

$$\int_{(a,b]} f(t)dY(r(t)) := f(t)Y(r(t))\Big|_{t=a}^{t=b} - \int_{(a,b]} Y(r(t))df(t). \tag{2.1}$$

The integral on the right-hand side exists for time scale deformations r that are left- or right-continuous and have bounded range. We realize

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that the above definition of random integrals is very "ancient," but it is sufficient for our purposes. Integrals over (a, c) are defined as limit in probability of (2.1) when $b \uparrow c$. For r decreasing, integrals over $[a, b)$ are, in fact, over $(c, d]$ by changing $r(t)$ to s in (2.1).

LEMMA 2.1. *Let Y be a $D_E[0, \infty)$-valued r.v. with stationary independent increments, $Y(0) = 0$ a.s. and r be strictly decreasing function. Then*
(a) $\hat{\mathscr{L}}\Big(\int_{(a,b]} f(t)dY(r(t))\Big)(y) = \exp\Big\{-\int_b^a [\log \hat{\mathscr{L}}(Y(1))(-yf(t))]dr(t)\Big\}$
for $y \in E'$.
(b) *If Y has values in the Hilbert space H and $\mathscr{L}(Y(1)) = [\tilde{x}, R, M]$, $r(t) = t^\beta$, $-2 < \beta < 0$, and $f(t) = t$, then for $s > 0$*

$$\mathscr{L}\Big(\int_{[e^{-s},1)} tdY(t^\beta)\Big) = [\tilde{x}_s, R_s, M_s]$$

where

$$\text{(i)} \quad \tilde{x}_s := \tilde{x}\int_{e^{-s}}^1 tdt^\beta - \beta\int_{e^{-s}}^1 \int_H x\frac{t^\beta(1-t^2)}{1+\|x\|^2t^2}\frac{\|x\|^2}{1+\|x\|^2}M(-dx)dt\,;$$

(Bochner integral)

$$\text{(ii)} \quad R_s := -R\int_{e^{-s}}^1 t^2dt^\beta = -\frac{\beta}{\beta+2}(1-e^{-s(\beta+2)})R\,;$$

$$\text{(iii)} \quad M_s(A) := -\int_{e^{-s}}^1 M(-t^{-1}A)dt^\beta \qquad \textit{for } A \in \mathscr{B}(H\setminus\{0\})\,.$$

Proof. (a) The proof is similar to the one of Lemma 2.2 in Jurek (1988). Two minus signs in the formula (a) are due to the fact that r is a decreasing function.

(b) The formula (i)-(iii) follow from (a) and the form of the characteristic functions of infinitely divisible measures on Hilbert spaces cf. Parthasarathy (1967), Chapter VI.

Remark 2.1. (1) In the case of a Banach space only the shift $\tilde{x}_s$ has slightly different form. Gaussian covariance operator R_s and Lévy spectral measure M_s are as in Hilbert space case.

(b) We will use $\tilde{x}_\infty$ and M_∞ as the limits of (i) and (iii) when $s \to \infty$, provided the limits exist. Of course, $R_\infty = -\beta(\beta+2)^{-1}R$.

LEMMA 2.2. *For a r.v. Y as is in Lemma* 2.1 (b), *we have the following*:

(a) *If $-1 < \beta < 0$, then $\tilde{x}_\infty := \lim_{s\to\infty} \tilde{x}_s$ exists (in the norm of H) when* $\int_{\|x\|>1} \|x\|^{-\beta}dM(x) < \infty$;

(b) *if* $-2 < \beta \leq -1$, $\tilde{x} = 0$ *and* M *is symmetric then* $\tilde{x}_s = \tilde{x}_\infty = 0$ *for* $s > 0$.

Proof. Let $F(s) := M(\|x\| > s)$ for $s \geq 0$. Then x_∞ exists if

$$(1)\quad \|\tilde{x}\| \int_0^1 t^\beta dt < \infty \quad \text{and} \quad (2)\quad -\int_0^1 \int_0^\infty \frac{s^3}{1+s^2} \frac{t^\beta(1-t^2)}{1+s^2t^2} dF(s)dt < \infty ,$$

cf. the formula (i) in Lemma 2.1(b). The integral (2) can be written as the sum of the following two:

$$I_1 := -\int_0^\infty \frac{s^{2-\beta}}{1+s^2} g(s)dF(s) \,; \qquad I_2 := -\int_0^\infty \frac{s^{-\beta}}{1+s^2} h(s)dF(s) \,,$$

where

$$g(s) := \int_0^s \frac{v^\beta}{1+v^2} dv \,, \qquad h(s) := \int_0^s \frac{v^{\beta+2}}{1+v^2} dv \qquad \text{for } s > 0 \,.$$

Note that $g(s)$ exists (for some $s > 0$) if and only if $-1 < \beta < 0$. Moreover, $\lim_{s\to\infty} g(s) \leq \int_0^1 (v^\beta/(1+v^2))dv + \int_1^\infty (dv/(1+v^2)) < \infty$. Since $-t^2dF(t)$ is a finite positive measure on $[0, 1]$ and

$$I_1 = -\int_0^1 \frac{t^{-\beta}}{1+t^2} g(t)t^2dF(t) - \int_1^\infty \frac{t^2}{1+t^2} g(t)t^{-\beta}dF(t)$$

we conclude that I_1 is finite because $\int_{\|x\|>1} \|x\|^{-\beta} M(dx) < \infty$. For the integral I_2, at first we should observe that

$$\lim_{s\to 0} \frac{h(s)}{s^{2+\beta}} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{h(s)}{s^2} = 0 \,.$$

This together with

$$I_2 = -\int_0^1 \frac{s^{-(2+\beta)}}{1+s^2} h(s)s^2dF(s) - \int_1^\infty \frac{h(s)}{1+s^2} s^{-\beta}dF(s)$$

gives that I_2 is finite since $\int_{\|x\|>1} \|x\|^{-\beta} M(dx) < \infty$. Thus part (a) is proved. Part (b) is obvious and therefore the lemma is proved completely.

LEMMA 2.3. *Let* N *and* N_∞ *be two Borel measures on* $H \setminus \{0\}$ *related each to the other by formula*

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$$N_\infty(A) = (-\beta)\int_0^1 N(-t^{-1}A)t^{\beta-1}dt,$$

where β *is a constant from the interval* $(-2, 0)$. *Then* N_∞ *is a Lévy spectral measure if and only if* N *is a Lévy spectral measure and* $\int_{\|x\|>1}\|x\|^{-\beta}\cdot N(dx) < \infty$.

Proof. Recall that in Hilbert spaces a measure is a Lévy spectral measure if and only if it is finite outside every neighborhood of zero and integrates $\|x\|^2$ on the unit ball around zero; cf. Parthasarathy (1967), Theorem VI.4.10. This together with the following equalities

$$\begin{aligned} N_\infty(\|x\| \geq \varepsilon) &= -\beta\varepsilon^\beta\int_\varepsilon^\infty N(\|x\| \geq v)v^{-(\beta+1)}dv \\ &= -\beta\varepsilon^\beta\int_{\|x\|>\varepsilon}\int_\varepsilon^{\|x\|} v^{-(\beta+1)}dvN(dx) \\ &= \varepsilon^\beta\int_{\|x\|\geq\varepsilon}\|x\|^{-\beta}N(dx) - N(\|x\|\geq\varepsilon), \end{aligned}$$

$$\begin{aligned} \int_{\|x\|\leq 1}\|x\|^2N_\infty(dx) &= -\beta\int_0^1\int_{\|z\|\leq t^{-1}}\|z\|^2N(dz)t^{\beta+1}dt \\ &= -\beta\int_{\|z\|\leq 1}\|z\|^2\int_0^1 t^{\beta+1}dtN(dz) \\ &\quad -\beta\int_{\|z\|\leq 1}\|z\|^2\int_0^{\|x\|^{-1}} t^{\beta+1}dtN(dz) \\ &= -\beta(\beta+2)^{-1}\Big[\int_{\|z\|\leq 1}\|z\|^2N(dz) + \int_{\|z\|>1}\|z\|^{-\beta}N(dz)\Big], \end{aligned}$$

concludes the proof.

§ 3. Proofs

Proof of Theorem 1.1. Lemmas 2.2 and 2.3 combined with Theorem VI.5.5 in Parthasarathy (1967) show for $-1 < \beta < 0$, $[x_t, R_t, M_t] \Rightarrow [x_\infty, R_\infty, M_\infty]$ as $t \to \infty$ if $E[\|Y(1)\|^{-\beta}] < \infty$, i.e., (1) $\Rightarrow$ (2). Here we also use the fact for infinitely divisible measure ν with Lévy spectral measure N and subadditive function f ($f(t+s) \leq K(f(t)+f(s))$ for all $s, t > 0$ and a constant K)

$$\int_E f(\|x\|)\nu(dx) < \infty \quad \text{if and only if} \quad \int_{\|x\|\geq 1} f(\|x\|)N(dx) < \infty;$$

cf. for instance deAcosta (1980). Implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Suppose that for each sequence $t' \to \infty$ there exists a subsequence $t'' \to \infty$ and a r.v. ξ such that $\mathscr{L}(Z^{\beta}(t'') + y_{t''}) \Rightarrow \xi$ as $t'' \to \infty$. Then the ξ's are infinitely divisible measures with the Lévy spectral measure M_{∞}. So, by Lemma 2.3 we get $\int_{\|x\|\geq 1} \|x\|^{-\beta} M(dx) < \infty$, which completes the proof of part (a).

The proof of part (b) is contained in Lemmas 2.2 and 2.3 together with reasoning as it is in part (a).

Remark 3.1. If $-2 < \beta \leq -1$ and there exist $y_t \in H$ such that $(\mathscr{L}(Z^{\beta}(t) + y_t))_{t \geq 0}$ is conditionally compact as $t \to \infty$ then we have $\boldsymbol{E}[\| Y(1)\|^{-\beta}] < \infty$.

The most crucial part of the proof of Theorem 1.2 is given in the construction presented in Lemma 3.1. In fact, the construction in question is a further extension of the one given in the proof of Theorem 1.2(a) in Jurek (1988).

LEMMA 3.1. *Let* $-2 < \beta < 0$, $\mu \in \mathrm{ID}(E)$ *and for each* $t > 0$ *there exists* $\mu_t \in \mathrm{ID}(E)$ *such that*

$$\mu = T_{e^{-t}} \mu^{*e^{-\beta t}} * \mu_t \,. \tag{3.1}$$

Then there exists a $D_E[0, \infty)$*-valued r.v.* Y *with stationary independent increments such that* $Y(0) = 0$ *a.s. and for* $t > 0$

$$\mu_t = \mathscr{L}\Big(\int_{[e^{-t},1)} u dY(u^{\beta})\Big) \,.$$

Proof. As in the case of $\beta > 0$ (cf. Jurek (1988)) we construct a $D[0, \infty)$-valued r.v. Z with independent increments such that $Z(0) = 0$ and

$$\mathscr{L}(Z(t)) = \mu_t \qquad \text{for } t > 0 \,. \tag{3.2}$$

Furthermore, the process $\tilde{Y}$ defined as follows

$$\tilde{Y}(t) := \int_{(0,t]} e^{s} dZ(s) \tag{3.3}$$

has independent increments and for $0 \leq s \leq t$ and $h \in \boldsymbol{R}$ such that $s + h \geq 0$ we obtain

$$\mathscr{L}(\tilde{Y}(t+h) - \tilde{Y}(s+h)) = \mathscr{L}(\tilde{Y}(t) - \tilde{Y}(s))^{*e^{-\beta h}} \,, \tag{3.4}$$

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cf. formula (3.8) in Jurek (1988). Now let Y_1 be a $D[1, \infty)$-valued r.v. with independent increments, $Y_1(1) = 0$ and such that for $0 \le v \le w$

$$\mathscr{L}(Y_1(e^{-\beta w}) - Y_1(e^{-\beta v})) := \mathscr{L}(\tilde{Y}(v) - \tilde{Y}(w)) . \tag{3.5}$$

Hence and from (3.4) we get for $c > 0$ and such that $ce^{-\beta v} \ge 1$

$$\begin{aligned}\mathscr{L}[Y_1(ce^{-\beta w}) - Y_1(ce^{-\beta v})] &= \mathscr{L}[\tilde{Y}(v + \beta^{-1}\log c^{-1}) - \tilde{Y}(w + \beta^{-1}\log c^{-1})] \\ &= \mathscr{L}(\tilde{Y}(v) - \tilde{Y}(w))^{*c} = \mathscr{L}(Y_1(e^{-\beta w}) - Y_1(e^{-\beta v}))^{*c} .\end{aligned}$$

Putting $c := e^{\beta v}$ we obtain $\mathscr{L}(Y_1(e^{-\beta(w-v)})) = \mathscr{L}(Y_1(e^{-\beta w}) - Y_1(e^{-\beta v}))^{*e^{\beta v}}$ or equivalently for $1 \le s \le t$ we have

$$\mathscr{L}(Y_1(t/s)) = \mathscr{L}(Y_1(t) - Y_1(s))^{*s^{-1}} . \tag{3.6}$$

Since Y_1 has independent increments and $Y_1(1) = 0$, therefore for $1 \le s \le a \le b$ we have

$$\mathscr{L}(Y_1(b/s))^{*s} = \mathscr{L}(Y_1(a/s) - Y_1(s/s))^{*s} * \mathscr{L}(Y_1(b/s) - Y_1(a/s))^{*s} .$$

This together with (3.6) gives

$$\mathscr{L}(Y_1(b) - Y_1(s)) = \mathscr{L}(Y_1(a) - Y_1(s)) * \mathscr{L}(Y_1(b/s) - Y_1(a/s))^{*s} ,$$

and consequently

$$\mathscr{L}(Y_1(b) - Y_1(a)) = \mathscr{L}(Y_1(b/a) - Y_1(a/s))^{*s} \qquad \text{for } 1 \le s \le a \le b .$$

Putting $f(a, b) := \log \hat{\mathscr{L}}(Y_1(b) - Y_1(a))$ for $1 \le a \le b$ we obtain $f(a, a) = 0$, $f(a, b) + f(b, c) = f(a, c)$ for $1 \le a \le b \le c$ and $sf(a/s, b/s) = f(a, b)$ for $1 \le s \le a \le b$. Furthermore putting $g(v) := f(1, v) = \log \hat{\mathscr{L}}(Y_1(v))$, $v \ge 1$ we have

$$sg(v/s) = f(s, v) = f(1, v) - f(1, s) = g(v) - g(s) \qquad \text{for } 1 \le s \le v$$

or equivalently

$$g(st) = sg(t) + g(s) \qquad \text{for all } s, t \ge 1 .$$

Hence $sg(t) + g(s) = g(ts) = tg(s) + g(t)$, i.e., $(s - 1)g(t) = (t - 1)g(s)$ and therefore $g(t) = (t - 1)g(2)$ for all $t \ge 1$. Since Y_1 has independent increments and $Y_1(1) = 0$ we conclude that the increments of Y_1 are also stationary. Finally, $Y(t) := Y_1(t + 1)$ for $t \ge 0$ gives a $D_E[0, \infty)$-valued r.v. with independent and stationary increments and $Y(0) = 0$. Moreover, taking into account (3.2), (3.3) and (3.5) we get

$$\mu_t = \mathscr{L}(Z(t)) = \mathscr{L}\Big(\int_{(0,t]} e^{-s} d\tilde{Y}(s)\Big) = \mathscr{L}\Big(-\int_{(0,t]} e^{-s} dY_1(e^{-\beta s})\Big)$$
$$= \mathscr{L}\Big(\int_{[e^{-t},1)} u dY_1(u^\beta)\Big) = \mathscr{L}\Big(\int_{[e^{-t},1)} u dY(u^\beta)\Big),$$

which completes the proof of Lemma 3.1.

Proof of Theorem 1.2. At first, let us note that for a strictly stable measure γ_β with the exponent $-\beta$ and a positive constant c we have $T_c\gamma_\beta = \gamma_\beta^{*c^{-\beta}}$. Since

$$T_c\mathscr{L}\Big(\int_{(0,1)} t dY(t^\beta)\Big) = \mathscr{L}\Big(\int_{(0,c)} t dY(c^{-\beta} t^\beta)\Big) = \mathscr{L}\Big(\int_{(0,c)} t dY(t^\beta)\Big)^{*c^{-\beta}}$$

we infer that measures of the form (1.5) belong to the class $\mathscr{U}_\beta$ with $\mu_c = \mathscr{L}\Big(\int_{[c,1)} t dY(t^\beta)\Big)$ in (1.4) for $0 < c < 1$.

Conversely, if μ satisfies (3.1), we infer that $(\mu_t)_{t\geq 0}$ is shift conditionally compact; cf. Parthasarathy (1967), Theorem III.2.2. If $-1 < \beta < 0$, then Theorem 1.1 (a) gives that $\mu_t \Rightarrow \mathscr{L}\Big(\int_{(0,1)} t dY(t^\beta)\Big)$ as $t \to \infty$. Consequently, the first factor in (3.1) $T_{e^{-t}}\mu^{*e^{-\beta t}}$ converges, say to γ_β, as $t \to \infty$. But for $a > 0$

$$T_a\gamma_\beta = \lim_{s\to\infty} (T_{e^{-s}}\mu^{*e^{-\beta s}})^{*a^{-\beta}} = \gamma_\beta^{*a^{-\beta}}$$

which shows that γ_β is a strictly stable measure with the exponent $-\beta$.

Assuming that μ is symmetric in (3.1) we have that both factors are symmetric and conditionally compact. In fact, both converge because of Theorem 1.1 (b). Consequently, similar arguments apply for $-2 < \beta \leq 1$ as they did for $-1 < \beta < 0$.

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