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ZERO SQUARE NEAR-RINGS

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Abstract

The purpose of this paper is to provide examples and explore properties of a wide variety of zero square (left) near rings. Among the main results are complete classifications of (i) finite Abelian groups which are the additive group of a zero square near-ring and (ii) finite non-Abelian groups which support 3-nilpotent distributive zero square near-rings.

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1. Introduction and preliminaries

Zero square near-rings having both distributive properties were considered by Heatherly in [3]. He gave examples, explored nilpotency and properties of the additive groups of such near-rings, and raised the question of whether all zero square near-rings are right distributive. The present author [5] answered Heatherly's question by giving an example of a non-distributive zero square near-ring on the dihedral group of order eight. More recently, Feigelstock [1] has provided several examples of both Abelian and non-Abelian zero square near-rings which are not right distributive.

In this paper we will show that there is an abundance of zero square near-rings (distributive, pseudo-distributive, and neither) having a wide variety of additive groups. Complete classifications are given for finite Abelian groups which are the additive group of a zero square near-ring, and for non-Abelian groups which support 3-nilpotent distributive zero square near-rings. We also show that any zero square near-rings with cyclic addition is 3-nilpotent, and

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we provide several necessity conditions for a non-distributive distributively generated zero square near-ring.

Throughout the paper ZS *near-ring* will denote a left near-ring, which is not a ring, in which $x^2 = 0$ for all x and $xy \neq 0$ for some xy. These basic properties of such a near-ring are trivial to verify.

LEMMA 1.1. *If N is a ZS near-ring, then*

(i) *N is zero-symmetric;*
(ii) *$xyx = 0$ for all $x, y \in N$;*
(iii) *If $x \neq 0$ and $y \neq 0$, then $xy \neq x$ and $xy \neq y$.*

If N is a near-ring and $x \in N$, x is called a *right-distributive element* if $(a+b)x = ax + bx$ for every $a, b \in N$. N is a *distributive near-ring* if all its elements are right distributive, and N is *distributively generated* (d.g.) if N^+ is generated by a set of right distributive elements [7]. N is *pseudo-distributive* if $(ab + cd)x = abx + cdx$ and $ab + cd = ab$ for all $a, b, c, d, x \in N$ [4]. A near-ring is *Abelian* if its additive group is Abelian.

2. Abelian zero square near-rings

It is well known that distributive or distributively generated near-rings with Abelian additive groups are rings. But pseudo-distributive near-rings which are not rings can be Abelian. In this section we classify finite Abelian groups which support non-pseudo-distributive ZS near-rings. We also show that every ZS near-rings with cyclic addition is 3-nilpotent, and give examples of both pseudo-distributive and non-pseudo-distributive ZS near-rings defined on cyclic and non-cyclic groups.

THEOREM 2.1. *The cyclic group of order n is the additive group of a ZS near-ring if and only if $n = p^2 m$ for some prime p and $m > 1$.*

PROOF. (i) Let N be a ZS near-ring of order n with $N^+ = \langle x \rangle$. Since $N^2 \neq 0$ and $x(mx) = 0$ for every $m \in Z$, there exist distinct positive integers $j, k < n$ such that $(jx)x = kx$. Then $(jx)(jx) = jkx = 0$. So $n | jk$. Also notice that

$$[(jx)(jx)]x = 0 = (jx)[(jx)x] = (jx)(kx) = k^2 x.$$

This implies $n|k^2$. But $n \nmid k$; hence n is not square-free.

Now suppose $n = p^2$. Since $p|j, p \mid k, j < p^2$, and $k < p^2$, we can write $j = ap$ and $k = bp$ for positive integers $a, b < p$. Therefore, there

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exists a positive integer $c < p$ such that $bc \equiv 1 \pmod{p}$. Let $cb = rp + 1$ for $0 \leq r < p$. Then

$$(apx)(cax) = cabpx = (rp + 1)(apx) = rp^2ax + apx = apx,$$

which gives us the contradiction

$$[(apx)(cax)](cax) \neq (apx)[(cax)(cax)].$$

(ii) Let $n = p^2m$ for some prime p and some integer $m > 1$, and let $N^+ = \langle x \rangle$ be the cyclic group of order n. Define multiplication on N^+ by

$$(px)(jx) = jpmx \text{ and } (rx)(jx) = 0 \text{ for } r \neq p.$$

It is routine to check that $(N, +, \cdot)$ is a ZS near-ring.

The near-rings constructed in part (ii) above are pseudo-distributive and 3-nilpotent. We do not know whether all ZS near-rings with cyclic addition are pseudo-distributive, but they are all 3-nilpotent.

THEOREM 2.2. *Every ZS near-ring with cyclic addition is 3-nilpotent.*

PROOF. Let N be a ZS near-ring of order n with $N^+ = \langle x \rangle$. Suppose $(jx)(kx)(vx) \neq 0$ for positive integers $j, k, v < n$. Then $(jx)x = tx \neq 0$ and $(kx)x = mx \neq 0$. Hence $(jx)(tx) = t^2x = 0$ and $(kx)x = mx \neq 0$. Hence $(jx)(tx) = t^2x = 0$ and $(kx)(mx) = m^2x = 0$, which implies $n | t^2$ and $n | m^2$. Therefore, $n | tm$, giving us the contradiction

$$(jx)[(kx)(vx)] = (jx)(vmx) = tvmx = 0.$$

We now consider Abelian ZS near-rings with non-cyclic addition.

LEMMA 2.3. *Let p and q be distinct primes. If $N \cong Z_{p^a} \oplus Z_{p^2}$ for $1 \leq a \leq 2$, $N \cong Z_p \oplus Z_{pq}$, or $N \cong Z_p \oplus Z_p \oplus Z_p$, then N is the additive group of a ZS near-ring.*

PROOF. (i) If $N = \langle x \rangle \oplus \langle y \rangle$ where $\langle x \rangle$ has order p^α and $\langle y \rangle$ has order p^2, define multiplication on N by

$$(x + y)(jx + ky) = kpy \quad \text{and} \quad ab = 0 \text{ if } a \neq (x + y).$$

(ii) If $N = \langle x \rangle \oplus \langle y \rangle$ where $\langle x \rangle$ has order p and $\langle y \rangle$ has order pq, let t be the smallest positive integer such that $p | (q + t)$. Define multiplication on N by

$$(x + y)(jx + ky) = t(j - k)x - q(j - k)y \quad \text{and} \quad ab = 0 \text{ if } a \neq (x + y).$$

(iii) If $N = \langle x\rangle \oplus \langle y\rangle \oplus \langle z\rangle$ where each summand has order p, define multiplication on N by

$$x(jx + ky + mz) = kz \quad \text{and} \quad ab = 0 \text{ if } a \neq x.$$

It is routine to verify that for each of these multiplications $(N, +, \cdot)$ is a ZS near-ring.

Theorem 2.4. *A non-cyclic Abelian group N is the additive group of a ZS near-ring if and only if N is not isomorphic to $Z_p \oplus Z_p$ for some prime p.*

Proof. (i) If N is not isomorphic to $Z_p \oplus Z_p$, then N has a direct summand G such that $G \cong Z_{p^2m}$ for $m > 1$, $G \cong Z_{p^\alpha} \oplus Z_{p^2}$ for $1 \leq \alpha \leq 2$, $G \cong Z_p \oplus Z_{pq}$, or $G \cong Z_p \oplus Z_p \oplus Z_p$. It follows from Theorem 2.1 and Lemma 2.3 that G supports a ZS near-ring. Therefore, the direct sum of the ZS near-ring on G and the zero ring on N/G is a ZS near-ring with additive group isomorphic to N.

(ii) Suppose N is a zero square near-ring and $N^+ = \langle a\rangle \oplus \langle b\rangle$, where each summand has order p.

If $(ja)a = ra + sb$ for some positive integer $j < p$ and non-negative integers $r, s < p$, then $(ja)(ja) = jra + jsb = 0$. Therefore, $r = s = 0$. Now suppose $(ja)b = ra + sb$. Then

$$(ja)(ra + sb) = sra + s^2b = 0.$$

Hence $r = s = 0$. By a similar argument, it can be shown that $(jb)b = (jb)a = 0$. But since N does not have zero multiplication, there exist positive integers $j, k < p$ and non-negative integers $v, w < p$ such that $(ja + kb)(va + wb) \neq 0$. It follows that $(ja + kb)a = ma + nb \neq 0$ or $(ja + kb)b = ra + sb \neq 0$ for non-negative integers $m, n, r, s < p$.

If $ma + nb = 0$, then

$$(ja + kb)^2 = (ja + kb)(kb) = kra + ksb = 0$$

which implies $r = s = 0$. This contradiction gives us $ma + nb \neq 0$. The supposition that $ra + sb = 0$ results in the same contradiction. Therefore, $ma + nb \neq 0$ and $ra + sb \neq 0$.

If $p = 2$, then $j = k = m = n = r = s = 1$. But this implies that $(a + b)a = a + b$, which is impossible. So $p > 2$.

Notice that

$$(ja + kb)(ma + nb) = (m^2r + nr)a + (mn + ns)b = 0,$$

and

$$(ja + kb)^2 = (jm + kr)a + (jn + ks)b = 0.$$

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Hence $m+s \equiv 0 \pmod{p}$, $m^2+nr \equiv 0 \pmod{p}$, and $jm+kr \equiv 0 \pmod{p}$.

Since $p > 2$, there exists a positive integer $c < p$ such that $c \not\equiv kn^{-1} \pmod{p}$. So

$$(jm + kr) \equiv c(nr + m^2) \equiv 0 \pmod{p},$$

and

$$jm - cm^2 \equiv cnr - kr \equiv (j - cm)m \equiv (cm - j)s \equiv (cn - k)r \pmod{p}.$$

Therefore, $(cm - j)r^{-1} \equiv (cn - k)s^{-1} \not\equiv 0 \pmod{p}$.

Let $(j - cm)r^{-1} \equiv d \equiv (k - cn)s^{-1} \pmod{p}$. Now notice that

$$\begin{aligned}(ja + kb)(ca + db) &= (cm + dr)a + (cn + ds)b \\ &= [cm+(j - cm)r^{-1}r]a+[cn+(k - cn)s^{-1}s]b=ja+kb.\end{aligned}$$

But this contradicts Lemma 1.1, hence N is not a ZS near-ring.

3. Non-Abelian ZS near-rings

First we consider distributive ZS near-rings. For any distributive near-ring N, $A = \{a \in N \mid ax = xa = 0 \text{ for all } x \in N\}$ is an ideal containing N'. Heatherly [3] noted that whenever N^2 is not contained in A, then N/A is a non-trivial zero square ring; hence the limitations of order and nilpotency for zero square rings [8] are inherited by these near-rings. The commutator near-rings constructed by Heatherly on nilpotent-class-two groups and by Feigelstock [1] on generalized nil-2 groups are distributive ZS near-rings with $N^2 \subseteq A$. The next several results show that such near-rings can be defined on a wide variety of additive groups including all finite nilpotent groups and dihedral groups of order $8n$ for $n \geq 1$.

DEFINITION 3.1. A finite Abelian group G will be called kq-non-cyclic for some prime q and some positive integer k if, when G is written as the direct sum of cyclic groups of prime power order, at least k of the summands are q-groups.

THEOREM 3.2. *A finite non-Abelian group N is the additive group of a 3-nilpotent distributive ZS near-ring if and only if N has a normal subgroup A which contains N', there exists a prime p such that N/A is $2p$-non-cyclic, and $p \mid |A|$.*

PROOF. (i) Let N be a non-Abelian group with properties described in the theorem, and let $t \in A$ be such that $o(t) = p$. Also let B and C be two summands of N/A which are p-groups, $N/A = B \oplus C \oplus G$, $x \in B$ such that

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$px \in A$, and $y \in C$ such that $py \in A$. Then every element of N can be uniquely written as $jx + ky + g + a$ for positive integers $j, k \leq p$, $g \in G$, and $a \in A$. Define multiplication in N by

$$(j_1x + k_1y + g_1 + a_1)(j_2x + k_2y + g_2 + a_2) = (j_1k_2 - j_2k_1)t.$$

It is routine to verify that $(N, +, \cdot)$ is a 3-nilpotent distributive ZS near-ring.

(ii) Let N be a finite 3-nilpotent distributive ZS near-ring. Since N is non-trivial, there exists $x, y \in N$ such that $xy \neq 0$; hence there is a prime p such that $p|(o(x), o(y))$. It follows that $p||A|$, $x \notin A$, and $y \notin A$. Suppose $x = jy + a$ for some integer j and some $a \in A$. Then $xy = (jy + a)y = 0$. This contradiction implies that $x \notin (jy + A)$ for every integer j. Therefore, x and y are in different cyclic summands of N/A, and each of these summands has order divisible by p. Hence N/A is $2p$-non-cyclic.

Corollary 3.3. *Every finite nilpotent group is the additive group of a distributive ZS near-ring.*

Proof. Let N be a finite nilpotent group. Then $N = S \oplus G$ where S is a non-Abelian Sylow p-subgroup. Let A be the Frattini subgroup of S. Then $S' \subseteq A$, $S' \neq 0$, and S/A is elementary abelian of order p^m with $m > 1$ [2, 6]. It follows that S/A is $2p$-non-cyclic; so S is the additive group of a distributive ZS near-ring. The direct product of this near-ring and the zero ring on G is a distributive ZS near-ring with additive group N.

Corollary 3.4. *A dihedral group of order $2n$ supports a distributive ZS near-ring if and only if $4|n$.*

Proof. (i) Let N be a dihedral group of order $2n$ where $4|n$. Then N' has an element of order two and $N/N' \cong Z_2 \oplus Z_2$.

(ii) Let N be a dihedral group of order $2n$ where $4 \nmid n$. If n is even, $N/N' \cong Z_2 \oplus Z_2$, but N' has no element of order two. IF n is odd, $N/N' \cong Z_2$. In either case N is not the additive group of a distributive ZS near-ring.

The following example shows that there are non-abelian pseudo-distributive ZS near-rings which are not distributive.

Example 3.5. Let N be a dihedral group of order $4k$ such that $N = \langle a, b \rangle$ where $2ka = 2b = 0$. Every element of N can be uniquely written

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as $ja + mb$ for integers j, m where $0 \leq j < 2k$ and $0 \leq m \leq 1$. So the following multiplication is well-defined on N

$$a(ja + b) = ka \text{ and } xy = 0 \text{ otherwise.}$$

It is routine to verify that with this multiplication N is a pseudo-distributive ZS near-ring. It is not right distributive since $(a + b)b \neq ab + b^2$.

Finally, we consider Feigelstock's question [1]: are there distributively generated ZS near-rings which are not distributive? Although the question remains open, the next theorem places several necessary conditions on such a near-ring.

The following lemmas are stated for reference; the proofs are trivial.

LEMMA 3.6. *If N is a near-ring with N^+ generated by a set of right distributive elements whose products commute additively, then N is distributive.*

LEMMA 3.7. *If N is a near-ring and $a, b \in N$ with b right distributive, then $(-a)b = a(-b) = -(ab)$.*

LEMMA 3.8. *If N is a ZS near-ring with right distributive elements a and b, then $ab = -(ba)$.*

THEOREM 3.9. *If N is a non-distributive ZS near-ring which is generated additively by a set D of right distributive elements, then*

(i) *D contains at least three elements;*
(ii) *if $|N|$ is odd, then N is 3-nilpotent;*
(iii) *for every $a, b, c \in D$, $ca + (ab + cb) = (ab + cb) + ca$.*

PROOF. (i) Suppose $D = \{a, b\}$. Then the only products of elements of D are $0, ab$, and ba. Since $ab = -ba$ (Lemma 3.8), all products in D commute additively. Therefore, by Lemma 3.6, N is distributive.

(ii) Suppose N is k-nilpotent for $k > 3$. Then there exist $a, b, c \in D$ such that $abc \neq 0$. Also

$$(a + bc)(a + bc) = bca + abc = 0,$$

and

$$(ab + c)(ab + c) = cab + abc = 0.$$

Hence

$$bca = cab = (ca)b = (-b)ca = -(bca).$$

Thus bca has additive order two, which contradicts the fact that $|N|$ is odd.

(iii) Let $a, b, c \in D$. Then

$$(a + b + c)(a + b + c) = ba + ca + ab + cb + ac + bc = 0.$$

Hence $ca + ab + cb = ab + cb + ca$.

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