# On certain infinite integrals involving Struve functions and parabolic cylinder functions

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The object of the present note is to obtain a number of infinite integrals involving Struve functions and parabolic cylinder functions.

1. G. N. Watson[1] has proved that

$$\int_0^\infty e^{-\frac{1}{4}x^2} x^m D_n(x)\,dx = \frac{\sqrt{\pi}\,2^{\frac{1}{2}(n-m-1)}\,\Gamma(m+1)}{\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)} \qquad (R(m) > -1). \quad (1)$$

From (1)

$$\int_0^\infty e^{-\frac{1}{4}x^2} x^m\,{}_pF_q(a_1,\ldots,a_p;\ c_1,\ldots,c_q;\ x^2y^2)\,D_n(x)\,dx$$

$$= \frac{\sqrt{\pi}\,2^{\frac{1}{2}(n-m-}\ \ \Gamma(m+1)}{\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)}\,{}_{p+2}F_{q+1}\left[\begin{matrix}a_1,\ldots,a_p,\frac{1}{2}m+\frac{1}{2},\ \frac{1}{2}m+1\,;\\ c_1,\ldots,c_q,\frac{1}{2}m-\frac{1}{2}n+1\,;\ 2y^2\end{matrix}\right] \quad (2)$$

follows provided that the integral is convergent and term-by-term integration is permissible. A great many interesting particular cases of (2) are easily deducible: the following will be used in this paper.

$$\int_0^\infty \sqrt{xy}\,x^s e^{-\frac{1}{4}x^2}\,\mathbf{H}_{s+\frac{1}{2}}(xy)\,D_{2s+2}(x)\,dx = (-1)^s y^{s+1} e^{-\frac{1}{4}y^2} D_{2s+1}(y) \qquad (3)$$

$$\int_0^\infty \sqrt{xy}\,x^{s+\frac{1}{2}} e^{-\frac{1}{4}x^2} J_s(xy)\,D_{2s+2}(x)\,dx = (-1)^{s+1} y^{s+\frac{1}{2}} e^{-\frac{1}{4}y^2} D_{2s+2}(y) \qquad (4)$$

$$\int_0^\infty e^{-\frac{1}{4}x^2} D_{2s}(x)\cos xy\,dx = \sqrt{\frac{\pi}{2}}\,(-1)^s y^{2s} e^{-\frac{1}{4}y^2} \qquad (5)$$

(4) is already known.[2] Here and later $s$ is a non-negative integer.

2. The following integrals are obtained by integration with respect to $y$ from 0 to $\infty$ and inversion of the order of integrations.

Let us divide both sides of (5) by $(y^2+z^2)^s$, integrate, and use

$$\int_0^\infty t^{2s}(t^2+a^2)^{-s} e^{-\frac{1}{2}t^2}\,dt = 2^{s-\frac{1}{2}}\,\Gamma(s+\tfrac{1}{2})\,e^{\frac{1}{4}a^2} D_{-2s}(a). \qquad (6)$$

We get for $s>0$

$$\int_0^\infty x^{s-\frac{1}{2}} e^{-\frac{1}{4}x^2} K_{s-\frac{1}{2}}(xz)\,D_{2s}(x)\,dx = \sqrt{\frac{\pi}{2}}\,(-1)^s\,\Gamma(2s)\,z^{s-\frac{1}{2}} e^{\frac{1}{4}z^2} D_{-2s}(z). \quad (7)$$

Multiplying (4) by $K_{s+\frac{1}{2}}(yz)$ and integrating

$$\int_0^\infty x^{2s+1}(x^2+z^2)^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}D_{2s+2}(x)\,dx = \Gamma(2s+2)\,z^{2s+1}e^{\frac{1}{4}z^2}D_{-2s-2}(z). \quad (8)$$

Multiplying (5) by $e^{-yz}$, integrating, and using a well-known integral representation[3] of parabolic cylinder functions we obtain[1]

$$\int_0^\infty z(x^2+z^2)^{-1}e^{-\frac{1}{4}x^2}D_{2s}(x)\,dx = \sqrt{\frac{\pi}{2}}\,\Gamma(2s+1)(-1)^s e^{\frac{1}{4}z^2}D_{-2s-1}(z). \quad (9)$$

From this formula it follows that

$$\int_0^\infty x^{2s}(x^2+z^2)^{-1}e^{-\frac{1}{4}x^2}D_{2s}(x)\,dx = \sqrt{\frac{\pi}{2}}\,\Gamma(2s+1)\,z^{2s-1}e^{\frac{1}{4}z^2}D_{-2s-1}(z), \quad (10)$$

since $x^{2s}/(x^2+z^2) = (-)^s z^{2s}/(x^2+z^2)$ + an even polynomial of degree $2s-2$ in $x$, and the contribution of that polynomial vanishes on account of the orthogonal property of parabolic cylinder functions.

Multiplying (3) by $y\,K_{s+\frac{1}{2}}(yz)$, integrating, and using (10) and the known result[4]

$$\int_0^\infty x\,\mathbf{H}_{s+\frac{1}{2}}(xz)\,K_{s+\frac{1}{2}}(xy)\,dx = z^{s+\frac{3}{2}}y^{-s-\frac{3}{2}}(y^2+z^2)^{-1},$$

we obtain

$$\int_0^\infty x^{s+\frac{3}{2}}e^{-\frac{1}{4}x^2}K_{s+\frac{1}{2}}(xy)\,D_{2s+1}(x)\,dx = \sqrt{\frac{\pi}{2}}\,(-1)^s\,\Gamma(2s+3)\,y^{s-\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2s-3}(y). \quad (11)$$

3. To conclude this paper, a few integrals will be evaluated with the help of the operational calculus.

We write $\qquad f(p) \doteqdot h(t)$

if

$$f(p) = p\int_0^\infty e^{-pt}h(t)\,dt.$$

The following results are well known[5]:—

$$f\left(\frac{p}{x}\right) \doteqdot h(xt) \quad (12) \qquad\qquad p(p^2+x^2)^{-\frac{1}{2}} \doteqdot J_0(xt) \quad (13)$$

$$p\,I_s(y\sqrt{p})\,K_s(x\sqrt{p}) \doteqdot \frac{1}{2t}\exp\left(-\frac{x^2+y^2}{4t}\right)I_s\left(\frac{xy}{2t}\right) \quad (14)$$

$$\exp\left(-\frac{x^2+y^2}{4p}\right)I_s\left(\frac{xy}{2p}\right) \doteqdot J_s(x\sqrt{t})\,J_s(y\sqrt{t}) \quad (15)$$

$$\Gamma(s)\,p^{s+1}e^{\frac{1}{4}p^2}D_{-s}(p) \doteqdot \frac{d^s}{dt^s}(t^{s-1}e^{-\frac{1}{4}t^2}), \quad (s>0). \quad (16)$$

Goldstein[6] has proved that if $\phi(p) \doteqdot f(t)$ and $\psi(p) \doteqdot g(t)$ then

$$\int_0^\infty \phi(t)\,g(t)\,t^{-1}\,dt = \int_0^\infty f(t)\,\psi(t)\,t^{-1}\,dt. \quad (17)$$

In (8) let us put $z = p$, multiply by $p$, and interpret by means of (13) and (16), thus obtaining

$$\int_0^\infty x^{2s+1} e^{-\frac{1}{4}x^2} J_0(xz) D_{2s+2}(x)\, dx = \frac{d^{2s+1}}{dz^{2s+1}} (z^{2s+1} e^{-\frac{1}{2}z^2}). \qquad (18)$$

Apply (17) to (14) and (15) to find

$$\int_0^\infty I_s(z\sqrt{t}) J_s(y\sqrt{t}) J_s(z\sqrt{t}) K_s(y\sqrt{t})\, dt$$

$$= \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}(y^2+z^2)t} I_s^2(\tfrac{1}{2}yzt)\, dt = \frac{(-1)^{-s-\frac{1}{2}}}{\pi yz} Q_{s-\frac{1}{2}}\left(-\frac{y^4+z^4}{2y^2z^2}\right) \qquad (19)$$

$$(y \pm z > 0).$$

In the integral

$$\int_0^\infty x(x^4+4k^4)^{-\frac{1}{2}} J_0(xy)\, dx = J_0(ky) K_0(ky)$$

we put $k = \sqrt{\tfrac{1}{2}p}$, multiply by $p$, and have on interpretation

$$p J_0(y\sqrt{\tfrac{1}{2}p})\, K_0(y\sqrt{\tfrac{1}{2}p}) \doteqdot (2t)^{-1} J_0\left(\frac{y^2}{4t}\right). \qquad (20)$$

Combining this with $p/(p+1) \doteqdot e^{-t}$ in the manner of (17),

$$\int_0^\infty x e^{-2x^2} J_0(xy) K_0(xy)\, dx = \frac{\pi}{16}\{\mathbf{H}_0(\tfrac{1}{4}y^2) - Y_0(\tfrac{1}{4}y^2)\} \qquad (21)$$

can be derived.

From (15) and (20) we deduce that

$$\int_0^\infty J_0(x\sqrt{t}) J_0(y\sqrt{t}) J_0(z\sqrt{t}) K_0(z\sqrt{t})\, dt$$

$$= \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}(x^2+y^2)t} J_0(\tfrac{1}{2}z^2t) I_0(\tfrac{1}{2}xyt)\, dt = \pi^{-1}(xyz^2i)^{-\frac{1}{2}} Q_{-\frac{1}{2}}\left(\frac{x^4+y^4-2x^2y^2+4z^4}{8xyz^2i}\right). \qquad (22)$$

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