MAXIMAL IDEAL SPACES OF BANACH ALGEBRAS OF DERIVABLE ELEMENTS

R. J. LOY*

(Received 20 March 1969)

Communicated by G. B. Preston

Let A be a commutative Banach algebra, D a closed derivation defined on a subalgebra Δ of A, and with range in A. The elements of Δ may be called derivable in the obvious sense. For each integer $k \geqq 1$, denote by Δ_k the domain of D^k (so that $\Delta_1 = \Delta$); it is a simple consequence of Leibniz's formula that each Δ_k is an algebra. The classical example of this situation is $A = C(0, 1)$ under the supremum norm with D ordinary differentiation, and here $\Delta_k = C^k(0, 1)$ is a Banach algebra under the norm $\|\cdot\|_k$:

$$\|x\|_k = \sum_{n=0}^{k} \frac{1}{n!} \sup_{t\in[0,1]} |x^{(n)}(t)|.$$

Furthermore, the maximal ideals of Δ_k are precisely those subsets of Δ_k of the form $M \cap \Delta_k$ where M is a maximal ideal of A, and $\overline{M \cap \Delta_k} = M$, the bar denoting closure in A. In the present note we show how this extends to the general case.

If A is a commutative Banach algebra then $\|\cdot\|_A$, $\nu_A(\cdot)$, $\mathcal{M}(A)$ will denote the norm, spectral radius and maximal ideal space of A respectively. The author is indebted to the referee for the present proof of the following result.

THEOREM 1. *Let A, B be commutative Banach algebras, with B a dense subalgebra of A in the norm topology of A. Suppose that there is a constant K such that $\nu_B(x) \leqq K\nu_A(x)$ for $x \in B$. Then the map $\Gamma : \mathcal{M}(A) \to \mathcal{M}(B) : M \mapsto M \cap B$ is a homeomorphism of $\mathcal{M}(A)$ onto $\mathcal{M}(B)$ (and so $\nu_B(x) = \nu_A(x)$ for $x \in B$).*

PROOF. If ψ is a multiplicative linear functional on A then $\psi | B$ is clearly such a functional on B. Conversely, if ϕ is a multiplicative linear functional on B, the given inequality shows that ϕ is continuous in the norm topology of A, and so has a unique continuous extension, also multiplicative linear, to all of A. From the correspondence between multiplicative linear functionals and maximal modular ideals it follows that Γ is bijective. That Γ is a homeomorphism is an immediate con-

* This paper is based on a portion of the author's doctoral thesis, Monash University 1968, written under the supervision of Professor J. B. Miller and supported by a General Motors-Holden's Limited Research Fellowship.

sequence of the fact that B is dense in A. The last statement is clear from the form of Γ.

We now turn to the situation at hand.

LEMMA 1. *Let A be a Banach algebra with norm $\|\cdot\|$, D a closed derivation defined on a subalgebra Δ of A, with range in A. Then for each integer $k \geqq 1$, Δ_k is a Banach algebra under the norm $\|\cdot\|_k$:*

$$\|x\|_k = \sum_{n=0}^{k} \frac{1}{n!} \|D^n x\|.$$

PROOF. As was remarked above each Δ_k is certainly an algebra, and an application of Leibniz's formula shows that $\|\cdot\|_k$ is a norm on Δ_k. If $\{x_n\} \subseteq \Delta_k$ is Cauchy under $\|\cdot\|_k$ then $\{D^j x_n\}$ is Cauchy in A for $0 \leqq j \leqq k$. Setting $y_j = \lim_n D^j x_n$, the closure of D shows that $y_j = D^j y_0$, whence $y_0 \in \Delta_k$ and $\|x_n - y_0\|_k \to 0$ as $n \to \infty$.

LEMMA 2. *Let A be a commutative normed algebra, D a derivation defined on a subalgebra Δ of A, with range in A. Denote by $v_k(\cdot)$ the spectral radius in Δ_k calculated from $\|\cdot\|_k$. Then if $x \in \Delta_k$, $v_k(x) = v_A(x)$.*

PROOF. It is clear that $v_k(x) \geqq v_A(x)$ for all $x \in \Delta_k$. Now for $j < n$ and $x \in \Delta_j$,

$$D^j x^n = \sum_{i=1}^{j} u_{i,j} x^{n-i}$$

where the $u_{i,j}$ are polynomials in $D^r x$, $1 \leqq r \leqq j$, of degree $\leqq j$, the scalars concerned being polynomials in n of degree $\leqq j$.[1] To see this, note that the formula is true for $j = 1$, since $Dx^n = nx^{n-1}Dx$. Supposing by way of induction that it holds for $j = m-1$, we have

$$D^m x^n = \sum_{i=1}^{m-1} \{D(u_{i,m-1})x^{n-i} + (n-i)u_{i,m-1} x^{n-i-1} Dx\},$$

which is of the desired form.

Thus if $x \in \Delta_k$ and $n > k$,

$$\begin{aligned} \|x^n\|_k &= \|x^n\| + \sum_{j=1}^{k} \frac{1}{j!} \|\sum_{i=1}^{j} u_{i,j} x^{n-i}\| \\ &\leqq \|x^{n-k}\| \left\{ \|x^k\| + \sum_{j=1}^{k} \sum_{i=1}^{j} \frac{1}{j!} \|u_{i,j}\| \cdot \|x^{k-i}\| \right\} \\ &\leqq Kn^k \|x^{n-k}\| \end{aligned}$$

[1] The exact form is

$$\frac{D^j x^n}{j!} = \sum_{i_1+\cdots+i_n=j} \frac{D^{i_1} x}{i_1!} \cdots \frac{D^{i_n} x}{i_n!}.$$

https://doi.org/10.1017/S1446788700006686 Published online by Cambridge University Press

for some constant K, by the properties of the elements $u_{i,j}$. But this means $v_k(x) \leqq v_A(x)$.

Our main result is an immediate consequence of Lemmas 1 and 2, and Theorem 1.

THEOREM 2. *Let A be a commutative Banach algebra, D a closed derivation on a subalgebra Δ of A, with range in A. Suppose that Δ_k is dense in A for some integer $k \geqq 1$. Then the map $\Gamma_j : \mathscr{M}(A) \to \mathscr{M}(\Delta_j) : M \mapsto M \cap \Delta_j$ is homeomorphism of $\mathscr{M}(A)$ onto $\mathscr{M}(\Delta_j)$, $1 \leqq j \leqq k$.*

COROLLARY 1. *If A has an identity e then $e \in \Delta$.*

PROOF. Theorem 2 shows that $\mathscr{M}(\Delta)$ is compact, and so by Silov's theorem there is an idempotent $f \in \Delta$ with $\hat{f} \equiv 1$ on $\mathscr{M}(\Delta)$, and hence on $\mathscr{M}(A)$. But this means the idempotent $e-f$ is quasi-nilpotent, and hence zero.

COROLLARY 2. *If Δ is dense in A and D has non-empty resolvent set then Γ_j is a homeomorphism for each $j \geqq 1$.*

PROOF. By Lemma VIII.2.9 of [1] Δ_j is dense in A for each $j \geqq 1$.

REMARK. In the situation of Theorem 2 define, for $\alpha > 0$,

$$\Delta_{\infty,\alpha} = \left\{ x \in \bigcap_{k \geqq 1} \Delta_k : ||x||_{\infty,\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \cdot ||D^n x|| < \infty \right\}.$$

An argument similar to that of Lemma 1 shows that $\Delta_{\infty,\alpha}$ is a Banach algebra under $||\cdot||_{\infty,\alpha}$, however $\mathscr{M}(A)$ and $\mathscr{M}(\Delta_{\infty,\alpha})$ are not homeomorphic in general, even when $\Delta_{\infty,\alpha}$ is dense in A. Indeed, in the classical situation mentioned at the beginning of this paper, $\mathscr{M}(A) = [0, 1]$, while $\mathscr{M}(\Delta_{\infty,\alpha})$ is homeomorphic to the closed unit disc.

Reference

[1] N. Dunford and H. T. Schwartz, *Linear operators*, I (Interscience, New York, 1958).

Department of Mathematics
Carleton University
Ottawa 1, Canada

https://doi.org/10.1017/S1446788700006686 Published online by Cambridge University Press