LINEAR MAPPINGS BETWEEN TOPOLOGICAL VECTOR SPACES

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1. Introduction

If A and B are locally convex topological vector spaces, and B has certain additional structure, then the space $L(A,B)$ of all continuous linear mappings of A into B is characterized, within isomorphism, as the inductive limit of a family of spaces, whose elements are functions, or measures. The isomorphism is topological if $L(A,B)$ is given a particular topology, defined in terms of the seminorms which define the topologies of A and B. The additional structure on B enables $L(A,B)$ to be constructed, using the duals of the normed spaces obtained by giving A the topology of each of its seminorms separately.

The representation theorems lead to explicit representations of $L(A,B)$, in terms of functions, or measures, depending on two variables, if A and B are certain function spaces. Simple proofs are obtained for some known cases—when A or B is $C(P)$, the space of continuous complex functions on a compact Hausdorff space P (Dunford and Schwartz [4] give a representation which includes this case), and when $A = L^p(P)$ $(1 < p < \infty)$ (for which Cac [2] has given a representation)—but by different methods from these authors. But in addition, explicit representations, which appear to be new, are obtained for certain pairs of spaces which are not Banach spaces; when A or B are spaces of Schwartz distributions or test functions [7], having compact support. For example, a continuous linear mapping from Schwartz test functions into $C(P)$ may be identified with a suitable indexed family of Schwartz distributions.

2. Calibrations and structured spaces

If A and B are convex spaces (locally convex Hausdorff topological vector spaces), let $L(A,B)$ denote the space of all continuous linear mappings from A into B. Denote by $C(W)$ the space of all bounded continuous complex functions on the Hausdorff space W, with the uniform norm. The topology of a convex

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space A can be specified by a (non-unique) *calibration*, namely a set of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$; similarly let $\{\|\cdot\|_\gamma : \gamma \in \Gamma\}$ be a calibration for B.

The topology of A is unchanged by adjoining to the given calibration for A the maximum of each finite subset of the seminorms. The resulting calibration will be called *saturated*; it has the property (Bourbaki [1], page 97) that Λ is a directed set with pre-ordering $\geqq$, where for nets $\{x_\alpha\}$ in A,

$$\|x_\alpha\|_\mu \to 0 \text{ and } \mu \geqq \lambda \Rightarrow \|x_\alpha\|_\lambda \to 0, \tag{1}$$

or equivalently

$$\mu \geqq \lambda \Leftrightarrow \exists k = k(\lambda,\mu) : \|x\|_\lambda \leqq k\,\|x\|_\mu \qquad (\forall x \in A). \tag{2}$$

REMARKS. If $\mu \geqq \lambda$ and $\lambda \geqq \mu$, then the seminorms $\|\cdot\|_\mu$ and $\|\cdot\|_\lambda$ are (topologically) equivalent.

Let A be a convex space whose calibration is saturated. Denote by A_λ the factor space A/σ, where σ is the equivalence relation $x\,\sigma\,y$ iff $\|x-y\|_\lambda = 0$, and A_λ has the topology given by the corresponding quotient seminorm $\|\cdot\|_\lambda$. Denote by $A_\lambda^\sim$ the completion of A, and by $A_\lambda^{\sim\prime}$ the dual of $A_\lambda^\sim$.

DEFINITION. A convex space B will be called *structured* if its elements are bounded functions from a set W into a Banach space H, and if the topology of B is specified by seminorms $\|\cdot\|_\gamma$ $(\gamma \in \Gamma)$ of the form

$$\|y\|_\gamma = \sup_{w \in W} |(K_\gamma y)(w)| \qquad (y \in B, \gamma \in \Gamma) \tag{3}$$

where $K_\gamma : B \to B$ is a linear mapping (*not* necessarily continuous), $|\cdot|$ denotes the norm in H, and the set $\{K_\gamma : \gamma \in \Gamma\}$ includes the identity mapping, say for $\gamma = 0$.

EXAMPLES. Let $\boldsymbol{D}(I)$ denote the space of infinitely differentiable complex functions x, having support in the interval I in Euclidean n-space, with topology given by either of the equivalent sets of seminorms:

$$\|x\|_\lambda = \sup_{t \in I} |x^{(\lambda)}(t)| \tag{4}$$

$$\|x\|'_\lambda = \max_{j \leqq \lambda} \|x\|_j . \tag{5}$$

Here $\lambda \in \Lambda_s$, the set of n-tuples $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of non-negative integers, ordered by $\lambda \leqq \lambda'$ iff $\lambda_j \leqq \lambda'_j$ for all j, and $x^{(\lambda)}$ denotes the partial derivative of x of order $(\lambda_1, \cdots, \lambda_n)$. Let $\boldsymbol{E}(I)$ denote the space of the restrictions to I of infinitely differentiable complex functions on n-space, with topology given by (4) or (5). Then $\boldsymbol{D}(I)$ and $\boldsymbol{E}(I)$ are structured, in terms of the calibration (4); the equivalent calibration (5) is saturated.

Any Banach space B is structured, since each $y \in B$ may be represented, by its natural mapping into the second dual space B'', as a complex function on the unit sphere in B' (or, using Choquet's theorem, as a function on the set of extreme points of the unit ball in B'); then (3) is immediate, with K as the identity mapping, and W the domain of the functions.

3. Natural topology for $L(A, B)$

Let $T \in L(A,B)$, where A and B are convex spaces, and the calibration of A is saturated. Since T is continuous, for each $\gamma \in \Gamma$ there are λ_i, δ', r such that

$$\| x \|_{\lambda_i} < \delta' \qquad (i = 1,2,\cdots,r(\gamma)) \Rightarrow \| Tx \|_{\lambda} < 1.$$

Since Λ is a directed set, there is $\lambda \in \Lambda$ with $\lambda \geqq \lambda_i$ $(i = 1,2,\cdots,r)$. Then, by (1), there is δ such that

$$\| x \|_{\lambda} < \delta \Rightarrow \| Tx \|_{\gamma} < 1. \tag{6}$$

The values of $\lambda = \lambda(\gamma, T)$ determine, for each $T \in L(A,B)$, a (non-unique) function $\Delta\colon \Gamma \to \Lambda$, which will be called an *index function* for T. The set $S(\Gamma,\Lambda)$ of all functions from Γ into Λ is partially ordered by

$$\Delta_\beta \geqq \Delta_\alpha \Leftrightarrow \Delta_\beta(\gamma) \geqq \Delta_\alpha(\gamma) \qquad (\text{all } \gamma \in \Gamma); \tag{7}$$

denote also $\Delta_\beta > \Delta_\alpha \Leftrightarrow \Delta_\beta \geqq \Delta_\alpha$ and $\Delta_\beta \neq \Delta_\alpha$. From (1), if Δ is an index function for T, then so also is any $\Delta' \geqq \Delta$. If, in particular, A is countably semi-normed, then there exists a *minimal* (in terms of $\geqq$) index function for T; denote it by Δ^T_{min}.

Denote by $M(\Delta)$ the subspace of $L(A,B)$ consisting of those $T \in L(A,B)$ for which there is an index function $\leqq \Delta \in S(\Gamma,\Lambda)$. Now

$$\Delta_\alpha \leqq \Delta_\beta \Rightarrow M(\Delta_\alpha) \subset M(\Delta_\beta); \tag{8}$$

denote by $i_{\alpha\beta}$ this embedding of $M(\Delta_\alpha)$ in $M(\Delta_\beta)$.

Since $T \in L(A,B)$, each of the seminorms

$$\| T \|_{\gamma,\mu} = \sup\{\| Tx \|_\gamma \colon \| x \|_\mu \leqq 1\} \qquad (\gamma \in \Gamma) \tag{9}$$

is finite, if $\mu = \Delta(\gamma)$ for some index function Δ of T. Topologise $M(\Delta)$ by the seminorms $\| T \|_{\gamma,\Delta(\gamma)}$ $(\gamma \in \Gamma)$. If Δ_α and Δ_β are index functions of T, with $\Delta_\alpha \leqq \Delta_\beta$, let $\lambda = \Delta_\alpha(\gamma)$ and $\mu = \Delta_\beta(\gamma)$, for given $\gamma \in \Gamma$; since $\mu \geqq \lambda$,

$$\| x \|_\lambda \leqq k \| x \|_\mu,$$

with k given by (2); hence

$$\{x\colon \| x \|_\mu \leqq 1\} \subset \{x\colon \| x \|_\lambda \leqq k\};$$

therefore

$$\| T \|_{\gamma\,\mu} \leqq k^{-1} \| T \|_{\gamma,\lambda} \qquad (\mu \geqq \lambda). \tag{10}$$

Consequently, $i_{\alpha\beta}$ is continuous.

Since also

$$\Delta_\alpha \leqq \Delta_\beta \leqq \Delta_\delta \Rightarrow i_{\alpha\delta} = i_{\beta\delta} \circ i_{\alpha\beta}, \tag{11}$$

the family $\{M(\Delta_\alpha); i_{\alpha\beta}\}$ of spaces and mappings is an inductive spectrum over $S(\Gamma,\Lambda)$ (Dugundji [3], page 420). The inductive limit space of this spectrum is the quotient space $\Sigma_\alpha M(\Delta_\alpha)/\equiv$, where Σ_α denotes free union over $S(\Gamma,\Lambda)$ and $\equiv$ denotes the equivalence relation

$$T_\alpha \in M(\Delta_\alpha) \equiv T_\beta \in M(\Delta_\beta)$$

iff there exists $\delta \geqq \alpha,\beta$ such that

$$i_{\alpha\delta}T_\alpha = i_{\beta\delta}T_\beta.$$

It will be convenient to call the topology of this inductive limit space the *natural topology* for $L(A,B)$. This topology is locally convex (Robertson and Robertson [6], page 79, Prop. 4), and, for given topologies for A and B, it is clearly independent of the particular choice of calibrations for B, or for A so that (1) and (2) hold. If A and B are normed spaces, the natural topology is the operator norm topology.

The natural topology is a topology of uniform convergence; it could, of course, be expressed in terms of neighbourhoods instead of seminorms, but this does not offer any obvious simplification.

4. Representation theorems

Let A be a convex space whose calibration is saturated; let B be a convex space whose elements are functions whose domain is a set W. A subspace M of $L(A,B)$ is *represented* by a vector space Q, whose elements are functions (or measures, or distributions) g whose domain is $X \times W$ (where X is a given set) if there is a bijection ϕ of M onto Q/ρ, where ρ is as equivalence relation on Q, and a bilinear form $F[\cdot,\cdot]$ such that

$$(Tx)(w) = F[x, g(\cdot, w)], \tag{12}$$

where $x \in A$, $T \in M$, $w \in W$, and g denotes a representative of the equivalence class $[g] = \phi(T) \in Q/\rho$. The equivalence relation ρ will not be mentioned if it is the identity. The representation is *topological* if also M and Q are topological vector spaces, and ϕ maps the topology of M onto that of Q/ρ.

As an example of (12), consider A as a space of real-valued functions on a measure space Y, and T defined by

$$(Tx)(w) = \int_Y x(y)g_T(y,w)\,d\mu(y) = F[x, g_T(\cdot, w)].$$

If each subspace $M(\Delta_\alpha)$ of $L(A,B)$ is topologically represented by a topological vector space $Q(\Delta_\alpha)$ then, since the representation is a topological isomorphism, there is a bijection ϕ^* of the inductive limit space, M^* say, of the $M(\Delta_\alpha)$ onto the inductive limit space, Q^* say, of the $Q(\Delta_\alpha)$; and ϕ^* maps the topology of M^* onto that of Q^*, since E^* does not change the values of the seminorms $\|T\|_{\gamma,\mu}$. The space Q^* will then be called an *inductive representation* of M^*, or of $\mathbf{L}(A,B)$.

THEOREM 1. *Let B be a structured space, of functions which map W into a Banach space H; let B have calibration $\{\|\cdot\|_\gamma : \gamma \in \Gamma\}$. Let A be any convex space, whose calibration $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ is saturated. For each $\lambda \in \Lambda$, let V_λ be a Banach space of functions (or complex measures, or distributions) defined on a set X, and σ_λ an equivalence relation on V_λ, such that a congruence (an isometric isometry) between $\mathbf{L}(A_\lambda^\sim, H)$ and V_λ/σ_λ is established by*

$$f(x) = F_\lambda[x, f^*], \tag{13}$$

where $x \in A_\lambda^\sim$, $f \in \mathbf{L}(A_\lambda^\sim, H)$, $f^* \in V_\lambda$, and F_λ is a bilinear form, which may depend on λ.

Then $\mathbf{L}(A,B)$ is inductively represented by the inductive limit of a family of spaces $U^*(\Delta)$, where $\Delta \in S(\Gamma,\Lambda)$, and $U^*(\Delta)$ is a subspace of

$$(V_{\Delta(0)}/\sigma_{\Delta(0)}) \times W.$$

If $T \in \mathbf{L}(A,B)$, and Δ is an index function for T, then

$$(Tx)(w) = F_{\Delta(0)}[x, g(\cdot, w)]; \tag{14}$$

$$T_{\gamma,\Delta(\gamma)} = \sup_{w \in W} \|K^*_{\gamma,\Delta} g(\cdot, w)\|; \tag{15}$$

where $x \in A$, $w \in W$, $g(\cdot, w) \in V_{\Delta(0)}$, and

$$K^*_{\gamma,\Delta} : V_{\Delta(0)} \to V_{\Delta(\gamma)}$$

is a linear mapping determined by K_γ. The representation is topological if $\mathbf{L}(A,B)$ has its natural topology and $U^*(\Delta)$ is topologised by the seminorms $\|T\|_{\gamma,\Delta(\gamma)}$ $(\gamma \in \Gamma)$.

REMARKS. If $H = \mathbf{C}$, the complex field, then each f in the Banach space $A_\lambda^{\sim\prime}$ may be represented as a complex function on the unit sphere of $A_\lambda^{\sim\prime\prime}$ (or on the set of extreme points of the unit ball in $A_\lambda^{\sim\prime\prime}$, using Choquet's theorem.) In this sense, (13) is trivial. In various particular cases (see later theorems) V_λ can be given explicitly as a space of complex functions or measures.

Not all $\Delta \in S(\Gamma,\Lambda)$ need contribute to the inductive limit.

If the V_λ are function spaces then, for each Δ, the subspace $M(\Delta)$ of $\mathbf{L}(A,B)$ is isomorphic to a space of functions $W \to V_{\Delta(0)}$, for which the seminorms (15) are of the form (3); hence each subspace $M(\Delta)$ is also a structured space.

If A is countably normed, and, for each T, $\Delta^T_{min}(\gamma)$ is independent of γ, then $L(A,B)$ is inductively represented by the inductive limit of a *sequence* of spaces U^*_λ $(\lambda = 0, 1, \cdots)$, where U^*_λ is a subspace of $(V_\lambda/\sigma_\lambda) \times W$. In particular, if A is a normed space, then $L(A,B)$ is represented by a subspace U of $(V_0/\sigma_0) \times W$, with the topology defined by the seminorms

$$(16) \qquad \| T \|_\gamma = \sup_{w \in W} \| K^*_{\gamma,\Delta} g(\cdot, w) \| \qquad (\gamma \in \Gamma, \; g(\cdot, w) \in V_0).$$

If A is a convex space with the Mackey topology (so in particular if A is barrelled), then the space of all linear mappings of A into B which are continuous in the given topology of A and the weak topology of B coincides with $L(A,B)$, so is also represented by Theorem 1. For if T is continuous from A with strong topology to B with weak topology, then T is continuous from A with weak topology to B with weak topology ([6], page 39, Prop. 13); so if A has its Mackey topology, T is continuous from A with strong topology to B with strong topology; the converse is immediate.

PROOF OF THEOREM 1. Let Δ be an index function for $T \in L(A,B)$; let $\gamma \in \Gamma$; let $\lambda = \Delta(\gamma)$. For fixed γ, define the linear mapping $f_w : A \to H$ by $f_w = (K_\gamma T.)(w)$. Since

$$(17) \qquad \begin{aligned} \sup_{w \in W} |f_w(x-y)| &= \sup_{w \in W} |(K_\gamma T(x-y))(w)| \\ &= \| T(x-y) \|_\gamma \\ &\leqq \| T \|_{\gamma,\lambda} \| x-y \|_\lambda, \end{aligned}$$

f_w defines a unique element (also written f_w) of $L(A_\lambda, H)$. Since

$$(18) \qquad \sup_{w \in W} \sup_{\|x\|_\lambda \leqq 1} |f_w(x)| = \sup_{\|x\|_\lambda \leqq 1} \| Tx \|_\gamma = \| T \|_{\gamma,\lambda} < \infty \quad \text{since } \lambda = \Delta(\gamma),$$

the mappings f_w $(w \in W)$ are equicontinuous on A_λ.

By continuity, f_w can be extended, without increase of norm, to a continuous mapping $f^*_w : A^\sim_\lambda \to H$. By (13),

$$(19) \qquad f^*_w(x) = F_\lambda[x, g_{\gamma,\lambda}(\cdot, w)];$$

where $x \in A^\sim_\lambda$, and $g_{\gamma,\lambda}(\cdot, w)$ is written for the function (or complex measure or distribution) f^* corresponding to $w \in W$. Thus, for $x \in A \subset A^\sim_\lambda$, and Δ any index function for T,

$$(20) \qquad (K_\gamma Tx)(w) = F_{\Delta(\gamma)}[x, g_{\gamma,\Delta(\gamma)}(\cdot, w)].$$

From (18), with $\lambda = \Delta(\gamma)$,

$$(21) \qquad \| T \|_{\gamma,\lambda} = \sup_{w \in W} \| f_w \| = \sup_{w \in W} \| g_{\gamma,\lambda}(\cdot, w) \| \quad (\gamma \in \Gamma),$$

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where $\| g_{\gamma,\lambda}(\cdot, w)\|$ denotes the norm in V_λ, since the mapping $f \to f^*$ in (13) is an isometry.

Equation (20) defines a linear mapping ψ_γ of $K_\gamma T$ onto $[g_{\gamma,\Delta(\gamma)}(\cdot,\cdot)]$, the equivalence class in

$$(V_{\Delta(\gamma)}/\sigma_{\Delta(\gamma)}) \times W$$

of which $g_{\gamma,\Delta(\gamma)}(\cdot,\cdot)$ is a representative. Since $F_{\Delta(\gamma)}$ is a bilinear form, and the mapping $f \to [f^*]$ defined by (13) is a bijection, ψ_γ has zero kernel, so ψ_γ^{-1} exists. Denote by σ^* the canonical mapping of $V_{\Delta(0)}$ into

$$V_{\Delta(0)}/\sigma_{\Delta(0)};$$

denote by e_γ any linear embedding of $V_{\Delta(\gamma)}/\sigma_{\Delta(\gamma)}$ into $V_{\Delta(\gamma)}$. Define

$$K^*_{\gamma,\Delta}: V_{\Delta(0)} \to V_{\Delta(\gamma)}$$

by

$$K^*_{\gamma,\Delta} = e_\gamma \circ \psi_\gamma \circ K_\gamma \ \ \psi_0^{-1} \circ \sigma^*. \tag{22}$$

Then $K^*_{\gamma,\Delta}$ maps $g_{0,\Delta(0)}(\cdot,\cdot)$ onto $g_{\gamma,\Delta(\gamma)}(\cdot,\cdot)$. This, with (20), proves (14), writing g for $g_{0,\Delta(0)}$.

Denote by $Z(\Delta_\alpha)$ the subspace of $V_{\Delta_\alpha(0)} \times W$ consisting of those functions $g_{0,\Delta_\alpha(0)}(\cdot,\cdot)$ for which all the seminorms (21) are finite, with the convex topology determined by these seminorms. Since these seminorms are finite for each $T \in \boldsymbol{L}(A,B)$ for which Δ_α is an index function, there is, by (20), a linear injection

$$j_{\alpha\beta}: M(\Delta_\alpha) \to Z(\Delta_\beta)$$

for each Δ_α and $\Delta_\beta \geqq \Delta_\alpha$ in $S(\Gamma,\Lambda)$. Let $U(\Delta_\alpha) = j_{\alpha\alpha}M(\Delta_\alpha)$, with the relative topology of $Z(\Delta_\alpha)$; $U(\Delta_\alpha)$ is, in general, a *proper* subspace of $Z(\Delta_\alpha)$, since the finiteness of all the seminorms (15) does *not* imply that $Tx \in B$ for *all* $x \in A$.

Since $j_{\alpha\alpha}$ is a bijection onto $U(\Delta_\alpha)$, there is a linear injection $\phi_{\alpha\beta} = j_{\alpha\beta} \circ j_{\alpha\alpha}^{-1}$: $U(\Delta_\alpha) \to Z(\Delta_\beta)$ which, by (11), satisfies $\phi_{\alpha\delta} = \phi_{\beta\delta} \circ \phi_{\alpha\beta}$ whenever $\Delta_\alpha \leqq \Delta_\beta \leqq \phi_\Delta$. Since $j_{\alpha\alpha}$ does not change the seminorms (15), $j_{\alpha\alpha}$ is continuous. Since $j_{\alpha\beta} = j_{\beta\beta} \circ i_{\alpha\beta}$ and $i_{\alpha\beta}$ is continuous, $j_{\alpha\beta}$ is a continuous mapping onto $U(\Delta_\beta)$; hence $\phi_{\alpha\beta}: U(\Delta_\alpha) \to U(\Delta_\beta)$ is continuous. Therefore the family $\{U^*(\Delta_\alpha); \phi_{\alpha\beta}\}$, where $U^*(\Delta_\alpha) = U(\Delta_\alpha)/\sigma_{\Delta_\alpha(0)}$ is an inductive spectrum over $S(\Gamma,\Lambda)$. From (15) and the definition of natural topology for $\boldsymbol{L}(A,B)$, $\boldsymbol{L}(A,B)$ is inductively represented by the inductive limit of this spectrum.

THEOREM 2. *Let the spaces A and B satisfy the hypotheses of Theorem* 1; *let $\Delta \in S(\Gamma,\Lambda)$; define the mapping $T: A \to B$ by* (14), *where $g(\cdot, w) \in V_{\Delta(0)}$, $w \in W$. Let g be such that $Tx \in B$ whenever $x \in A$. For each $\gamma \in \Gamma$, assume that*

$$(K_\gamma Tx)(w) = F_{\Delta(\gamma)}[x, K^*_{\gamma,\Delta} g(\cdot, w)], \tag{23}$$

where $K^*_{\gamma,\Delta}: V_{\Delta(0)} \to V_{\Delta(\gamma)}$ is a linear mapping satisfying

$$\text{(24)} \qquad \sup_{w \in W} \left\| K^*_{\gamma,\Delta} g(\cdot, w) \right\| < \infty .$$

Then $T \in \mathbf{L}(A,B)$, and Δ is an index function for T.

PROOF. Since T maps A linearly into B, it suffices to show that T is continuous. From (9) and (3),

$$\begin{aligned} \| T \|_{\gamma,\Delta(\gamma)} &= \sup_{\|x\|_{\Delta(\gamma)} \leqq 1} \ \sup_{w \in W} \left| (K_\gamma T x)(w) \right| \\ &= \sup_{w \in W} \ \sup_{\|x\|_{\Delta(\gamma)} \leqq 1} \left| F_{\Delta(\gamma)}[x, K^*_{\gamma,\Delta} g(\cdot, w)] \right|, \text{ by (23)} \\ &= \sup_{w \in W} \left\| K^*_{\gamma,\Delta} g(\cdot, w) \right\| \qquad \text{by (13)} \\ &< \infty \qquad \text{by (24).} \end{aligned}$$

5. Representations of particular spaces

Let A and B satisfy the hypotheses of Theorem 1; define T by (14). Suppose that (*i*) A is such that V_λ and F_λ are known explicitly, and (*ii*) the subspace $U^*(\Delta)$ of

$$(V_{\Delta(0)}/\sigma_{\Delta(0)}) \times W$$

for which T maps A *onto* B (rather than onto a superspace of B) can be characterized. Then the representation of $\mathbf{L}(A,B)$ can be given explicitly. Theorems 3 to 7 give examples; in them, all functions (unless stated otherwise) are complex-valued, I and J are compact real intervals, P and Q are compact Hausdorff spaces, and V denotes total variation (of a measure). If σ_λ is not mentioned, it is the identity.

THEOREM 3. *$\mathbf{L}(C(P), C(Q))$ is isometric and isomorphic to a space of finite Radon measures $g(\cdot, w)$ on P, where $w \in Q$, such that $g(\cdot, w)$ is weak*-continuous in $w \in Q$, and $\sup_{w \in Q} Vg(\cdot, w)$ is finite. Then $T \in \mathbf{L}(A,B)$ if and only if*

$$\text{(25)} \qquad (Tx)(w) = \int_P x(v)\, dg(v,w) \qquad (x \in C(P), w \in Q)$$

$$\text{(26)} \qquad \| T \| = \sup_{w \in Q} Vg(\cdot, w)$$

PROOF. In Theorem 1, set $A = C(P)$, $B = C(Q)$; $A' = \mathbf{L}(A_\lambda^\sim, \mathbf{C})$, where $\| \cdot \|_\lambda$ is the uniform norm, is congruent to the space V of finite Radon measures on P, and

$$f(x) = F[x, f^*] = \int_P x\, df^* .$$

So (14) and (15) give (25) and (26), with (26) finite; and the requirement that T maps into $C(Q)$ is that g satisfies

$$(27) \qquad \lim_{w \to w_0} \int_P x(\cdot)\,dg(\cdot\,,w) = \int_P x(\cdot)\,dg(\cdot\,,w_0)$$

$(w, w_0 \in Q)$, i.e. the weak*-continuity of $g(\cdot\,,w)$ in w. Conversely, if T is defined by (25), and (26) and (27) hold, then $T \in \boldsymbol{L}(A,B)$ by Theorem 2, since by (27), T maps into $C(Q)$.

THEOREM 4. *If* $1 < p < \infty$ *and* μ *is a measure on* P, *then* $\boldsymbol{L}(L^p_\mu(P), C(Q))$ *is isomorphic and isometric to a space of functions* $g(v \cdot w)$ $(v \in P,\ w \in Q)$ *defined by the properties*:

$$(28) \qquad \sup_{w \in Q} \| g(\cdot\,,w)\|_q < \infty \quad (p^{-1} + q^{-1} = 1;\ \|\cdot\|_q \text{ is the } L^q_\mu(P)\text{-norm})$$

$$(29) \qquad \int_E g(\cdot\,,w)\,d\mu(\cdot) \text{ is continuous in } w \in Q, \text{ for each measurable subset } E \subset P.$$

Then $T \in \boldsymbol{L}(L^p_\mu(P), C(Q))$ *iff*

$$(30) \qquad (Tx)(w) = \int_P x(v)\,g(v,w)\,d\mu(v) \quad (x \in L^p_\mu(P), w \in Q)$$

and $\| T \|$ *is given by the left side of the inequality* (28).

PROOF. In Theorem 1, set $A = L^p_\mu(P)$, $B = C(Q)$; $A' = \boldsymbol{L}(A^\sim_\lambda, \boldsymbol{C})$ is congruent to $L^q_\mu(P)$, with $F[x,f^*] = \int_P x(v) f^*(v)\,d\mu(v) (x \in A)$. So $\boldsymbol{L}(A,B)$ is congruent to a subspace of $L^q_\mu(P) \times Q$, and (14) and (15) give (30) and (28); and (29) follows on substituting the characteristic function of E for $x(\cdot)$ in (30), and requiring that $Tx \in C(Q)$.

Conversely it suffices, by Theorem 2, to show that (28), (29) and (30) imply $Tx \in C(Q)$ if $L^p_\mu(P)$. There is a simple function $\bar{x}$ such that $\| x - \bar{x} \|_p < \varepsilon/(4k)$, where k is the supremum in (28). Let $h(v,w) = g(v,w) - g(v,w_0)$, where $w, w_0 \in Q$. Since $\bar{x}$ is a simple function, (29) requires that $\left|\int_P \bar{x} h\,d\mu\right| < \varepsilon/2$ if $w \in N(w_0)$, a suitable neighbourhood of w_0, depending on ε. Then

$$\begin{aligned} |(Tx)(w) - (Tx)(w_0)| &= \left| \int_P (x - \bar{x}) h\,d\mu + \int_P \bar{x} h\,d\mu \right| \\ &\leqq \|(x - \bar{x})\|_p \| h \|_q + \left| \int_P \bar{x} h\,d\mu \right| \\ &< \varepsilon/(4k) \cdot 2k + \varepsilon/2. \end{aligned}$$

So $T \in C(Q)$.

THEOREM 5. *If* $1 < p < \infty$ *and* μ *is a measure on* P, *then* $\boldsymbol{L}(L^p_\mu(P), E(J))$ *is inductively represented by a space of functions* $g_r(v,w)$ $(v \in P;\ w \in J;\ r = 0, 1, 2 \cdots)$ *having the properties*:

(31) $$\sup_{w \in J} \| g_r(\cdot, w) \|_q < \infty \quad (p^{-1} + q^{-1} = 1; \ \|\cdot\|_q \text{ is the } L^q_\mu(P)\text{-norm})$$

(32) $$[g_r(\cdot, w) - g_r(\cdot, w_0)]/[w - w_0] \to g_{r+1}(\cdot, w_0)$$

in the weak L^p_μ topology on P, as $w \to w_0$. $(w, w_0 \in J)$. Then $T \in \boldsymbol{L}(L^p_\mu(P), E(J))$ iff (for $x \in L^p_\mu(P)$; $w \in J$; $r = 0, 1, 2, \cdots$)

(33) $$(D^r T x)(w) = \int_P x(v) g_r(v, w) d\mu(v),$$

where D is the derivative operator.

REMARKS. The seminorms $\| T \|_r$ (see (16)) equal the expressions on the left of (31), for $r = 0, 1, \cdots$. The Theorem remains true for J replaced by $(-\infty, \infty)$.

PROOF. Let $T \in \boldsymbol{L}(L^p_\mu(P), \boldsymbol{E}(J))$. For $r = 0, 1, 2, \cdots$, the map

$$D: \boldsymbol{E}(J) \to \boldsymbol{E}(J)$$

is continuous; since also $\boldsymbol{E}(J) \subset C(J)$,

$$D^r(T) \in \boldsymbol{L}(L^p_\mu(P), C(J)).$$

So (31) and (33) follow from (28) and (30) of Theorem 5. From (33), if $x \in L^p_\mu(P)$,

(34) $$\frac{(D^r T x)(w) - (D^r T x)(w_0)}{w - w_0} = \int_P x(v) \left[\frac{g_r(v, w) - g_r(v, w_0)}{w - w_0} \right] d\mu(v)$$

Since $Tx \in \boldsymbol{E}(J)$, the left side of (34) $\to (D^{r+1} T x)(w_0)$ as $w \to w_0$; and (32) follows, using (33). From (3) with $K_r = D^r$, (16), and (31), the natural topology for $\boldsymbol{L}(L^p_\mu(P), \boldsymbol{E}(J))$ is that given by the sequence of seminorms $\| T \|_r$ given by the expressions in (31).

Conversely, define T by (33) with $r = 0$, and assume (31) and (32); by Theorem 2, it is required only to verify that $Tx \in \boldsymbol{E}(J)$ if $x \in L^p_\mu(P)$. If (33) holds for some $r \geqq 0$, then so does (34); by (32),

$$\text{the right side of (34)} \to \int_P x(v) g_{r+1}(v, w_0) d\mu(v)$$

as $w \to w_0$ in J; hence so does the left side; so (33) holds for $r + 1$, and, by induction, for all r; so Tx is infinitely differentiable. Now

$$\begin{aligned} \| T \|_r &= \sup_{w \in J} \sup_{\|x\|_r \leqq 1} \left| (D^r T x)(w) \right| \\ & \sup_{w \in J} \sup_{\|x\|_r \leqq 1} \| x \| \cdot \| g_r(\cdot, w) \| \quad \text{by (33)} \\ & \sup_{w \in J} \| g_r(\cdot, w) \\ & < \infty \qquad \text{by (31).} \end{aligned}$$

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So $Tx \in \mathbf{E}(J)$.

REMARKS. Let P be a compact convex subset of Euclidean n-space: let $A = \mathbf{E}(P)$ (see (4) and (5)); let $f \in A'$. Then (compare (6)) there is a seminorm $\|\cdot\|_\lambda$ of A such that f is continuous in $\|\cdot\|_\lambda$. So f extends by continuity to a continuous linear functional on $A_\lambda^\sim$. To each $x \in A$, attach (uniquely) the set of functions $\{x^{(q)}: q \leqq \lambda\}$; this defines an injection j of A_λ into the direct sum S_λ of finitely many (s, say) copies of $C(P)$. Norm S by

$$\max_{q \leqq \lambda} \sup_{t \in P} \left| x^{(q)}(t) \right|.$$

Then $f^\sim(y) = f(j^{-1}y)$ $(y \in jA_\lambda^\sim)$ determines a functional $f^\sim$ on $jA_\lambda^\sim$ with the same norm, $p(f)$ say, that f has as an element of $A_\lambda^{\sim\prime}$. The Hahn-Banach theorem extends $f^\sim$ to a continuous linear functional on S_λ, with the same norm. Then by the Riesz representation theorem, there is a (complex) measure on P^s, represented by measures f_q^* on P, corresponding to the direct summands of S_λ, such that

$$f(x) = \sum_{q \leqq \lambda} \int_P x^{(q)}(v)\, df_q^*(v) \tag{35}$$

$$p(f) = \sum_{q \leqq \lambda} Vf_q^*. \tag{36}$$

This proof is adapted from the representation [5] for Schwartz distributions with compact support P. If f is such a distribution, then it is well known that

$$f(x) = \sum_{q \leqq \lambda'} \int_N x^{(q)}(v) df_q^*(v) \qquad (x \in \mathbf{E}(I)) \tag{37}$$

where N is an arbitrary neighbourhood of P, and I is an interval of $\mathbf{R}^n$, containing P; here the measures f_q^* depend, in general, on the choice of N. However, if P is compact *convex* then Schwartz [7] shows that N may be replaced by P in (37), provided λ' is replaced by λ, where λ/λ' depends on P but not on f.

It follows that, within a topological isomorphism, $E(P)'$ is the space of Schwartz distributions with support in P, and $f \in \mathrm{P}(\mathbf{E})'$ iff f has a representation (35), (36), for some $\lambda \in \Lambda_s$. It is convenient to identify f with the vector $\{f_r^* : r \leqq \lambda\}$ of measures.

THEOREM 6. *Let P be a compact convex subset of Euclidean n-space; let Q be a compact Hausdorff space. Then $\mathbf{L}(E(P), C(Q))$ is topologically represented by a space of elements $g(\cdot,\cdot)$, where for each $w \in Q$,*

$$g(\cdot, w) = \{g_r(\cdot, w): r \leqq \lambda\}$$

is a Schwartz distribution with support in P. If $T \in \mathbf{L}(E(P), C(Q))$ and Δ is the minimal index function for T, then for $\lambda = \Delta(0)$, $w \in Q$,

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$$(38)\qquad (Tx)(w) = \sum_{r \leqq \lambda} \int_P x^{(r)}(v)\,dg_r(v,w) \qquad (x \in E(P));$$

$$(39)\qquad \| T \|_{0,\Delta(0)} = \sup_{w \in Q} p(g(\cdot\,,w)),$$

where $p(\cdot)$ *is defined in* (36);

(40) $(g \cdot, w)$ *is weak *-continuous in* $w \in Q$ *with respect to* $E(P)$.

Conversely, if T *is defined by* (38), *and* (39) *and* (40) *hold, then* $T \in L(E(P), C(Q))$.

REMARK. (40) means that, if $\langle \cdot, \cdot \rangle$ denotes evaluation of a distribution, then for each $x \in E(P)$,

$$\langle g(\cdot\,,w), x(\cdot) \rangle$$

is continuous in $w \in Q$.

PROOF. Set $A = E(P)$, $\Lambda = \Lambda_s$, and $B = C(Q)$ in Theorem 1: if $x \in E(P)$ and $f \in A_\lambda^{\sim\prime}$, then (35) and (36) hold; therefore (38) and (39) follow from Theorem 1; (40) is precisely the condition that T maps into $C(Q)$. The converse is immediate from Theorem 2.

THEOREM 7. *Let* P *and* Q *satisfy the conditions for* P *as in Theorem* 6. *Then* $L(E(P), E(Q))$ *is topologically represented by a space of sequences*

$$\{g_\gamma(\cdot\,,\cdot) \colon \gamma \in \Lambda_s\},$$

where for each $w \in Q$, $g_\gamma(\cdot\,,w)$ *is a Schwartz distribution with support in* P. *If*

$$T \in L(E(P), E(Q))$$

and Δ *is the minimal index function for* T, *then for* $\lambda = \Delta(0)$, $x \in E(P)$, $w \in Q$, $\gamma \in \Lambda_s$,

$$(41)\qquad (D^\gamma Tx)(w) = \sum_{r \leqq \lambda} \int_P x^{(r)}(v)\,dg_{\gamma,r}(v,w) = \langle g_\gamma(\cdot\,,w), x(\cdot) \rangle$$

where

$$g_\gamma(\cdot\,,\cdot) = \{g_{\gamma,r}(\cdot\,,\cdot) \colon r \leqq \lambda\};$$

$$(42)\qquad \| T \|_{\gamma,\Delta(\gamma)} = \sup_{w \in Q} p(g_\gamma(\cdot\,,w) < \infty;$$

(43) $g_\gamma(\cdot\,,w)$ *is weak-*-continuous in* $w \in Q$ *with respect to* $E(P)$;

(44) $g_{\gamma+1}(\cdot\,,w) = (\partial/\partial w)\,g_\gamma(\cdot\,,w)$, *the derivative taken in the weak-* sense on* $E(P)$.

Conversely, if T *is defined by* (41) *with* $\gamma = 0$, *and the* g_γ *satisfy* (42), (43), (44), *and* $\lambda \in \Lambda_s$, *then* $T \in L(E(P), E(Q))$.

REMARK. If w has components w_j, then $\partial g/\partial w$ means the vector with components $\partial g/\partial w_j$; and $h(w,w_0)/(w-w_0)$ means the vector with components $h(w,w_0)/(w-w_0)_j$.

PROOF. If $T \in \mathbf{L}(\mathbf{E}(P),\mathbf{E}(Q))$, and $\gamma \in \Lambda_s$, then $D^\gamma T \in \mathbf{L}(\mathbf{E}(P),C(Q))$ (where D is the differentiation operator). Hence Theorem 6 applies, and (38), (39), (40) prove (41) (for each γ), (42), (43): except that λ may depend on γ as well as on T. From (41), for $w, w_0 \in Q$,

$$(45) \qquad \frac{(D^\gamma Tx)(w)-(D^\gamma Tx)(w_0)}{w-w_0} = \frac{\langle g_\gamma(\cdot,w),x(\cdot)\rangle - \langle g_\gamma(\cdot,w_0),x(\cdot)\rangle}{w-w_0}$$

Since T maps into $\mathbf{E}(Q)$, $(D^\gamma Tx)(w_0)$ exists, so the left side of (45) converges to it as $w \to w_0$, hence so does the right side. Let $\{w_n\} \to w_0$ in Q; let (41) hold for given γ and λ; then

$$\phi_n = (g_\gamma(\cdot,w_n) - g_\gamma(\cdot,w_0))/(w_n - w_0)$$

is a continuous linear mapping from A_λ (where $A = \mathbf{E}(P)$), convergent as $w \to w_0$ to $g_{\gamma+1}(\cdot,w_0)$; by the uniform boundedness principle, $g_{\gamma+1}(\cdot,w_0)$ is also continuous on A_λ; hence λ is independent of γ, and (44) holds. The converse is proved as in Theorem 6.

COROLLARY. *The space $\mathbf{L}(\mathbf{E}(Q)',\mathbf{E}(P)')$ of all continuous linear mappings from Schwartz distributions with support in Q to Schwartz distributions with support in P, where P and Q satisfy the hypotheses of Theorem 7, has the following representation. Let*

$$U \in \mathbf{L}(\mathbf{E}(Q)',\mathbf{E}(P)');$$

let $x \in \mathbf{E}(P)$; let $f \in \mathbf{E}(Q)'$; by (37), *f may be specified in terms of* (*complex*) *measures h_q on Q by*

$$(46) \qquad f(y) = \sum_{q \leq \mu} \int_Q y^{(q)}(w)\, dh_q(w) \qquad (y \in \mathbf{E}(Q)).$$

Then

$$(47) \qquad (Uf)(x) = \sum_{r \leq \lambda} \int_P x^{(r)}(v)\, d_v \left[\int_Q \sum_{q \leq \mu} g_{q,r}(v,w)\, dh_q(w) \right]$$

where the measures $g_{q,r}(\cdot,w)$ satisfy (42), (43), (44). *And conversely, if U is defined by* (47) *then $U \in \mathbf{L}(\mathbf{E}(Q)',\mathbf{E}(P)')$.*

PROOF. Since $\mathbf{E}(P)$ and $\mathbf{E}(Q)$ are reflexive metrisable convex spaces,

$$U \in \mathbf{L}(\mathbf{E}(Q)',\mathbf{E}(P)')$$

iff U is the adjoint of an element $T \in \mathbf{L}(\mathbf{E}(P),\mathbf{E}(Q))$; and

$$(Uf)(x) = f(Tx).$$

Then (47) follows from (41).

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References

[1] N. Bourbaki, *Espaces vectoriels topologiques*, Ch. I & II (1st. edn.).
[2] N. P. Cac, 'Linear Transformations on some functional spaces,' *Proc. Lond. Math. Soc.* (3) **16** (1966), 705–776.
[3] J. Dugundji, *Topology* (Allyn & Bacon, Boston, 1967).
[4] N. Dunford and J. T. Schwartz, *Linear Operators, Part* 1. (Interscience, New York, 1958).
[5] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. 2 (Dunod, Paris, 1964).
[6] A. P. Robertson and W. Robertson, *Topological Vector Spaces* (Cambridge University Press, 1964).
[7] L. Schwartz, *Théorie des distributions* (Hermann, Paris, new edition 1966).

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