ELEMENTARY PROOFS OF THE EXTREMAL PROPERTIES OF THE EIGENVALUES OF THE STURM-LIOUVILLE EQUATION

Paul R. Beesack

(received July 2, 1959)

1. Introduction. If the Sturm-Liouville eigenvalue problem

$$(1.1) \qquad (py')' + (q + \lambda g)y = 0, \; y(a) = y(b) = 0,$$

is first approached from the standpoint of differential equations theory - as opposed, say, to the calculus of variations, or the theory of integral equations - the extremal properties of the eigenvalues seem to be generally regarded as lying beyond the scope of the theory. Thus, neither in the standard work of Bôcher [1], nor in the recent work of Coddington and Levinson [2] is any mention made of this topic. Collatz [3, 166-8] gives an elementary proof of the minimum property of the least positive eigenvalue of (1.1), and a brief indication of how this argument can be extended to the higher eigenvalues. The purpose of this paper is to consolidate this elementary approach, and to extend it to cover the singular cases where either the interval is infinite, or one or more of the coefficients are singular at the end-points.

2. The principal integral inequality. The extremal properties of the eigenvalues of the Sturm-Liouville equation can all be based on the following theorem.

THEOREM 2.1. Let $p(x)$ and $G(x)$ be continuous, and $p(x) > 0$ for $a<x<b$. Suppose the self-adjoint equation

$$(2.1) \qquad (py')' + Gy = 0$$

has a solution $y(x) \neq 0$ on $a<x<b$, and that

Can. Math. Bull. vol. 3, no. 1, Jan. 1960

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

$$(2.2)\quad (x-a)y'(x)/y(x) = O(1) \text{ as } x\to a+,\ (b-x)y'(x)/y(x) = O(1) \text{ as } x\to b-.$$

If both of the conditions

$$(2.3)_1\qquad p(x) = O(x-a) \text{ or } p(x)\int_a^x p^{-1}(t)dt = O(x-a) \text{ as } x\to a+,$$

$$(2.3)_2\qquad p(x) = O(b-x) \text{ or } p(x)\int_x^b p^{-1}(t)dt = O(b-x) \text{ as } x\to b-,$$

hold, then

$$(2.4)\qquad \overline{\lim}_{a'\to a,\ b'\to b} \int_{a'}^{b'} G(x)u^2(x)dx \leq \int_a^b p(x)u'^2(x)dx$$

holds for any function u(x) whose derivative is (absolutely) integrable on $a \leq x \leq b$, and for which

$$(2.5)\qquad u(a) = u(b) = 0,\ \int_a^b pu'^2dx < \infty.$$

Before proving the theorem, we point out that it is valid if a or b, or both, are infinite, provided the order conditions are modified to read

$$x\cdot y'(x)/y(x) = O(1) \quad \text{as } |x| \to \infty,$$

and

$$p(x) = O(x) \quad \text{or} \quad p(x)\int_{-\infty}^x p^{-1}(t)dt = O(x) \text{ as } x\to -\infty,$$

with a similar condition at b, if b is infinite. In the case that a is finite, the first of conditions (2.2) will be assured if y(x) is continuous on $a \leq x \leq b$, and if $\lim_{x\to a+} y'(x)$ exists.

To prove the theorem, set $h = y'/y$, and note that h(x) satisfies the Riccati equation

$$(2.6)\qquad (ph)' = -G - ph^2,\quad a < x < b.$$

Now for any a', b' with $a < a' < b' < b$, consider the integral

$$(2.7)\qquad I(a',b') = \int_{a'}^{b'} p(u' - hu)^2dx$$

$$= \int_{a'}^{b'} pu'^2dx - 2\int_{a'}^{b'} phuu'dx + \int_{a'}^{b'} ph^2u^2dx,$$

where u(x) is any <u>admissible function</u>, i.e., any function satisfying the hypotheses following (2.4). These integrals all exist since $p(x)$ and $h(x)$ are continuous on $a' \leq x \leq b'$. Also, since $p(x) > 0$ on $a < x < b$, it follows that

(2.8) $\quad I(a',b') > 0$ unless $u'(x) \equiv h(x)u(x), \quad a' \leq x \leq b'$,

that is, unless $u(x) \equiv cy(x)$ on $a' \leq x \leq b'$. Moreover if $I(a',b') > 0$ for any fixed a', b', then we clearly have

$$\lim_{a' \to a,\ b' \to b} I(a',b') > 0 ,$$

(possibly $+\infty$). Integrating by parts the second term in the above expansion of $I(a',b')$, and making use of (2.6), (2.7), we obtain

$$I(a',b') = \int_{a'}^{b'} pu'^2 dx - \int_{a'}^{b'} Gu^2 dx - phu^2 \Big|_{a'}^{b'}$$

Hence

(2.9) $$\int_{a'}^{b'} Gu^2 dx \leq \int_{a'}^{b'} pu'^2 dx - phu^2 \Big|_{a'}^{b'} ,$$

where equality holds if, and only if, $u(x) \equiv cy(x)$.

We shall now prove that

(2.10) $\quad \lim_{a' \to a} phu^2 = 0$, and $\lim_{b' \to b} phu^2 = 0$.

We deal with the case $a' \to a$, the other proof being identical. If the first of conditions $(2.3)_1$ holds, then using this and (2.2) we obtain

$$|phu^2| \leq k_1 k_2 u^2$$

for x near $a+$, so that (2.10) follows from (2.5). If the second of conditions $(2.3)_1$ holds, we use the fact that

$$u(x) = \int_a^x u'(t)dt.$$

Hence,

$$|u(x)| \leq \int_a^x |u'| \, dt = \int_a^x (p^{\frac{1}{2}}|u'|).(p^{-\frac{1}{2}})dt$$

$$\leq \left(\int_a^x pu'^2 dt\right)^{\frac{1}{2}} \left(\int_a^x p^{-1} dt\right)^{\frac{1}{2}} ,$$

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

so

$$u^2(x) \leq k_3(x) \int_a^x p^{-1} dt ,$$

where $k_3(x) \to 0$ as $x \to a+$ by the second of conditions (2.5). Thus, using $(2.3)_1$ we obtain

$$|phu^2| \leq k_1 k_4 k_3(x) ,$$

so (2.10) also follows in this case.

Using (2.10), the inequality (2.4) now follows from (2.9) on letting $a' \to a$ and $b' \to b$. By the remarks following (2.8), equality can hold in (2.4) only if $u(x) \equiv cy(x)$. However, $cy(x)$ may not be admissible.

COROLLARY 2.1.1. Equality holds in (2.4) if, and only if, $u(x) \equiv cy(x)$, where $c = 0$ unless $y'(x)$ is absolutely integrable on $a \leq x \leq b$, and

$$y(a) = y(b) = 0, \quad \int_a^b py'^2 dx < \infty. \tag{2.11}$$

COROLLARY 2.1.2. If either $y(x)$ is admissible, or if $G(x) \geq 0$ (but $G \not\equiv 0$), and $y(x)$ is continuous on $a \leq x \leq b$ with $y(a) = y(b)$, and either

$$\int_a^{b'} py'^2 dx = \infty, \text{ or } \int_{a'}^b py'^2 dx = \infty, \tag{2.12}$$

and

$$\overline{\lim}_{x \to a} |pyy'| < \infty, \quad \overline{\lim}_{x \to b} |pyy'| < \infty, \tag{2.13}$$

then the inequality (2.4) is best possible in the sense that the unit constant on the right side of (2.4) cannot be replaced by any smaller factor.

The corollary is obviously valid if $y(x)$ is admissible; hence we suppose $G(x) \geq 0$, $y(a) = y(b)$, and

$$\int_a^{b'} py'^2 dx = \infty ,$$

as well as the conditions (2.13). In this case, let $u(x)$ be defined by

$$u(x) = \begin{cases} 0, & a \leq x \leq a', \\ y(x) - y(a'), & a' \leq x \leq b', \\ 0, & b' \leq x \leq b, \end{cases}$$

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

where a' and b' will be assigned later, and in such a way that $y(a') = y(b')$. This $u(x)$ is an admissible function, and using (2.9) and (2.1) we have

$$\int_{a'}^{b'} Gu^2 dx = \int_{a'}^{b'} G\,[y(x)-y(a')]^2 dx$$

$$= \int_{a'}^{b'} Gy^2 dx - 2y(a') \cdot \int_{a'}^{b'} Gy dx + y^2(a') \cdot \int_{a'}^{b'} G^2 dx$$

$$= \int_{a'}^{b'} py'^2 dx - phy^2 \Big|_{a'}^{b'} + 2y(a') \cdot \int_{a'}^{b'} (py')' dx + y^2(a') \cdot \int_{a'}^{b'} G^2 dx \, .$$

Hence,

$$\int_{a}^{b} Gu^2 dx \geq \int_{a'}^{b'} py'^2 dx - pyy' \Big|_{a'}^{b'} + 2y(a')\,(py')\Big|_{a'}^{b'}$$

$$= \int_{a}^{b} pu'^2 dx - p(a')y(a')y'(a') - p(b')y(b')y'(b') + 2y(a')p(b')y'(b').$$

Given δ $(0 < \delta < 1)$, this last expression will exceed $(1-\delta)\int_a^b pu'^2 dx$, provided

$$\delta \int_{a'}^{b'} py'^2 dx > p(a')y(a')y'(a') + p(b')y(b')y'(b') - 2y(a')p(b')y'(b').$$

We will shortly show that, given a' sufficiently close to a, there exists $b' < b$ such that $y(a') = y(b')$. Assuming this for now, our result will be proved if we can choose a' so that

$$\delta \int_{a'}^{b'} py'^2 dx > p(a')y(a')y'(a') - p(b')y(b')y'(b').$$

Now, the right side of this inequality does not exceed

$$|p(a')y(a')y'(a')| + |p(b')y(b')y'(b')|$$

which, according to (2.13), remains finite as $a' \to a$. Since the left side diverges to $+\infty$ as $a' \to a$, the result follows.

It remains only to prove the assertion relating to $y(a') = y(b')$. Since $y(x)$ is continuous on $a \leq x \leq b$, and $y'(x)$ exists on $a < x < b$, with $y(a) = y(b)$, it follows that there exists α, $a < \alpha < b$ such that $y'(\alpha) = 0$. Moreover, assuming $y(x) > 0$ on $a < x < b$, it follows from the fact that $G(x) \geq 0$ that we have

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

$$y'(x) \geq 0 \quad \text{for} \quad a < x \leq \alpha\ , \ y'(x) \leq 0 \ \text{for} \ \alpha \leq x < b.$$

By the first of conditions (2.12), we cannot have $y(x) \equiv y(\alpha)$ for x near a, so that $y(x)$ is strictly increasing for $a < x < a + \varepsilon$. Hence, if $a < a' < a + \varepsilon$ we have $y(a) < y(a') < y(\alpha)$. It then follows from $y(\alpha) > y(a') > y(b)$ that there exists b' on $\alpha < x < b$ with $y(b') = y(a')$.

A similar proof shows that the corollary is valid when the second of conditions (2.12) holds.

Before leaving this section, we want to point out that hypotheses (2.2), (2.3) could be discarded provided the class of admissible functions for (2.4) satisfied the following conditions:

$$(2.14) \quad \begin{cases} u(x) \in C, \ a < x < b; \\ u'^2(x) \text{ is integrable for every interval } [a', b'] \subset (a, b); \\ \lim_{x \to a+} phu^2 = \lim_{x \to b-} phu^2 < \infty; \\ \int_a^b pu'^2 dx < \infty . \end{cases}$$

For this class of admissible functions, (2.4) would be a strict inequality unless $u(x) \equiv cy(x)$, where $c = 0$ unless

$$(2.15) \quad \lim_{x \to a+} pyy' = \lim_{x \to b-} pyy', \text{ and } \int_a^b py'^2 dx < \infty.$$

Moreover, corollary 2.1.2 remains valid for the admissible class (2.14).

3. <u>Extremal</u> <u>Properties</u>. We consider the Sturm-Liouville problem

$$(3.1) \qquad (py')' + (q + \lambda g)y = 0, \quad y(a) = y(b) = 0,$$

where we assume that p, q, g are continuous for $a < x < b$ $(-\infty \leq a < b \leq \infty)$, and $p(x) > 0$ on $a < x < b$. Throughout this section we shall assume the existence of a sequence $\lambda_1, \lambda_2, \ldots$ of eigenvalues of (3.1), and a corresponding sequence $y_1, y_2, \ldots$ of eigenfunctions, each of which has only a finite number of zeros on $a < x < b$. The simple boundary conditions of (3.1) may often be replaced by less stringent conditions (which we shall note later in connection with some singular problems).

THEOREM 3.1. Suppose that $p(x)$ satisfies conditions (2.3), that the eigenfunctions $y_n(x)$ of (3.1) satisfy condition (2.2) and that $y_n'(x)$ is absolutely integrable on $a \leq x \leq b$, with $\int_a^b py_n'^2 dx < \infty$. If $\lambda_m \neq \lambda_n$, then

$$\int_a^b g y_m y_n dx = 0.$$

Proof. Using the fact that y_m, y_n satisfy (3.1) for $\lambda = \lambda_m$, λ_n respectively, we obtain

$$(3.2) \qquad (\lambda_n - \lambda_m) \int_{a'}^{b'} g y_n y_m dx = y_n(py_m') \Big|_{a'}^{b'} - y_m(py_n') \Big|_{a'}^{b'}$$

in the usual way. We now prove, more or less as in Theorem 2.1, that

$$(3.3) \qquad \lim_{a' \to a} p(a')y_n(a')y_m'(a') = 0,$$

with a corresponding result as $b' \to b$, whence the desired conclusion will follow from (3.2).

The proof of (3.3) makes essential use of the boundary conditions $y_n(a) = y_n(b) = 0$, as well as conditions (2.2) and (2.3). On the other hand, the orthogonality conclusion of the theorem follows from (3.2) even if some or all of these conditions are not satisfied, provided

$$(3.4) \qquad \lim_{x \to a+} p(y_n y_m' - y_m y_n') = \lim_{x \to b-} p(y_n y_m' - y_m y_n') = 0.$$

THEOREM 3.2. Let p, q, g, y_n, λ_n, satisfy the hypotheses assumed in the preceding theorem. Suppose y_n has consecutive zeros at $x_0, x_1, \ldots, x_{k+1}$, where $a = x_0 < x_1 < \ldots < x_{k+1} = b$. Let $u(x)$ be any function satisfying the conditions:

(α) $u'(x)$ is absolutely integrable on $a \leq x \leq b$;

(β) $u(x_i) = 0$, $0 \leq i \leq k+1$;

(γ) $\int_a^b pu'^2 dx < \infty$, $\int_a^b |q| u^2 dx < \infty$, $0 < \int_a^b gu^2 dx < \infty$.

Then

$$(3.5) \qquad \lambda_n \leq \left(\int_a^b pu'^2 dx - \int_a^b qu^2 dx\right) \Big/ \int_a^b gu^2 dx.$$

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

Moreover, equality can hold in (3.5) only if

$$(3.6) \qquad u(x) \equiv c_i y_n(x), \quad x_{i-1} \leq x \leq x_i, \quad 1 \leq i \leq k+1.$$

To prove this result, we will apply theorem 2.1 to the successive subintervals $x_{i-1} < x < x_i$. By our assumptions, the hypotheses of this theorem are satisfied at $x_o = a$ and $x_{k+1} = b$. At an interior x_i, (2.2) is also satisfied since $y_n'(x_i)$ exists. (cf. the remark preceding the proof of theorem 2.1.) Moreover, since $p(x)$ is continuous and positive at an interior x_{i-1}, we have

$$0 < k_{i-1} \leq p(x) \leq K_{i-1}$$

in a neighbourhood of x_{i-1}. Hence

$$p(x)\int_{x_{i-1}}^{x} p^{-1}(t)dt \leq K_{i-1}\int_{x_{i-1}}^{x} k_{i-1}^{-1}dt = O(x-x_{i-1}).$$

Similarly, $p(x)$ satisfies the second of conditions $(2.3)_2$ at x_i.

Since $u(x)$ also satisfies all the hypotheses of theorem 2.1 on the subintervals $x_{i-1} \leq x \leq x_i$, we have

$$\int_{x_{i-1}}^{x_i} \{q + \lambda_n g\}\, u^2dx \leq \int_{x_{i-1}}^{x_i} pu'^2dx, \quad 1 \leq i \leq k+1.$$

By Corollary 2.1.1 the equality sign holds here only if (3.6) is satisfied.

In accordance with the remarks concerning (2.14), (2.15) we may drop the boundary conditions $y_n(a) = y_n(b) = 0$, as well as the conditions (2.2) and (2.3) for $y_n(x)$ and $p(x)$ respectively, provided the class of admissible functions $u(x)$ satisfy the following conditions:

$$(3.7) \begin{cases} u(x) \in C, \quad a < x < b; \\ u'^2(x) \text{ is integrable for every interval } [a', b'] \subset (a,b); \\ \lim_{x \to a+} py_n'y_n^{-1}u^2 = \lim_{x \to b-} py_n'y_n^{-1}u^2 = 0; \\ u(x_i) = 0, \quad 1 \leq i \leq k; \\ \int_a^b pu'^2dx < \infty, \ \int_a^b |q|u^2dx < \infty, \ 0 < \int_a^b gu^2dx < \infty. \end{cases}$$

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

For the class of admissible functions satisfying (3.7), equality can hold in (3.5) only if u(x) satisfies (3.6) where $c_1 = c_{k+1} = 0$ unless

(3.8) $$\lim_{x \to a+} p y_n y_n' = \lim_{x \to b-} p y_n y_n' = 0.$$

An example of a singular problem which is included in this latter formulation is the Legendre equation

$$\{(1-x^2)y'\}' + \lambda y = 0, \quad y(\pm 1) \text{ bounded},$$

with eigenvalues $\lambda_n = n(n+1)$, $n \geq 0$, and corresponding eigenfunctions $P_n(x)$ having n zeros on $-1 < x < 1$. In this case, the third of conditions (3.7) is satisfied if u(x) is bounded near $x = \pm 1$, while condition (3.8) is also satisfied. Here our conclusion is that if u(x) is any admissible function (in particular, u(x) should be bounded near $x = \pm 1$, and u(x) should vanish where $P_n(x) = 0$), then

$$n(n+1) \leq \int_{-1}^{1} (1-x^2)u'^2 dx \Big/ \int_{-1}^{1} u^2 dx ,$$

equality holding only if u(x) is a (piecewise) multiple of $P_n(x)$ on the subintervals between successive zeros of $P_n(x)$.

THEOREM 3.3. Under the preceding hypotheses on p, q, g, and on the eigenfunctions $y_n(x)$ corresponding to the eigenvalue λ_n of (3.1), suppose in addition, that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$, and that $y_n(x)$ has $k_n \leq n-1$ zeros on the open interval $a < x < b$. Finally, we assume that

$$\int_a^b p y_i'^2 dx < \infty, \quad \int_a^b |q| y_i^2 dx < \infty, \quad 0 < \int_a^b g y_i^2 dx < \infty, \quad 1 \leq i \leq n.$$

Let U(x) be any function satisfying the hypotheses (α), (γ) of theorem 3.2, as well as the orthogonality conditions

(3.9) $$\int_a^b g y_i U dx = 0, \quad i = 1, 2, \ldots, n-1.$$

Then

(3.10) $$\lambda_n \leq \left(\int_a^b p U'^2 dx - \int_a^b q U^2 dx\right) \Big/ \int_a^b g U^2 dx \text{ for } n \geq 2.$$

Moreover, equality holds in (3.10) if, and only if $U(x) \equiv c y_n(x)$, $a \leq x \leq b$.

We begin the proof by constructing a function $u(x)$ which satisfies the hypotheses of theorem 3.2. To this end, set

$$(3.11) \qquad u(x) = \sum_{i=1}^{n-1} c_i y_i(x) - U(x).$$

Suppose the zeros of $y_n(x)$ are $a = x_o < x_1 < \ldots < x_k < x_{k+1} = b$, where $k \leq n-1$. Since $u(a) = u(b) = 0$ for any c_i, $u(x)$ will satisfy (β) if and only if the c_i can be chosen so that

$$(3.12) \qquad \sum_{i=1}^{n-1} c_i y_i(x_j) = U(x_j), \quad j = 1, 2, \ldots, k.$$

If all $U(x_j) = 0$, we may take all $c_i = 0$; in this case (3.10) is a consequence of theorem 3.2. Hence we may assume that (3.12) is a non-homogeneous system. We defer until later the proof that this system has a (necessarily non-trivial) solution.

With $u(x)$ - assumed defined by (3.11), (3.12) - satisfying (β), we observe that condition (α) of theorem 3.2 is also satisfied by $u(x)$ in virtue of our hypotheses concerning the $y_i(x)$. It remains to verify (γ); in doing so we shall note certain auxiliary results. From (3.11) we have

$$\int_a^b pu'^2dx = \sum_1^{n-1} c_i^2 \int_a^b py_i'^2dx + \int_a^b pU'^2dx +$$

$$+ 2 \sum_{i \neq j} c_i c_j \int_a^b py_i' y_j' dx$$

$$- 2 \sum_1^{n-1} c_i \int_a^b pU'y_i'dx.$$

However,

$$\int_{a'}^{b'} py_i'y_j'dx = py_i'y_j \Big|_{a'}^{b'} - \int_{a'}^{b'} y_j(py_i')'dx$$

$$= py_i'y_j \Big|_{a'}^{b'} + \int_{a'}^{b'} \{q + \lambda_i g\} y_i y_j dx.$$

As $a' \to a$ and $b' \to b$, the right side of this equation tends to $\int_a^b qy_iy_jdx$ by theorem 3.1 and equation (3.3). This latter integral exists as is easily seen by the Cauchy-Schwarz inequality. Thus we have

$$(3.13) \qquad \int_a^b p y_i' y_j' dx = \int_a^b q y_i y_j dx, \quad i \neq j.$$

Similarly, using (3.9) we obtain

$$(3.14) \qquad \int_a^b p U' y_i' dx = \int_a^b q U y_i dx, \qquad 1 \leq i \leq n-1.$$

(Here we used the conditions $U(a) = U(b) = 0$.) We now have

$$\int_a^b p u'^2 dx$$

$$(3.15) = \sum_1^{n-1} c_i^2 \int_a^b p y_i'^2 dx + \int_a^b p U'^2 dx + 2\sum_{i \neq j} c_i c_j \int_a^b q y_i y_j dx$$

$$- 2 \sum_1^{n-1} c_i \int_a^b q U y_i dx < \infty.$$

A similar expansion together with several applications of the Cauchy-Schwarz inequality also gives

$$\int_a^b |q| u^2 dx < \infty.$$

Note that

$$\int_a^b p u'^2 dx - \int_a^b q u^2 dx$$

$$(3.16) \qquad = \sum_1^{n-1} c_i^2 \left\{ \int_a^b p y_i'^2 dx - \int_a^b q y_i^2 dx \right\} + \int_a^b p U'^2 dx - \int_a^b q U^2 dx.$$

Finally, applying theorem 3.1 and (3.9), we obtain

$$(3.17) \qquad \int_a^b g u^2 dx = \sum_1^{n-1} c_i^2 \int_a^b g y_i^2 dx + \int_a^b g U^2 dx,$$

and hence

$$0 < \int_a^b g u^2 dx < \infty,$$

by our hypotheses on U and the y_i.

It now follows from theorem 3.2, (3.16), (3.17) that

$$\lambda_n \leq \frac{\sum_1^{n-1} c_i^2 \left\{ \int_a^b p y_i'^2 dx - \int_a^b q y_i^2 dx \right\} + \int_a^b p U'^2 dx - \int_a^b q U^2 dx}{\sum_1^{n-1} c_i^2 \int_a^b g y_i^2 dx + \int_a^b g U^2 dx} .$$

On the other hand, theorem 3.2 also gives

$$\int_a^b py_i'^2dx - \int_a^b qy_i^2dx = \lambda_i \int_a^b gy_i^2dx, \quad 1 \leq i \leq n-1,$$

so that

$$\lambda_n \leq \frac{\sum_1^{n-1} \lambda_i c_i^2 \cdot \int_a^b gy_i^2dx + \int_a^b pU'^2dx - \int_a^b qU^2dx}{\sum_1^{n-1} c_i^2 \int_a^b gy_i^2dx + \int_a^b gU^2dx}$$

$$= \frac{\sum_1^{n-1} \lambda_i A_i + N}{\sum_1^{n-1} A_i + D},$$

where all $A_i \geq 0$, $D > 0$. Using the fact that $\lambda_i - \lambda_n < 0$, and that at least one $A_i > 0$ (in the case we are considering), this inequality implies that

$$\lambda_n < N/D$$

proving (3.10) with strict inequality.

We now prove that the non-homogeneous system (3.12) has a solution. The matrix of coefficients of these equations is the $k \times (n-1)$ matrix A with elements a_{ij} given by

$$a_{ij} = y_j(x_i).$$

If $r(A) = k$, the system has a solution. Suppose $r(A) = r < k$. Then any $(r+1)$ columns of A are linearly dependent, and there are constants $c_1, \ldots, c_{r+1}$ not all zero such that

$$\sum_{i=1}^{r+1} c_i y_i(x_j) = 0, \quad 1 \leq j \leq k.$$

Define $v(x) = \sum_{i=1}^{r+1} c_i y_i(x)$. Then $v(a) = v(x_1) = \ldots = v(x_k)$ $= v(b) = 0$. As in the details following equation (3.12), one sees that $v(x)$ is an admissible function for the minimum problem (3.5) so that

$$\lambda_n \leq \left(\int_a^b pv'^2dx - \int_a^b qv^2dx\right) \Big/ \int_a^b gu^2dx = Q.$$

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

On the other hand, we now show that, in fact,

$$\lambda_1 \leq Q \leq \lambda_{r+1} \leq \lambda_k < \lambda_n;$$

so that r(A) = k follows from this contradiction. To see this one shows, precisely as in equations (3.13) - (3.17), that

$$Q = \sum_1^{r+1} c_i^2 \lambda_i \int_a^b g y_i^2 dx \Big/ \sum_1^{r+1} c_i^2 \int_a^b g y_i^2 dx$$

$$= \sum_1^{r+1} d_i \lambda_i ,$$

where $d_i = A_i \Big/ \sum_1^{r+1} A_j$, $A_i = c_i^2 \int_a^b g y_i^2 dx$, so that all $d_i \geq 0$, and $\sum_1^{r+1} d_i = 1$.

By hypothesis,

$$\lambda_1 \leq \lambda_i \leq \lambda_{r+1}, \quad 1 \leq i \leq r+1,$$

whence

$$\lambda_1 = \lambda_1 \sum_1^{r+1} d_i \leq \sum_1^{r+1} \lambda_i d_i \leq \lambda_{r+1} \sum_1^{r+1} d_i = \lambda_{r+1},$$

establishing our contradiction.

It only remains to discuss the possibility of equality in (3.10). We have shown this can only occur when U(x) also satisfies condition (β) of theorem 3.2. By this same theorem, equality can then hold in (3.10) only if

(3.18) $$U(x) \equiv c_i y_n(x), \quad x_{i-1} \leq x \leq x_i, \quad 1 \leq i \leq k+1.$$

We now show that the only such U(x) which also satisfy conditions (3.9) have all c_i equal. This will complete the proof of our theorem. Suppose then that U(x) satisfies both (3.18) and (3.9). Then

$$\sum_{i=1}^{k+1} c_i \int_{x_{i-1}}^{x_i} g y_j y_n dx = 0, \quad 1 \leq j \leq n-1.$$

Now,

$$\int_a^b g y_j y_n dx = \sum_{i=1}^{k+1} \int_{x_{i-1}}^{x_i} g y_j y_n dx = 0,$$

by theorem 3.1. Multiplying the last equation by c_1 and substracting from each of the preceding equations gives

(3.19) $$\sum_{i=2}^{k+1} (c_i - c_1) \int_{x_{i-1}}^{x_i} g y_j y_n dx = 0, \quad 1 \leq j \leq n-1.$$

Since $k \leq n-1$, this system of $n-1$ equations has only the trivial solution $c_i - c_1 = 0$, unless the determinant of coefficients of the first k equations is zero. However, if this determinant were zero there would exist constants $a_1, \ldots, a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j \int_{x_{i-1}}^{x_i} g y_j y_n dx = 0, \quad 2 \leq i \leq k+1.$$

Setting $u(x) = \sum_{j=1}^{k} a_j y_j(x)$, this implies

(3.20) $$\int_{x_{i-1}}^{x_i} g y_n u dx = 0, \quad 2 \leq i \leq k+1$$

Hence we have $\int_{x_1}^{b} g y_n u dx = 0$, and since $\int_a^b g y_n u dx = 0$ by Theorem 3.1, we also have

(3.21) $$\int_a^{x_1} g y_n u dx = 0.$$

According to (3.20) and (3.21) we now have

(3.22) $$\int_a^{x_i} g y_n u dx = 0, \quad 1 \leq i \leq k+1.$$

Now, as in theorem 3.1, we have

$$- p y_j y_n' \Big|_a^{x_i} = (\lambda_n - \lambda_j) \int_a^{x_i} g y_n y_j dx, \quad 1 \leq j \leq k, \quad 1 \leq i \leq k+1.$$

Using (3.3) this reduces to

$$- p(x_i) y_n'(x_i) y_j(x_i) / (\lambda_n - \lambda_j) = \int_a^{x_i} g y_n y_j dx.$$

Multiply this equation by a_j and sum over $1 \leq j \leq k$ to obtain

$$- p(x_i) y_n'(x_i) \sum_{j=1}^{k} a_j y_j(x_i) / (\lambda_n - \lambda_j) = \int_a^{x_i} g y_n u dx = 0, \quad 1 \leq i \leq k+1,$$

by (3.22). However, for $1 \leq i \leq k$, we have $p(x_i) > 0$, $y_n'(x_i) \neq 0$, hence

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

(3.23) $$\sum_{j=1}^{k} a_j y_j(x_i)/(\lambda_n - \lambda_j) = 0, \quad 1 \le i \le k$$

By hypothesis, not all $a_j/(\lambda_n - \lambda_j)$ are zero, and hence the determinant $|y_j(x_i)| = 0$. But this is impossible since we have already established that $r(\|y_j(x_i)\|) = k$.

As before, the boundary conditions $y_j(a) = y_j(b) = 0$ may be omitted, as well as condition (2.2) for $y_j(x)$, $1 \le j \le n$, and conditions (2.3) for $p(x)$, provided the $y_j(x)$ satisfy the remaining hypotheses of Theorem 3.3, together with the conditions

(3.24) $$\lim_{x \to a+} p y_n' y_n^{-1} y_j^2 = \lim_{x \to b-} p y_n' y_n^{-1} y_j^2 = 0, \quad 1 \le j \le n-1,$$

(3.25) $$\lim_{x \to a+} p y_i' y_j = \lim_{x \to b-} p y_i' y_j = 0, \quad 1 \le i, j \le n.$$

In this case, the minimum property (3.10) holds for the class of functions $U(x)$ satisfying the conditions (cf. (3.7)):

(3.26) $$\begin{cases} U(x) \in C, \; a < x < b; \\ U'^2(x) \text{ is integrable for every interval } [a', b'] \subset (a,b); \\ \lim_{x \to a+} p y_n' y_n^{-1} U^2 = \lim_{x \to b-} p y_n' y_n^{-1} U^2 = 0; \\ \lim_{x \to a+} p y_j' U = \lim_{x \to b-} p y_j' U = 0; \\ \int_a^b g y_j U dx = 0, \quad 1 \le j \le n-1; \\ \int_a^b p U'^2 dx < \infty, \quad \int_a^b |q| U^2 dx < \infty, \quad 0 < \int_a^b g U^2 dx < \infty. \end{cases}$$

For this class of admissible functions, equality holds in (3.10) if, and only if, $U(x) \equiv c y_n(x)$.

The Legendre eigenvalue problem, as stated previously, satisfies hypotheses (3.24), (3.25), and again the limit conditions of (3.26) are satisfied if $U(x)$ is bounded near $x = \pm 1$. Hence we conclude that if $U(x)$ is any (suitably integrable) function, continuous and bounded on $-1 < x < 1$, such that

$$\int_{-1}^{1} P_j(x) U(x) dx = 0, \quad 0 \le j \le n-1,$$

then

$$n(n+1) \leq \int_{-1}^{1}(1-x^2)U'^2dx \Big/ \int_{-1}^{1}U^2dx ,$$

equality holding only if $U(x) \equiv cP_n(x)$.

As a second singular problem, this time on an infinite interval, we consider the Hermite equation

$$(e^{-x^2}y')' + \lambda e^{-x^2}y = 0, \quad -\infty < x < \infty ,$$

with boundary condition $y = O(|x|^k)$ as $|x| \to \infty$, for some $k > 0$. Here the eigenvalues are $\lambda_n = 2(n-1)$, $n = 1, 2, \ldots$, with corresponding eigenfunctions the Hermite polynomials $y_n = H_{n-1}(x)$ having n-1 zeros on $-\infty < x < \infty$. Conditions (3.24), (3.25) are clearly satisfied, as are the pertinent conditions of (3.26) for functions U(x) satisfying the boundary conditions noted above.

For a final example, consider the equation

$$y'' + \{\lambda - (m^2 - \tfrac{1}{4})x^{-2}\}y = 0, \; y(0) = y(1) = 0, \quad (m \geq \tfrac{1}{2}).$$

The eigenvalues are given [4, p. 325] by $\lambda_n = k_n^2$, $n = 1, 2, \ldots$, where k_n is the n^{th} positive zero of the Bessel function $J_m(x)$. The corresponding eigenfunction is $y_n = x^{\frac{1}{2}}J_m(k_n x)$ with n-1 zeros on $0 < x < 1$. In this case the boundary conditions are satisfied by all y_n, as indeed are all the hypotheses of theorem 3.3. We conclude that

$$k_n^2 \leq \left\{\int_0^1 U'^2dx + \int_0^1 (m^2 - \tfrac{1}{4})x^{-2}U^2dx\right\} \Big/ \int_0^1 U^2dx$$

whenever $\int_0^1 x^{\frac{1}{2}}J_m(k_i x)U(x)dx = 0$ for $i = 1, 2, \ldots, n-1$,

equality holding only if $U = x^{\frac{1}{2}}J_m(k_n x)$.

4. Maximum-minimum properties of the eigenvalues. Theorem 3.2 requires a precise knowledge of the zeros of the n^{th} eigenfunction $y_n(x)$. If, however, we assume that the λ_i and y_i satisfy the additional hypotheses of theorem 3.3, we may avoid this requirement as we now show. Let

$$\mathcal{K}_n = \{u(x) \mid u(x) \text{ has no more than } (n-1) \text{ zeros on } a < x < b; \; u(a) = u(b) = 0\},$$

https://doi.org/10.4153/CMB-1960-010-3 Published online by Cambridge University Press

$$\mathcal{K}_n(u) = \{v(x) \mid v \text{ satisfies } (\alpha), (\gamma) \text{ of theorem 3.2; } v = 0 \text{ if } u = 0\}.$$

Set

(4.1) $$d_n(u) = \inf_{v \in \mathcal{K}_n(u)} \left\{ \left(\int_a^b pv'^2 dx - \int_a^b qv^2 dx \right) \Big/ \int_a^b gv^2 dx \right\}.$$

Then[1)]

(4.2) $$\lambda_n = \sup_{u \in \mathcal{K}_n} d_n(u).$$

To prove this result, we note that since y_n is assumed to have $k \leq n-1$ zeros on $a < x < b$, we have

(4.3) $$\lambda_n = d_n(y_n) \leq \sup_{u \in \mathcal{K}_n} d_n(u),$$

by theorem 3.2. To prove the opposite inequality, let $u \in \mathcal{K}_n$, and suppose the zeros of u are $a, x_1', x_2', \ldots, x_\alpha', b$, where $\alpha \leq n-1$. We now construct a function $v \in \mathcal{K}_n(u)$ whose Rayleigh quotient appearing in (4.1) does not exceed λ_n. This will prove $d_n(u) \leq \lambda_n$ for all $u \in \mathcal{K}_n$, whence the opposite inequality to (4.3) follows. In fact, it suffices to take

$$v(x) = \sum_1^n c_i y_i(x).$$

As in the proof of theorem 3.3, $v \in \mathcal{K}_n(u)$ if the c_i (not all zero) can be chosen so that

$$\sum_{i=1}^n c_i y_i(x_j') = 0, \qquad 1 \leq j \leq \alpha \ (\leq n-1).$$

Such a solution always exists for this homogeneous system. But then, precisely as in the case of the function v(x) of theorem 3.3, it follows that

$$\lambda_1 \leq \left(\int_a^b pv'^2 dx - \int_a^b qv^2 dx \right) \Big/ \int_a^b gv^2 dx \leq \lambda_n,$$

completing the proof of (4.2).

1) This result is attributed by R. Courant [4, p. 463, footnote] to K. Hohenemser.

A second maximum-minimum characterization of λ_n is due to R. Courant [4, p. 406]. We formulate it as follows: Let $\mathcal{L}_n$ be the class of all (n-1)-tuples $\{v_1(x), \ldots, v_{n-1}(x)\}$ of functions such that $\int_a^b g v_i^2 dx < \infty$, $1 \leq i \leq n-1$. Let

$$\mathcal{L}_n(v_1, \ldots, v_{n-1}) = \{u(x) \mid u \text{ satisfies } (\alpha), (\gamma) \text{ of theorem 3.2;}$$

$$u(a) = u(b) = 0;$$

$$\int_a^b g u v_i dx = 0, \quad 1 \leq i \leq n-1\},$$

and set

$$(4.4) \quad d_n(v_1, \ldots, v_{n-1})$$

$$= \inf_{u \in \mathcal{L}_n(v_1, \ldots, v_{n-1})} \left\{ \left(\int_a^b p u'^2 dx - \int_a^b q u^2 dx \right) \Big/ \int_a^b g u^2 dx \right\}.$$

Then

$$(4.5) \qquad \lambda_n = \sup_{(v_1, \ldots, v_{n-1}) \in \mathcal{L}_n} d_n(v_1, \ldots, v_{n-1}).$$

For, by theorem 3.3 we have $d_n(y_1, \ldots, y_{n-1}) = \lambda_n$, so that

$$\lambda_n \leq \sup_{(v_1, \ldots, v_{n-1}) \in \mathcal{L}_n} d_n(v_1, \ldots, v_{n-1}).$$

Now for any $(v_1, \ldots, v_{n-1}) \in \mathcal{L}_n$, define the function

$$u(x) = \sum_1^n c_i y_i(x),$$

where the c_i are any non-trivial solution of the n-1 homogeneous equations

$$\int_a^b g u v_j dx = \sum_{i=1}^n c_i \int_a^b g v_j y_i dx = 0, \quad 1 \leq j \leq n-1.$$

Then $u \in \mathcal{L}_n(v_1, \ldots, v_{n-1})$, and as before $d_n(v_1, \ldots, v_{n-1}) \leq \lambda_n$, completing the proof of (4.5).

REFERENCES

1. M. Bôcher, Leçons sur les méthodes de Sturm, (Paris, 1917).

2. E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, (New York, 1955).

3. L. Collatz, Eigenwertprobleme und ihre numerische Behandlung, (New York, 1948).

4. R. Courant, and D. Hilbert, Methods of Mathematical Physics, vol. 1, (New York, 1953).

McMaster University